

Modern Methods in the Calculus of Variations: L^p Spaces, First edition
 I. Fonseca and G. Leoni,
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For the original text we use the color **Red**, for corrections the color **Green**, and for improvements and additions the color **Blue**. Names in brackets refer to the persons who called the error to our attention (to the best of our recollection) or suggested improvements and additions.¹

CHAPTER 1:

p. 8 In Exercise 1.10 the sentence "satisfying property (ii) of Proposition 1.7" should be removed. [Pietro Siorpaes]

Actually we have the following proposition. [Michael Klipper, G.L. see also the reference [Coh93]]

Proposition 1 *Let X be a nonempty set, let $\mathfrak{M} \subset \mathcal{P}(X)$ be an algebra, and let $\mu : \mathfrak{M} \rightarrow [0, \infty]$ be a finitely additive measure such that*

$$\lim_{n \rightarrow \infty} \mu(E_n) = 0 \tag{1}$$

for every decreasing sequence $\{E_n\} \subset \mathfrak{M}$ such that $\bigcap_{n=1}^{\infty} E_n = \emptyset$. Then μ is countably additive.

Proof. Let $\{F_n\} \subset \mathfrak{M}$ be a sequence of mutually disjoint sets such that $\bigcup_{n=1}^{\infty} F_n \in \mathfrak{M}$, and define

$$E_n := \bigcup_{k=n+1}^{\infty} F_k \in \mathfrak{M}.$$

Then by finite additivity we have

$$\begin{aligned} \mu\left(\bigcup_{k=1}^{\infty} F_k\right) &= \mu\left(\bigcup_{k=1}^n F_k\right) + \mu\left(\bigcup_{k=n+1}^{\infty} F_k\right) \\ &= \sum_{k=1}^n \mu(F_k) + \mu(E_n). \end{aligned}$$

Since $\{E_n\} \subset \mathfrak{M}$ is a decreasing sequence and $\bigcap_{n=1}^{\infty} E_n = \emptyset$, by (1) we have that $\mu(E_n) \rightarrow 0$ as $n \rightarrow \infty$, and so letting $n \rightarrow \infty$ in the previous identity yields

$$\mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} \mu(F_k).$$

■

¹G.L. would like to thank all his students in the class Math 21-720, Measure theory and integration, Fall 2007, for their useful comments. The style of this file is inspired by <http://www.hss.caltech.edu/kcb/IDA-Errata.pdf>

- p. 13 and 18 In the decomposition of Propositions 1.22 and 1.26 there's no uniqueness in general. Consider the Lebesgue measure $\mathcal{L}^1 : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty]$ and for every Borel set $E \subset \mathbb{R}$ define

$$\mu(E) := \begin{cases} 0 & \text{if } E \text{ is countable,} \\ \infty & \text{otherwise.} \end{cases}$$

Then $\mu : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty]$ is a measure, and

$$\mathcal{L}^1 + \mu = 0 + \mu.$$

Since \mathcal{L}^1 and the measure identically equal to zero are both nonatomic and σ -finite (and in particular semifinite), we have no uniqueness. [Pietro Siorpaes, Daniel Spector]

- p. 58 After formula (1.42), a cleaner argument would be: Define

$$\bar{u}(x) := \begin{cases} v(x) & \text{for } x \in E, \\ u(x) & \text{elsewhere.} \end{cases}$$

Then \bar{u} is admissible for $\nu_{ac}(X)$, and so

$$\nu_{ac}(X) \geq \int_X \bar{u} d\mu = \int_X u d\mu + \int_E (v - u) d\mu > \int_X u d\mu,$$

where we have used (1.42). [Hang Yu]

- p. 62 To prove uniqueness at the end of Step 1, a simpler argument is the following. Since

$$\infty > \nu(E) = \int_E u d\mu = \int_E v d\mu$$

for every $E \in \mathfrak{M}$, we have that

$$\int_E (v - u) d\mu = 0$$

for every $E \in \mathfrak{M}$, which implies that $u(x) = v(x)$ for μ a.e. $x \in X$ (see, e.g., Remark 1.88).

