Modern Methods in the Calculus of Variations: L^p Spaces, First edition
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In this file we collect some additional material and comments that we receive from readers.

- The following modification of Theorem 2.112 for $p = 1$ was proposed and written by Nguyen Huy Chieu. With respect to the original proof, it has more details and is more precise in terms of precise representatives. See also the file "Corrections and Improvements" for additional comments.

Theorem. *Let $(\Omega, \mathcal{A}, \mu)$ be a measured space with a $\sigma-$finite measure, and let E be a separable Banach space. The following holds.*

(i) If $L \in (L_1(\Omega; E))^*$ then there exists a unique $v \in L^w_\infty(\Omega; E^*)$ such that

$$L(u) = \int_\Omega \langle v(\omega), u(\omega)\rangle d\mu \tag{0.1}$$

for all $u \in L_1(\Omega; E)$. Moreover, $\|L\| = \|v\|_{L^w_\infty(\Omega;E^*)}$.

(ii) Every functional of the form (0.1), where $v \in L^w_\infty(\Omega; E^*)$, is linear and continuous on $L_1(\Omega; E)$.

Proof. (i) Let $L \in (L_1(\Omega; E))^*$. Since E is separable, there exists $\{e_n\} \subset E\backslash\{0\}$ such that $\overline{\{e_n\}} = E$. For each $n \in I\!N$,

$$L_n(u) := L(ue_n) \quad \forall u \in L_1(\Omega) = L_1(\Omega; I\!R)$$

is linear and continuous and

$$\|L_n\|_{(L_1(\Omega))^*} = \sup_{u\in L_1(\Omega)\backslash\{0\}} \frac{|L(ue_n)|}{\|u\|_{L_1(\Omega)}} \le \|L\|_{(L_1(\Omega;E))^*}\|e_n\|_E < \infty. \tag{0.2}$$

By the Riesz representation in $L_1(\Omega)$, there exists $v_{e_n} \in L_\infty(\Omega)$ such that

$$L(ue_n) = \int_\Omega v_{e_n}(\omega)u(\omega)d\mu \quad \forall u \in L_1(\Omega). \tag{0.3}$$

We have

$$\|L_n\|_{(L_1(\Omega))^*} = \|v_{e_n}\|_{L_\infty(\Omega)} \le \|L\|_{(L_1(\Omega;E))^*}\|e_n\|_E.$$

Similarly, for each $\alpha, \beta \in \mathbb{Q}$ and $i, j \in I\!N$, there exists a unique $v_{\alpha e_i+\beta e_j} \in L_\infty(\Omega)$ such that

$$L(u(\alpha e_i + \beta e_j)) = \int_\Omega v_{\alpha e_i+\beta e_j}(\omega)u(\omega)d\mu \quad \forall u \in L_1(\Omega).$$

We have

$$\|v_{\alpha e_i+\beta e_j}\|_{L_\infty(\Omega)} \le \|L\|_{(L_1(\Omega;E))^*}\|\alpha e_i + \beta e_j\|_E.$$

Since $v_{\alpha e_i+\beta e_j}$ is unique and L is linear,

$$\alpha v_{e_i}(\omega) + \beta v_{e_j}(\omega) = v_{\alpha e_i+\beta e_j}(\omega) \quad \mu - a.e., \tag{0.4}$$

which implies that

$$\|\alpha v_{e_i} + \beta v_{e_j}\|_{L_\infty(\Omega)} \le \|L\|_{(L_1(\Omega;E))^*}\|\alpha e_i + \beta e_j\|_E. \tag{0.5}$$

Put

$$\tilde{\Omega} = \Big\{\omega \in \Omega \ : \ |\alpha v_{e_i}(\omega) + \beta v_{e_j}(\omega)| \le \|L\|_{(L_1(\Omega;E))^*}\|\alpha e_i + \beta e_j\|_E,$$
$$\alpha v_{e_i}(\omega) + \beta v_{e_j}(\omega) = v_{\alpha e_i + \beta e_j}(\omega) \quad \forall i,j \in \mathbb{N}, \forall \alpha,\beta \in \mathbb{Q}\Big\}.$$

