21-268 Problem Set 14

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section

April 26 2017

Problem 1

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[a]

The boundary is the curve $\{(x, y, 0) : x^2 + y^2 = 1\}$, or the unit circle, since when $z = 0$ we have the disk $x^2 + y^2 \leq 1$, and depending on the sign of D:

If $D = 0$, then $z = 0$ anyway, so the curve is just the disk. Assuming D is nonzero, if D is positive, then as z increases x and y shrink given that $z = D(1 - x^2 - y^2)$, or $x^2 + y^2 + \frac{z}{D} = 1$; the third term is nonnegative, since if $\frac{z}{D} < 0$ then $x^2 + y^2$ is greater than 1, which cannot be; so since D is positive, z must be nonnegative, and larger z means smaller $x^2 + y^2$. A similar argument works with $D < 0$, where z must be nonpositive else we get $x^2 + y^2 > 1$, and as z gets more negative the quantity $\frac{z}{D}$ gets bigger, which reduces $x^2 + y^2$. So in all cases the surface is bounded by $x^2 + y^2 = 1$.

[b]

Using polar coordinates as a parametrization, we have

$$\vec{r} = \begin{bmatrix} \cos(t) \\ \sin(t) \\ 0 \end{bmatrix} \quad d\vec{r} = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{bmatrix} dt \quad \vec{F} = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{bmatrix}$$

For $t \in [0, 2\pi]$. So

$$\int \vec{F} \cdot d\vec{r} = \int_{2\pi}^{0} (\sin^2(t) + \cos^2(t)) dt = -2\pi$$

Since the orientation is clockwise, not anticlockwise, the integral bounds are reversed.

[c]

First let's calculate the curl of $\vec{F}$:

$$\det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y(z+1) & x(z+1) & 0 \end{bmatrix} = \begin{bmatrix} -\frac{\partial}{\partial z}(x(z+1)) \\ -\frac{\partial}{\partial z}(y(z+1)) \\ \frac{\partial}{\partial x}(x(z+1)) + \frac{\partial}{\partial y}(y(z+1)) \end{bmatrix} = \begin{bmatrix} -x \\ -y \\ 2z+2 \end{bmatrix}$$

Then we parametrize (x, y, z) using the following:

$$\vec{r}(s,t) = \begin{bmatrix} s\cos(t) \\ s\sin(t) \\ D(1-s^2) \end{bmatrix}$$

For $s \in [0,1]$ and $t \in [0, 2\pi]$. So the above curl becomes

$$\begin{bmatrix} -s\cos(t) \\ -s\sin(t) \\ 2D(1-s^2) \end{bmatrix} = \begin{bmatrix} -s\cos(t) \\ -s\sin(t) \\ 2D - 2Ds^2 + 2 \end{bmatrix}$$

And our normal vector is

$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = \begin{bmatrix} \cos(t) \\ \sin(t) \\ -2Ds \end{bmatrix} \times \begin{bmatrix} -s\sin(t) \\ s\cos(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 2Ds^2\cos(t) \\ 2Ds^2\sin(t) \\ s \end{bmatrix}$$

This is the correct direction, since s is nonnegative. The dot product between the cur and normal is

$$-2Ds^3\cos^2(t) - 2Ds^3\sin^2(t) + 2Ds - 2Ds^3 + 2s = -4Ds^3 + 2Ds + 2s$$

$$\int_0^{2\pi}\int_0^1 (-4Ds^3 + 2Ds + 2s)ds\,dt = \int_0^{2\pi} (-D + D + 1)dt = \int_0^{2\pi} dt = 2\pi$$

This is the opposite of 1b, as needed, since by the right-hand rule an upward vertical orientation gives a counterclockwise orientation around the boundary.

Problem 2

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[a]

Both surfaces are smooth and orientable, since their norms are defined and continuous everywhere. Additionally, these norms are all direct, since those on the bottom and top are pointing up, aligning with the right-hand rule on C. Furthermore, C is simple and closed, and since as stated it is their boundary and $\vec{F}$ is C^1, the hypotheses of the Kelvin-Stokes theorem are all fulfilled. That is,

$$\iint_{S_T} \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} = \int_C \vec{F} \cdot d\vec{r} = \iint_{S_B} \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma}$$

[b]

Take $S = S_T \cup S_B$. Note then that

$$\iint_S \vec{F} \cdot d\vec{\sigma} = \iint_{S_T} \vec{F} \cdot d\vec{\sigma} - \iint_{S_B} \vec{F} \cdot d\vec{\sigma}$$

Since the norm of S_B is inward. Now observe that S is closed and smooth, so the hypotheses of the divergence theorem are fulfilled, and thus

$$\iint_S \vec{F} \cdot d\vec{\sigma} = \iiint_R \vec{\nabla} \cdot \vec{F} dV$$

with R the region bounded by S. Since $\vec{F}$ is C^2, we can take both the divergence and the curl, ie $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F})$ is defined, with all the hypotheses regarding continuable differentiability still being fulfilled. So we have that

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{\sigma} = \iint_{S_T} (\vec{\nabla} \times \vec{F}) \cdot d\vec{\sigma} - \iint_{S_B} (\vec{\nabla} \times \vec{F}) \cdot d\vec{\sigma} = \iiint_R \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) dV$$

But by identity we know that the div of the cur is 0, so the rightmost integral is also 0. This implies that

$$\iint_{S_T} (\vec{\nabla} \times \vec{F}) \cdot d\vec{\sigma} - \iint_{S_B} (\vec{\nabla} \times \vec{F}) \cdot d\vec{\sigma} = 0$$

$$\iint_{S_T} (\vec{\nabla} \times \vec{F}) \cdot d\vec{\sigma} = \iint_{S_B} (\vec{\nabla} \times \vec{F}) \cdot d\vec{\sigma}$$