- p. 63 After formula (1.48), "and since $E_\sigma \cup F$ is admissible in the definition of T we arrive at a contradiction." should be replaced by "and since $E_n \cup F$ is admissible in the definition of T , we have that

$$T = \mu(E_\sigma) \geq \mu(E_n \cup F) = \mu(E_n) + \mu(F) \rightarrow \mu(E_\sigma) + \mu(F),$$

which yields a contradiction." Similarly, three lines after formula (1.48), " $E_\sigma \cup F$ is admissible" should be replaced by " $E_n \cup F$ is admissible". [Michael Klipper]

- p. 109 In the proof of Theorem 1.158 (Lebesgue differentiation theorem) the sentence "by the Besicovitch differentiation theorem" should be replaced by "by the Besicovitch differentiation theorem and Remark 1.154(ii)". [Paolo Piovano]
- p. 127 In Exercise 1.199 " $v = u\chi_X + c\chi_{\{\infty\}}$ " and " $\|v\|_{C(X^\infty)} = \max\{\|u\|_{C_0(X)}, |c|\}$ " should be replaced by " $(v - c)|_X = u$ " and " $\|v - c\|_{C(X^\infty)} = \|u\|_{C_0(X)}$ ", respectively.

CHAPTER 2:

- p. 141 Theorem 2.5 continues to hold for $0 < p < q < \infty$. Note that in (i) by sets of arbitrarily small positive measure we mean that for every $\varepsilon > 0$ there exists $E \in \mathfrak{M}$ such that $0 < \mu(E) \leq \varepsilon$, while in (ii) by sets of arbitrarily large finite we mean that for every $M > 0$ there exists $E \in \mathfrak{M}$ such that $M \leq \mu(E) < \infty$.
- p. 141 Theorem 2.5 can be extended to $q = \infty$. Indeed, we have

Theorem 2 *Let (X, \mathfrak{M}, μ) be a measure space. Suppose that $0 < p < \infty$. Then*

- (i) $L^p(X)$ is not contained in $L^\infty(X)$ if and only if X contains measurable sets of arbitrarily small positive measure;
- (ii) $L^\infty(X)$ is not contained in $L^p(X)$ if and only if $\mu(X) = \infty$.

Proof. (i) Assume that $L^p(X)$ is not contained in $L^\infty(X)$. Then there exists $u \in L^p(X)$ such that

$$\|u\|_{L^\infty} = \infty. \tag{2}$$

For each $n \in \mathbb{N}$ let

$$E_n := \{x \in X : |u(x)| > n\}.$$

Then

$$\mu(E_n) \leq \frac{1}{n^p} \int_X |u|^p d\mu \rightarrow 0$$

as $n \rightarrow \infty$. Thus, it suffices to show that $\mu(E_n) > 0$ for all n sufficiently large. If to the contrary, $\mu(E_n) = 0$ for some n , then we would have that $|u(x)| \leq n$ for μ a.e. $x \in X$, which would contradict (2). Hence, X contains measurable sets of arbitrarily small positive measure.

Conversely, assume that X contains measurable sets of arbitrarily small positive measure. Then it is possible to construct a sequence of pairwise disjoint sets $\{E_n\} \subset \mathfrak{M}$ such that $\mu(E_n) > 0$ for all $n \in \mathbb{N}$ and

$$\mu(E_n) \searrow 0.$$

Let

$$u := \sum_{n=1}^{\infty} c_n \chi_{E_n},$$

where $c_n \nearrow \infty$ are chosen such that

$$\sum_{n=1}^{\infty} c_n^p \mu(E_n) < \infty. \quad (3)$$

Then $u \in L^p(X) \setminus L^\infty(X)$.

(ii) If $\mu(X) < \infty$, then for any $u \in L^\infty(X)$, we have

$$\int_X |u|^p d\mu \leq (\|u\|_{L^\infty})^p \mu(X) < \infty, \quad (4)$$

and so $L^\infty(X) \subset L^p(X)$. Conversely, assume that $\mu(X) = \infty$. Then the function $u \equiv 1$ belongs to $L^\infty(X)$ but not to $L^p(X)$. ■

[Daniel Spector]