From (0.4) and (0.5) it follows that $\tilde{\Omega} \in \mathcal{A}$ and $\mu(\Omega \backslash \tilde{\Omega}) = 0$. Take any $e \in E$. Since $\{e_n\}$ is dense in E, there exists a sequence $\{e_{n_j}\}$ converging to e. For each $\omega \in \tilde{\Omega}$, $\{v_{e_{n_j}}(\omega)\}$ is a Cauchy sequence in $\mathbb{R}$. Hence $\{v_{e_{n_j}}(\omega)\}$ converges to some $v_e(\omega) \in \mathbb{R}$. Obviously, $v_e(\omega)$ does not depend on the particular sequence $\{e_{n_j}\}$ provided that $\{e_{n_j}\}$ converges to e. Since $\omega \in \tilde{\Omega}$,

$$|v_{e_{n_j}}(\omega)| \le \|L\|_{(L_1(\Omega;E))^*}\|v_{e_{n_j}}\|_E \quad \forall j \in \mathbb{N}.$$

Taking $j \to \infty$, we have

$$|v_e(\omega)| \le \|L\|_{(L_1(\Omega;E))^*}\|e\|_E \quad \forall \omega \in \tilde{\Omega}.$$

Hence $v_e \in L_\infty(\Omega)$. Take any $e, \tilde{e} \in E$. Suppose that sequences $\{e_{n_j}\}$ and $\{e_{\tilde{n}_j}\}$ converge, respectively, to e and $\tilde{e}$. We have

$$|v_{e_{n_j}}(\omega) - v_{e_{\tilde{n}_j}}(\omega)| \le \|L\|_{(L_1(\Omega;E))^*}\|e_{n_j} - e_{\tilde{n}_j}\|_E \quad \forall j \in \mathbb{N}, \ \forall \omega \in \tilde{\Omega}.$$

Letting $j \to \infty$, we have

$$|v_e(\omega) - v_{\tilde{e}}(\omega)| \le \|L\|_{(L_1(\Omega;E))^*}\|e - \tilde{e}\|_E \quad \forall \omega \in \tilde{\Omega}. \tag{0.6}$$

This implies that for each $\omega \in \tilde{\Omega}$ the function $E \ni e \mapsto v_e(\omega)$ is continuous. For any $\alpha, \beta \in \mathbb{R}$, there exists $(\alpha_j, \beta_j) \in \mathbb{Q} \times \mathbb{Q}$ such that $(\alpha_j, \beta_j) \to (\alpha, \beta)$ as $j \to \infty$. Take $\omega \in \tilde{\Omega}$. We have

$$\alpha_j v_{e_{n_j}}(\omega) + \beta_j v_{e_{\tilde{n}_j}}(\omega) = v_{\alpha_j e_{n_j} + \beta_j e_{\tilde{n}_j}}(\omega) \quad \forall j \in \mathbb{N}.$$

Note that the function $E \ni e \mapsto v_e(\omega)$ is continuous, taking $j \to \infty$, we have

$$\alpha v_e(\omega) + \beta v_{\tilde{e}}(\omega) = v_{\alpha e + \beta \tilde{e}}(\omega).$$

Hence for each $\omega \in \tilde{\Omega}$ the function $E \ni e \mapsto v_e(\omega)$ is linear and continuous. Replacing n by n_j in (0.3) and then taking $j \to \infty$ yields

$$L(uy) = \int_\Omega v_e(\omega)u(\omega)d\mu \quad \forall u \in L_1(\Omega). \tag{0.7}$$

Consider the function $v : \Omega \to E^*$ defined by

$$v(\omega) : \quad E \to \mathbb{R}, \quad e \mapsto v_e(\omega).$$

Since $\{e_n\}$ is dense in E and for each $\omega \in \tilde{\Omega}$, $v(\omega)(\cdot)$ is linear and continuous on E,

$$\|v(\omega)\|_{E^*} = \sup_n \frac{|\langle v(\omega), e_n\rangle|}{\|e_n\|_E} = \sup_n \frac{|v_{e_n}(\omega)|}{\|e_n\|_E} \leq \|L\|_{(L_1(\Omega;E))^*}$$

for all i $\omega \in \tilde{\Omega}$. For each $e \in E$ the mapping $\Omega \ni \omega \mapsto \langle v(\omega), e\rangle = v_e(\omega)$ is measurable, thus v is weak star measurable. We have

$$\|v\|_{L^w_\infty(\Omega;E^*)} = \operatorname*{ess\,sup}_\omega \|v(\omega)\|_{E^*} = \operatorname*{ess\,sup}_\omega \left(\sup_n \frac{|v_{e_n}(\omega)|}{\|e_n\|_E}\right) \leq \|L\|_{(L_1(\Omega;E))^*}.$$

