21-268 Problem Set 12

+24

April 11 2017

Problem 1

+7

[a]

We can parametrize C with $y = x = t$ for $t \in [0,1]$. . $\vec{r}(t) = (t,t)$, so the derivative is $(1,1)$ so the derivative factor can be ignored in both differentials. The integral becomes:

$$\int_0^1 2(t)(t^3)dt + \int_0^1 B(t^2)(t^2)dt = (2+B)\int_0^1 t^4 dt$$

$$(2+B)\left(\frac{t^5}{5}\right)_0^1 = (2+B)\frac{1}{5}$$

[b]

This time, the parametrization is $y = x^2 = t$, with $t \in [0,1]$, so

$$\vec{r}(t) = \begin{bmatrix} \sqrt{t} \\ t \end{bmatrix} \qquad \vec{r}\,'(t) = \begin{bmatrix} \frac{1}{2\sqrt{t}} \\ 1 \end{bmatrix}$$

The integral is then:

$$\int_0^1 2\sqrt{t}(t^3)\left(\frac{1}{2\sqrt{t}}\right)dt + \int_0^1 Bt(t^2)dt = (1+B)\int_0^1 t^3 dt$$

$$(1+B)\left(\frac{t^4}{4}\right)_0^1 = (1+B)\frac{1}{4}$$

[c]

Consider the function $F(x,y) = x^2y^3$. Then

$$\vec{\nabla}F = \begin{bmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xy^3 \\ 3x^2y^2 \end{bmatrix}$$

[d]

Since (P, Q) is the gradient of some scalar function $\mathbb{R}^2 \to \mathbb{R}$, this is a conservative vector field with F being the potential, thus these functions are path-independent and any curve from $(0,0)$ to $(1,1)$ will evaluate to the same thing.

Problem 2

+6

Note that the domain of this function is $\mathbb{R}^2$, which is simply connected, and C itself is a simple, closed curve; so the hypotheses of Green's theorem are satisfied (except for the fact that the curve is clockwise; we will have to remember to add a negative sign). Then noting that

$$\vec{F} = \begin{bmatrix} 2xy^3 \\ 2x + 3x^2y^2 \end{bmatrix} \qquad \vec{r} = \begin{bmatrix} x \\ y \end{bmatrix}$$

shows

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (2xy^3)dx + (2x + 3x^2y^2)dy$$

which by Green's theorem is

$$\int \int_R \left(\frac{\partial}{\partial x}(2x + 3x^2y^2) - \frac{\partial}{\partial y}(2xy^3) \right) dA$$

$$\int \int_R ((2 + 6xy^2) - (6xy^2))dA = \int \int_R 2dA$$

which is simply twice the area of the bounded region. Since it is made up of four rectangles of width 1, the area is $1+2+3+4 = 10$, so after doubling and applying the negative sign we find our answer to be -20.

Problem 3

+6

Splitting the terms up a bit, we want to show

$$\int \int_D \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} dA_{xy} = \int_C u \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right)$$

$$\int \int_D \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \right) dA_{xy} = \int_C u \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right)$$

So let $P = u\frac{\partial v}{\partial x}$ and $Q = u\frac{\partial v}{\partial y}$. Then

$$\frac{\partial P}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial xy} \qquad \frac{\partial Q}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial xy}$$

Thus note that the integrand of the left-hand side is equal to $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. The left-hand side is thus

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

which, by Green's theorem, equals

$$\int_C P dx + Q dy = \int_C u\frac{\partial v}{\partial x} dx + u\frac{\partial v}{\partial y} dy$$

which is what we needed to prove.

Problem 4

x5

Like in 2, we have

$$\int_C \vec{f} \cdot d\vec{r} = \int_C (2x + 2xy^2 e^{x^2 y})dx + (3y^2 + (1 + x^2 y)e^{x^2 y})dy$$

We can show that this integral is path-independent by claiming there is some f such that

$$\frac{\partial f}{\partial x} = 2x + 2xy^2 e^{x^2 y} \qquad \frac{\partial f}{\partial y} = 3y^2 + (1 + x^2 y)e^{x^2 y}$$

Note that $\frac{\partial}{\partial x}(e^{x^2 y}) = 2xye^{x^2 y}$. Integrating across the x-variable differential equation gives

$$f = x^2 + ye^{x^2 y} + g(y)$$

By inspection of the y-variable equation, we claim $g(y) = y^3$. So

$$f = x^2 + y^3 + ye^{x^2 y}$$

Since, by the chain rule, $\frac{\partial}{\partial y}(ye^{x^2 y}) = e^{x^2 y} + x^2 ye^{x^2 y} = (1 + x^2 y)e^{x^2 y}$, the y-partial of f is in fact the $\vec{j}$ term, like we need it to be. Since we have found the necessary f, we know not only that this integral is path-independent, but also that it evaluates to $f(c, d) - f(a, b)$ by the two-dimensional extension of the Fundamental Theorem of Calculus, so our answer is

$$c^2 + d^3 + de^{c^2 d} - a^2 - b^3 - be^{a^2 b}$$