- p. 150 In the statement of Vitali's convergence theorem "measurable" should be replaced by "integrable". [Hang Yu]
- p. 168 One line above Exercise 2.43, the "no" should be removed. [Daniel Spector]
- p. 171 In Exercise 2.45(ii) " $-p(u) \leq L(u) \leq p(u)$ " should be replaced by " $L(u) \leq p(u)$ ". [Pietro Siorpaes]
- p. 191 In formula (2.57) " $\frac{\rho}{1+\|\varphi\|_\infty}$ " should be replaced by " $\frac{\rho}{\alpha_N(1+\|\varphi\|_\infty)}$ ". [Rita Gonçalves Ferreira]
- p. 196 Line -6 "Theorem 2.99" should be replaced by "Theorem 2.11". [Rita Gonçalves Ferreira]
- p. 223 In the definition of the spaces $L^p((X, \mathfrak{M}, \mu); Y)$ and $L^\infty((X, \mathfrak{M}, \mu); Y)$ in Definition 2.109, one should consider, as usual, equivalence classes of functions.
- p. 223 Before Definition 2.111, one should add the following definition. "Let (X, \mathfrak{M}, μ) be a measure space and let Y be a Banach space. Two weakly star measurable functions $v, w : X \rightarrow Y'$ are equivalent if for every $y \in Y$, $v(x)(y) = w(x)(y)$ for μ a.e. $x \in X$. Note that the set where the equality fails may depend on y ."
- p. 226 We would like to thank Nguyen Huy Chieu for pointing out that the proof of the identity

$$\|v(x)\|_{Y'} = \sup_n \frac{|v(x)(y_n)|}{\|y_n\|_Y} = \sup_n \frac{|v_{y_n}(x)|}{\|y_n\|_Y} \quad (5)$$

on Line -7 is far from trivial (see also the file Additional Material). The missing details are as follows:

We begin by proving that if Y is separable, then a weakly measurable function $v : X \rightarrow Y'$ belongs to $L_w^\infty(X; Y')$ if and only if there exists a constant $C \geq 0$ such that for every $y \in Y$,

$$|v(x)(y)| \leq C \|y\|_Y \quad (6)$$

for μ a.e. $x \in X$. Note that the set where the equality fails may depend on y . To prove the claim, assume that (6) holds and let $\{y_n\} \subset Y$ be dense in Y . For every $x \in X$, $v(x) \in Y'$, and so by the density of $\{y_n\}$ and the fact that $v(x) : Y \rightarrow \mathbb{R}$ is bounded,

$$\|v(x)\|_{Y'} = \sup_{y \in Y \setminus \{0\}} \frac{|v(x)(y)|}{\|y\|_Y} = \sup_n \frac{|v(x)(y_n)|}{\|y_n\|_Y}. \quad (7)$$

Since the functions $x \in X \mapsto v(x)(y_n)$ are measurable, it follows that the function $x \in X \mapsto \|v(x)\|_{Y'}$ is measurable. Moreover, by (6), for every $n \in \mathbb{N}$ there exist $E_n \in \mathfrak{M}$ with $\mu(E_n) = 0$ such that

$$|v(x)(y_n)| \leq C \|y_n\|_Y \quad \text{for all } x \in X \setminus E_n. \quad (8)$$

Let $E_\infty := \bigcup_{n=1}^\infty E_n$. Then $\mu(E_\infty) = 0$ and by (7), for all $x \in X \setminus E_\infty$,

$$\|v(x)\|_{Y'} \leq C.$$

This shows that $v \in L_w^\infty(X; Y')$ with

$$\|v\|_{L_w^\infty(X; Y')} \leq \inf \{C \geq 0 : \text{property (6) holds}\} =: M_\infty(v).$$

Since $\|v\|_{L_w^\infty(X; Y')}$ is an admissible constant in (6), it follows that $\|v\|_{L_w^\infty(X; Y')} = M_\infty(v)$.

To prove (5), define

$$g(x) := \sup_n \frac{|v_{y_n}(x)|}{\|y_n\|_Y}, \quad x \in X.$$

Fix $y \in Y \setminus \{0\}$. Since $\{y_n\}$ is dense in Y , we may find a subsequence $\{y_{n_j}\}$ converging to y . Using the fact that $v_{y_{n_j}} \rightarrow v_y$ in $L^p(X)$, there exist a subsequence, not relabeled, and a set $F_y \in \mathfrak{M}$ with $\mu(F_y) = 0$, such that $v_{y_{n_j}}(x) \rightarrow v_y(x)$ for all $x \in X \setminus F_y$ as $j \rightarrow \infty$. Hence,

$$\frac{|v(x)(y)|}{\|y\|_Y} = \lim_{j \rightarrow \infty} \frac{|v_{y_{n_j}}(x)|}{\|y_{n_j}\|_Y} \leq g(x),$$

or, equivalently,

$$|v(x)(y)| \leq g(x) \|y\|_Y \quad (9)$$

for all $x \in X \setminus F_y$.

