1) F and G are $\mathcal{C}^1$ at $(0,0,0)$ as sums of such functions ✓ and

$$\begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial z} \end{pmatrix}_{(0,0,0)} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

has determinant $2 \times 3 - 2 \times 1 = 4 \neq 0$ so yes.

✓

2) By the chain rule,

$$z_u = \frac{\partial f}{\partial x}(x,y)\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}(x,y)\frac{\partial y}{\partial w} = 2u\, z_x + 2w\, z_y$$

Similarly $z_w = 2w\, z_x + 2u\, z_y$

so $z_u^2 - z_w^2 = 2(u-w)(z_x - z_y) 2(u+w)(z_x + z_y)$

$\overset{\parallel}{(z_u - z_w)(z_u + z_w)}$

$$= 4(u^2 - w^2)(z_x^2 - z_y^2)$$

□

3) f is $\mathcal{C}^1$ at $(0,0)$ so $\begin{cases} a = \frac{\partial f}{\partial x}(0,0) = \sin(0) = 0 \\ b = \frac{\partial f}{\partial y}(0,0) = (1+0)\cos(0) = 1 \end{cases}$

and $f(\Delta x, \Delta y) - f(0,0) = \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$\Leftrightarrow (1+\Delta x)\sin(\Delta y) = \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ $\quad\left((\varepsilon_2 + 1)\Delta y = \sin(\Delta y) \right)$.

$\parallel$

$\sin(\Delta y) + \Delta x \sin(\Delta y)$

so that $\begin{cases} \varepsilon_1 = \sin(\Delta y) \xrightarrow[(\Delta x, \Delta y) \to (0,0)]{} 0 & ✓ \\ \varepsilon_2 = \underbrace{\frac{\sin(\Delta y)}{\Delta y}}_{\xrightarrow[\Delta y \to 0]{} 1} - 1 \xrightarrow[(\Delta x, \Delta y) \to (0,0)]{} 0 & ✓ \end{cases}$

4. a) Since $\cancel{y}^4 \geq 0$, $|x|^2 \leq (x^6+y^4)^{2/6}$

Since $x^6 \geq 0$, $|y|^3 \leq (x^6+y^4)^{3/4}$

so that $0 \leq |f(x,y)| \leq \dfrac{(x^6+y^4)^{2/6+3/4}}{x^6+y^4} = \underbrace{(x^6+y^4)^{1/12}}_{\substack{\longrightarrow 0 \\ (x,y)\to(0,0) \\ \text{by continuity}}}$

So by the squeeze theorem,

$$f(x,y) \xrightarrow[(x,y)\to(0,0)]{} 0$$

b) By continuity of sin and $(x,y) \mapsto (x+y)^{1/2}$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \sin(0) = 0.$$

5) If $x > 0$, $f(x,y) = x|y|^5$ so $\frac{\partial f}{\partial x}(x,y) = |y|^5$ exists for all $y \in \mathbb{R}$.

If $x < 0$, ——— $-x|y|^5$ ——— $-|y|^5$ ———

At $x = 0$, $\dfrac{f(\Delta x, y) - f(0,y)}{\Delta x} = \dfrac{|\Delta x||y|^5}{\Delta x} = \begin{cases} +|y|^5 & \text{if } \Delta x > 0 \\ -|y|^5 & \text{—— } < 0 \end{cases}$

so exists if and only if $y = 0$.

<u>Conclusion</u>: $\frac{\partial f}{\partial x}$ exists everywhere <u>except when $x=0$ and $y \neq 0$</u>

6) a) $\vec{\nabla} f_{(x,y)} = \begin{pmatrix} 2x(x^2-1) \\ 2(y-2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ if and only if

$$(x = 0 \text{ or } \pm 1) \text{ and } (y=2)$$

↳ critical points $\boxed{(0,2),\ (1,2),\ (-1,2)}$

b) $Hf(x,y) = \begin{pmatrix} 12x^2-4 & 0 \\ 0 & 2 \end{pmatrix}$

At $(0,2)$, $Hf(0,2) = \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix}$, eigenvalues $-4, 2$
↳ <u>saddle point</u>

At $(\pm 1, 2)$, $Hf(\pm 1, 2) = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$, <u>rel. min</u>.

c) $\Delta f(0,2) = -4+2 = -2$.