21-268 – Handout on divergence and rate of area change

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1 Introduction

We saw in class, through the formula

div $\vec{v}(x_0, y_0) =$

$$\lim_{h\to 0^+} \frac{1}{(2h)^2}\left(\int_{y_0-h}^{y_0+h} (v_1(x_0+h,y) - v_1(x_0-h,y))\,\mathrm{d}y + \int_{x_0-h}^{x_0+h} (v_2(x,y_0+h) - v_2(x,y_0-h))\,\mathrm{d}x\right)$$

and the associated picture, that the divergence measures the amount of stretch of small areas by $\vec{v}$ around (x_0, y_0).

Let's see this more precisely, by connecting it to the notion of *determinant*, as we saw a few weeks ago that the (absolute value of the) determinant of the Jacobian of a vector field tells us how much areas are multiplied by. We are going to see that the divergence is actually rather a *rate of increase* for the areas.

2 The flow of a vector field and area change

Let $\vec{v}(x, y) \in \mathbb{E}^2$ define a vector field. If we start from (x, y) and follow $\vec{v}(x, y)$ for a small time t, that defines a function (a flow over time)

$$\vec{F}_t(x, y) = (x, y) + t\vec{v}(x, y)$$

We know from the first lectures, that a good approximation for how much a small area R around (x_0, y_0) gets changed is to multiply R by

$$\left|\det(J\vec{F}_t(x_0, y_0))\right| \tag{1}$$

The connection is the following:

Theorem 2.1.

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(J\vec{F}_t(x_0, y_0))\Big|_{t=0} = \mathrm{div}\vec{v}(x_0, y_0)$$

In other words, the divergence of $\vec{v}$ at (x_0, y_0) is the initial rate of increase of the area multiplication factor of $\vec{F}_t$ near (x_0, y_0).

Proof.

$$J\vec{F}_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 + t\frac{\partial v_1}{\partial x} & t\frac{\partial v_1}{\partial y} \\ t\frac{\partial v_2}{\partial x} & 1 + t\frac{\partial v_2}{\partial y} \end{pmatrix}$$

so that the determinant of the above is

$$\left(1 + t\frac{\partial v_1}{\partial x}\right)\left(1 + t\frac{\partial v_2}{\partial y}\right) - t^2\left(\frac{\partial v_1}{\partial y}\right)\left(\frac{\partial v_2}{\partial x}\right)$$

Taking the derivative in t and evaluating at $t = 0$ it remains exactly

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$$

□

For instance, if $\vec{v}(x, y) = (x, y)$, its divergence is everywhere 2. Thus, for small times t the flow of $\vec{v}$ multiplies areas by a factor $1 + 2t$. On the other hand one can cook an example of a vector field with 0 divergence but such that JF_t does not have a zero determinant, for instance by picking

$$\vec{v}(x, y) = (x^2/2, -xy)$$

The flow of that vector field approximately multiplies areas by a factor of $1-(tx_0)^2$ near a point (x_0, y_0) after a small time t, but observe that initially this factor stays very close to 1 (has value 1 and a 0-derivative at $t = 0$).

Remark 2.2. *Of course, the result is still true in higher dimensions, but we do not have the necessary tool to provide a simple proof yet. It basically relies on the formula*

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(I_n + tM)_{t=0} = \mathrm{Trace}(M)$$

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