Let $E := \{x \in X : g(x) = 0\}$. Define the function

$$f(x) := \begin{cases} 0 & \text{if } x \in E, \\ \frac{v(x)}{g(x)} & \text{if } x \in X \setminus E. \end{cases}$$

It follows from (9) and the previous remark that $f \in L_w^\infty(X; Y')$ with $\|f\|_{L_w^\infty(X; Y')} \leq 1$. Since

$$\|f\|_{L_w^\infty(X; Y')} = \operatorname{esssup}_{x \in X} \|f(x)\|_{Y'},$$

there exists $E_0 \in \mathfrak{M}$ with $\mu(E_0) = 0$ such that $\|f(x)\|_{Y'} \leq 1$ for all $x \in X \setminus E_0$. Define

$$v_0(x) := g(x) f(x), \quad x \in X.$$

Let $y \in Y$. If $x \in E \setminus F_y$, then $v_{y_n}(x) = 0$ for all $n \in \mathbb{N}$, and so $v(x)(y) = 0$, while $v_0(x)(y) = g(x) f(x)(y) = 0$. On the other hand, if $x \in X \setminus E$, then $v_0(x)(y) = g(x) f(x)(y) = v(x)(y)$. Hence, v_0 is equivalent to v because $v_0(x) = v(x)$ for every $x \in X \setminus F_y$. Since $\|f(x)\|_{Y'} \leq 1$ for all $x \in X \setminus E_0$, we have that

$$\begin{aligned} \sup_{y \in Y \setminus \{0\}} \frac{|v_0(x)(y)|}{\|y\|_Y} &= \sup_{y \in Y \setminus \{0\}} \frac{|g(x) f(x)|}{\|y\|_Y} = g(x) \sup_{y \in Y \setminus \{0\}} \frac{|f(x)|}{\|y\|_Y} \\ &= g(x) \|f(x)\|_{Y'} \leq g(x) \end{aligned}$$

for every $x \in X \setminus E_0$. This shows that $\|v_0(x)\|_{Y'} \leq g(x)$ for all $x \in X \setminus E_0$. To prove the converse inequality, note that if $x \in E$, then there is nothing to prove since $g(x) = 0$ and $v_0(x) = 0$, while if $x \in X \setminus E$, then $v_0(x) = v(x)$, and so

$$\|v_0(x)\|_{Y'} = \|v(x)\|_{Y'} = \sup_{y \in Y \setminus \{0\}} \frac{|v(x)(y)|}{\|y\|_Y} \geq \sup_n \frac{|v(x)(y_n)|}{\|y_n\|_Y} = g(x).$$

Finally, we note that (2.103) continues to hold with $v_0(x)(y)$ in place of $v_y(x)$, precisely

$$L(uy) = \int_X u(x) v_0(x)(y) d\mu(x),$$

because $v_0(x)(y) = v(x)(y)$ for all $x \in X \setminus F_y$ and $\mu(F_y) = 0$.

- p. 226 Line -3 " $L_w^p(X; Y')$ " should be replaced by " $L_w^q(X; Y')$ ".
- p. 227 Lines 10 and 12 " $L_w^p(X; Y')$ " should be replaced by " $L_w^q(X; Y')$ ".

CHAPTER 4:

- p. 256 Line 20 " $z_1, \dots, z_k \in \text{ri}_{\text{aff}}(C)$ " should be replaced by " $z_1, \dots, z_k \in C$ ".
- p. 259 Line 4 " $t \leq f(z)$ " should be replaced by " $t \leq f(v_2)$ ". [Daniel Spector]
- p. 259 Line -1 " b^q " should be replaced by " b^p ". [Daniel Spector]
- p. 279 In Exercise 4.55, " $|z_1|$ " should be replaced by " $|z_2|$ ".
- p. 294 The proof of Proposition 4.75 only shows that $f = g$ in $\text{dom}_e f$. To prove that $f = g$ outside $\text{dom}_e f$, fix $v_0 \in V \setminus \text{dom}_e f$ and $t_0 \in \mathbb{R}$ and find L as in (i). Then

