The Pigeonhole Principle

Western PA ARML Spring 2017
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Warm-Up

Prove that given five points inside an equilateral triangle of side length 2, there are two points within distance at most 1 of one another.

Problems

Please submit a full proof for one of the following problems.

1. There are 10 people sitting at a table, and none of them are in their spot. Show that there is a way to rotate the table such that at least two people are in their spot.

2. Prove that if 38 integers are selected from the range 1-999, then two of them differ by less than 27.

3. Let $S \subseteq \{1, 2, \ldots, 2n\}$ be a subset of $n+1$ elements. Show that there are $a, b \in S$ such that the greatest common divisor of a, b is 1.

4. There are 100 senators, and each one hates exactly 3 other senators. What is the least number n for which the senators can always be split into n groups such that no one hates another member of his group? (Note: you must show that it is always possible with n groups, and not always possible with $n-1$ groups).

5. (USAMTS 2006) Each point in the plane is colored red, green, or blue. Show that there is a rectangle whose vertices are the same color.

6. Show that any sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ which is either strictly increasing or strictly decreasing. (For example, the sequence $4, 2, 5, 1, 3$ contains the strictly decreasing subsequence $4, 2, 1$).

7. Given a real number x and a natural number N, show that there are integers p, q with $1 \leq q \leq N$ such that $|x - \frac{p}{q}| < \frac{1}{qN}$. (Note: this is the Dirichlet approximation theorem, intuitively it says that any real number can be approximated ‘pretty well’ by a ’simple’ rational number, namely one with a ‘small’ denominator).

Homework

Please submit a full proof for one of the following problems at the beginning of next week's practice.

1. Let S be a set of n lattice points in the plane. How big does n need to be to guarantee that there is a pair of points whose midpoint has integer coordinates?
2. During a month with 30 days a baseball team plays at least a game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.
3. (Putnam 1958) Let $S \subseteq \{1, 2, \ldots, 2n\}$ be a subset of $n + 1$ elements. Show that there are $a, b \in S$ such that a divides b.