By Theorem 2.110(i) and the density of $\{e_n\}$ in E, the class $\mathcal{S}$ of integrable simple functions of the form

$$s = \sum_{i=1}^n \chi_{F_i} c_i e_i, \tag{0.8}$$

where $c_i \in I\!R$ v $F_i \in \mathcal{A}$ $(i = 1, ..., n)$, is dense in $L_1(\Omega; E)$. Hence

$$\|L\|_{(L_1(\Omega;E))^*} = \sup_{s\in\mathcal{S}\setminus\{0\}} \frac{|L(s)\|}{\|s\|_{L_1(\Omega;E)}}.$$

Take any $s \in \mathcal{S}$ of the form (0.8). By (0.3) and H**ö**lder's inequality,

$$\begin{aligned} |L(s)| &= \Big|\sum_{i=1}^n \int_{F_i} c_i v_{e_i} d\mu\Big| \leq \sum_{i=1}^n \int_{F_i} |c_i|\|e_i\|_E \tfrac{|v_{e_i}(\omega)|}{\|e_i\|_E} d\mu \\ &\leq \int_\Omega \left(\sum_{i=1}^n \chi_{F_i}(\omega)|c_i|\|e_i\|_E\right) \sup_k \tfrac{|v_{e_k}(\omega)|}{\|e_k\|_E} d\mu \\ &\leq \|s\|_{L_1(\Omega;E)} \operatorname*{ess\,sup}_\omega \left(\sup_k \tfrac{|v_{e_k}(\omega)|}{\|e_k\|_E}\right). \end{aligned}$$

Hence

$$\|L\|_{(L_1(\Omega;E))^*} = \sup_{s\in\mathcal{S}\setminus\{0\}} \frac{|L(s)|}{\|s\|_{L_1(\Omega;E)}} \leq \operatorname*{ess\,sup}_\omega \left(\sup_k \frac{|v_{e_k}(\omega)|}{\|e_k\|_E}\right) = \|v\|_{L^w_\infty(\Omega;E^*)},$$

and thus

$$\|L\|_{(L_1(\Omega;E))^*} = \|v\|_{L^w_\infty(\Omega;E^*)}.$$

Since for each $\omega \in \tilde{\Omega}$ the mapping $E \ni e \mapsto v_e(\omega)$ is linear,

$$\begin{aligned} L(s) &= \sum_{i=1}^n \int_{F_i} c_i v_{e_i}(\omega) d\mu = \sum_{i=1}^n \int_{F_i} v_{c_i e_i}(\omega) d\mu = \sum_{i=1}^n \int_{F_i} v_{s(\omega)}(\omega) d\mu \\ &= \int_\Omega v_{s(\omega)}(\omega) d\mu = \int_\Omega v(\omega)(s(\omega)) d\mu = \int_\Omega \langle v(\omega), s(\omega)\rangle d\mu. \end{aligned}$$

Take any $u \in L_1(\Omega; E)$. Then there exists $\{s_j\} \subset \mathcal{S}$ such that

$$\lim_{j\to\infty} \|s_j - u\|_{L_1(\Omega;E)} = 0.$$

We have

$$L(u) = \lim_{j\to\infty} L(s_j) = \lim_{j\to\infty} \int_\Omega v_{s_j(\omega)}(\omega) d\mu.$$

From (0.6) it follows that

$$|v_{s_j(\omega)}(\omega) - v_{u(\omega)}(\omega)| \leq \|L\|_{(L_1(\Omega;E))^*} \|s_j(\omega) - u(\omega)\|_E$$

for all $\omega \in \tilde{\Omega}$. Hence

$$L(u) = \int_\Omega v_{u(\omega)}(\omega) d\mu = \int_\Omega \langle v(\omega), u(\omega)\rangle d\mu,$$

and (i) is proved.

(ii) Let $v \in L^w_\infty(\Omega; E^*)$ and L is the functional of the form (0.1). We need to prove that L is linear and continuous on $L_1(\Omega; E)$. Indeed, since $v \in L^w_\infty(\Omega; E^*)$, there exists $M > 0$ such tha $\|v(\omega)\|_{E^*} \leq M$ $\mu - a.e.$. Hence

$$\begin{aligned} |L(u)| &= \Big| \int_\Omega \langle v(\omega), u(\omega)\rangle d\mu \Big| \leq \int_\Omega |\langle v(\omega), u(\omega)\rangle| d\mu \\ &\leq \int_\Omega \|v(\omega)\|_{E^*} \|u(\omega)\|_E d\mu \leq M\|u\| \end{aligned}$$

for all $u \in L(\Omega; E)$. Note that $\Omega \ni \omega \mapsto \langle v(\omega), u(\omega)\rangle$ is measurable and L is linear. Hence L is linear and continuous. The proof is complete. $\square$