$$\langle v', v \rangle_{V', V} + \alpha_0 t \geq \alpha + \varepsilon \quad \text{for all } (v, t) \in \text{epi } f \text{ and } \langle v', v_0 \rangle_{V', V} + \alpha_0 t_0 \leq \alpha - \varepsilon,$$

and so letting $t \rightarrow \infty$, we obtain $\alpha_0 \geq 0$. If $\alpha_0 > 0$, then we proceed as before to conclude that $g(v_0) \geq t_0$. If $\alpha_0 = 0$, we have

$$\langle v', v \rangle_{V', V} \geq \alpha + \varepsilon \quad \text{for all } v \in \text{dom}_e f \text{ and } \langle v', v_0 \rangle_{V', V} \leq \alpha - \varepsilon,$$

so that

$$0 \geq 2\varepsilon + \langle v', v_0 \rangle_{V', V} - \langle v', v \rangle_{V', V} \quad \text{for all } v \in \text{dom}_e f.$$

By part (i) there exist $w' \in V'$ and $c \in \mathbb{R}$ such that

$$f(v) \geq \langle w', v \rangle_{V', V} + c \quad \text{for all } v \in V.$$

Using the last two inequalities we obtain that for all $t > 0$ and for all $v \in \text{dom}_e f$,

$$\begin{aligned} f(v) &\geq \langle w', v \rangle_{V', V} + c \\ &\geq \langle w', v \rangle_{V', V} + c + t(2\varepsilon + \langle v', v_0 \rangle_{V', V} - \langle v', v \rangle_{V', V}) =: g_t(v). \end{aligned}$$

Since $f = \infty$ outside $\text{dom}_e f$, we have that $f(v) \geq g_t(v)$ for all $t > 0$ and for all $v \in V$. Thus g_t is an admissible affine function. Hence

$$\begin{aligned} \infty = f(v_0) &\geq \sup \{g(v_0) : g \text{ affine continuous, } g \leq f\} \\ &\geq g_t(v_0) = \langle w', v_0 \rangle_{V', V} + c + t2\varepsilon \rightarrow \infty \end{aligned}$$

as $t \rightarrow \infty$.

CHAPTER 5:

- p. 327 A simpler proof of Lemma 5.2. Without loss of generality we may assume that $L = 1$. Since $a_n \rightarrow \infty$, we may construct an increasing sequence $n_k \nearrow \infty$ such that $a_{n_k} \geq 4^k$ for all $k \in \mathbb{N}$. Define

$$b_n := \begin{cases} \frac{1}{2^k} & \text{if } n = n_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{n=1}^{\infty} b_n = \sum_{k=1}^{\infty} \frac{1}{2^k} \leq 1,$$

while

$$\sum_{n=1}^{\infty} a_n b_n \geq \sum_{k=1}^{\infty} a_{n_k} b_{n_k} \geq \sum_{k=1}^{\infty} \frac{4^k}{2^k} = \infty.$$

[Pietro Siorpaes]

CHAPTER 6:

- p. 396 In Step 3 of the proof of Theorem 6.18 the space X is taken to be \mathbb{R}^N , which is the case used in the remainder of the book. For a general metric space, one should use the fact that metric spaces are paracompact (see, e.g., Rudin, Mary Ellen, A new proof that metric spaces are paracompact. Proc. Amer. Math. Soc. 20 (1969), 603) to find a locally finite refinement and then construct a partition of unity subordinated to the refinement (see Michael, Ernest, A note on paracompact spaces. Proc. Amer. Math. Soc. 4 (1953), 831–838). [Giovanni Leoni]
- p. 429 The function g_v introduced in (6.50) belongs to $L^1(E)$ and so in Step 3 on page 431, in place of points of approximate continuity of g_{v_n} we can take Lebesgue points. Hence, (6.59) is not needed and (6.61) should be replaced by

$$\frac{1}{|B(x, r) \cap E|} \int_{B(x, r) \cap E} |g_{v_{n_x}}(y) - g_{v_{n_x}}(x)| dy \leq \varepsilon,$$

and (6.53) now reads

$$\int_{B(x, r) \cap E} g_{v_{n_x}}(y) dy \leq |B(x, r) \cap E| (g_{v_{n_x}}(x) + \varepsilon).$$

In turn the first six lines on page 432 are not needed and in lines -1 up to -5 the factor $(1 - \varepsilon)$ is not needed. [Irene Fonseca]

- p. 433 The definition of the function ψ_k in (6.65) should be changed. For $x \in E$ we set

$$\psi_k(x) := \begin{cases} \sup \{n \in \mathbb{N} : h(x, n) \leq -kn^p\} & \text{if there is } n \in \mathbb{N} \text{ such that} \\ 1 & \begin{array}{l} h(x, n) \leq -kn^p, \\ \text{otherwise,} \end{array} \end{cases} \quad (10)$$

Then after (6.66), we claim that without loss of generality, we may assume that for all $x \in B_k$ there is $n \in \mathbb{N}$ such that $h(x, n) \leq -kn^p$. To see this, we introduce the set

$$B'_k := \{x \in B_k : \text{there is } n \in \mathbb{N} \text{ such that } h(x, n) \leq -kn^p\}.$$

Since

$$\infty = \int_{B_k} (\psi_k)^p dx = \int_{B'_k} (\psi_k)^p dx + 1 |B_k \setminus B'_k|,$$

we have that $\int_{B'_k} (\psi_k)^p dx = \infty$, and so, by replacing B_k with B'_k , the claim is proved. [Irene Fonseca]

- p. 434 Line 5 (and identifying B_k with B'_k as in the previous item), to find the sets E_k we proceed as follows. Write

$$B_{k,m} := \{x \in B_k : \psi_k(x) = m\}.$$

Then

$$\sum_{m=1}^{\infty} m^p |B_{k,m}| = \sum_{m=1}^{\infty} \int_{B_{k,m}} (\psi_k)^p dx = \int_{B_k} (\psi_k)^p dx = \infty,$$

and so there exists $m_k \in \mathbb{N}$ such that

$$\infty > \int_{\bigcup_{i=1}^{m_k} B_{k,i}} (\psi_k)^p dx > \frac{1}{k^2}.$$

Hence, there exists a measurable set $E_k \subset \bigcup_{i=1}^{m_k} B_{k,i}$ such that

$$\int_{E_k} (\psi_k)^p dx = \frac{1}{k^2}.$$

[Irene Fonseca]

- p. 438 Lines -2 and -5 The \leq should be $<$. [Irene Fonseca]

CHAPTER 8:

- p. 520 In the definition of the function φ_t , " $s \leq t + 1$ " should be replaced by " $s \geq t + 1$ ".
- p. 521 We observe that property (8.6) in Definition 8.3 is redundant. Indeed, it follows from the latter part of the proof of Theorem 8.2 that (8.6) is a consequence of having

$$\lim_{n \rightarrow \infty} \int_E h(x) \varphi(v_n(x)) dx = \int_E h(x) \int_{\mathbb{R}^m} \varphi(z) d\nu_x(z) dx$$

for every $h \in L^1(E)$ and $\varphi \in C_0(\mathbb{R}^m)$. Actually it suffices to take $h = 1$ and $\varphi = \varphi_t$, where φ_t is introduced below (8.5).

- p. 523 In part (iii) of Theorem 8.6 the compact set K can be replaced by a closed set as in the reference [Ba89]. The proof in the book continues to work without changes. [Marco Barchiesi]

- p. 528 Line 18 " $\bigcup_{j=1}^{i-i}$ " should be replaced by " $\bigcup_{j=1}^{i-1}$ ". [Daniel Spector]

APPENDIX A1:

- p. 554 In the statement of Corollary A.21(ii), "base" should be replaced by "local base".
- p. 555 In Definition A.25 "a neighborhood that is convex" should be replaced by "a local base of convex neighborhoods".
- p. 555 Line 17 " $p_E : X \rightarrow \mathbb{R}$ " should be replaced by " $p_E : X \rightarrow [0, \infty]$ ".
- p. 557 In the statement of the analytic form of the Hahn–Banach theorem, " $L(x)$ " in the last line should be replaced by " $L_1(x)$ ". [Daniel Spector]
- p. 557 "In the statement of Theorem A.36 the functional L has the additional property that $E \cup F$ is not contained in the hyperplane $\{x \in X : L(x) = \alpha\}$."

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