[illegible] Andrew 10/14/16

Section F

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21-241 HW#6

1| Standard basis of $M_{nn}(\mathbb{R})$

1) Pick $1 \le r, s \le n$ and PT $(E_{ij}E_{k\ell})_{rs} = 0$ if $r \ne i$ or $s \ne \ell$.

Claim: If $r \ne i$ or $s \ne \ell$, then $(E_{ij}E_{k\ell})_{rs} = 0$

Proof: Assume $r \ne i$ or $s \ne \ell$.

$(E_{ij}E_{k\ell})_{rs} = \sum_{a=1}^{n} (E_{ij})_{ra}(E_{k\ell})_{as}$

Since $(E_{ij})_{ra} = 1$ when $r = i$ and $a = j$, equals 0 elsewhere, and $(E_{k\ell})_{as} = 1$ when $a = k$ and $s = \ell$, equals 0 elsewhere,

$\sum_{a=1}^{n} (E_{ij})_{ra}(E_{k\ell})_{as}$ is non-trivial only when $a = j$ interesting

$\sum_{a=1}^{n} (E_{ij})_{ra}(E_{k\ell})_{as} = (E_{ij})_{rj}(E_{k\ell})_{js}$

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Case 1: $r \ne i$

Then $(E_{ij})_{rj} = 0$

So $(E_{ij}E_{k\ell})_{rs} = (E_{ij})_{rj}(E_{k\ell})_{js} = 0$. ✓

Case 2: $r = i$. $\Rightarrow s \ne \ell$

Then $(E_{k\ell})_{js} = 0$

So $(E_{ij}E_{k\ell})_{rs} = (E_{ij})_{rj}(E_{k\ell})_{js} = 0$. ☐ ✓

2) What is $(E_{ij}E_{k\ell})_{i\ell}$? (dep. on j and k)

$(E_{ij}E_{k\ell})_{i\ell} = \sum_{a=1}^{n} (E_{ij})_{ia}(E_{k\ell})_{a\ell}$ has non-trivial value only when $a = j$

$= (E_{ij})_{ij}(E_{k\ell})_{j\ell}$

$= (E_{k\ell})_{j\ell}$

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$= 1$ if $j = k$, 0 if $j \ne k$ ✓

3) Claim: $E_{ij}E_{k\ell} = \delta_{jk}E_{i\ell}$ where $\delta_{jk} = \begin{cases} 1 & \text{iff } j = k \\ 0 & \text{otherwise} \end{cases}$

Proof: Let $r, s \in [n]$ be arbitrary. WTS $(E_{ij}E_{k\ell})_{rs} = (\delta_{jk}E_{i\ell})_{rs}$

Case 1: $(r \ne i$ or $s \ne \ell)$

By 1) above, $(E_{ij}E_{k\ell})_{rs} = 0$.

3

$(\delta_{jk}E_{i\ell})_{rs} = \delta_{jk}(E_{i\ell})_{rs} = \delta_{jk}(0) = 0$ since $r \ne i$ or $s \ne \ell$.

Thus, $(E_{ij}E_{k\ell})_{rs} = (\delta_{jk}E_{i\ell})_{rs} \Rightarrow (E_{ij}E_{k\ell}) = (\delta_{jk}E_{i\ell})$

Case 2: $\neg(r \ne i$ or $s \ne \ell) \iff r = i$ and $s = \ell$

continued on back. (sorry!) Don't be sorry. More space = better writing

Case 2: $r=i$ and $s=\ell$. WTS $(E_{ij}E_{k\ell})_{rs} = (\delta_{jk}E_{i\ell})_{rs}$

$(E_{ij}E_{k\ell})_{rs} = (E_{ij}E_{k\ell})_{i\ell}$

By 2) earlier, $(E_{ij}E_{k\ell})_{i\ell} = 1$ if $j=k$, 0 if $j\neq k$.

$(\delta_{jk}E_{i\ell})_{rs} = (\delta_{jk}E_{i\ell})_{i\ell} = \delta_{jk}(E_{i\ell})_{i\ell} = \delta_{jk} = 1$ if $j=k$, 0 if $j \neq k$.

Thus $(E_{ij}E_{k\ell})_{rs} = (\delta_{jk}E_{i\ell})_{rs} \Rightarrow E_{ij}E_{k\ell} = \delta_{jk}E_{i\ell}$. $\square$

2) For all $1 \leq i,j \leq n$ and $\lambda \in \mathbb{R}$, $A = I_n + (\lambda-1)E_{ii}$, $B = I_n + \lambda E_{ij}$

1) $C = I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}$

2) A corresponds to multiplying a row by a non-zero scalar. Specifically, multiplying row i by λ.

3 B corresponds to adding a multiple of one row to another. Specifically, adding λ times row j to row i.

C corresponds to interchanging two rows. Specifically, switching rows i and j.

3) $\left[\begin{array}{ccc|ccc} 1 & & & 1 & & \\ & \lambda & & & 1 & \\ & & 1 & & & 1 \end{array}\right] \xrightarrow{R_i \leftarrow \frac{1}{\lambda}R_i} \left[\begin{array}{ccc|ccc} 1 & & & 1 & & \\ & 1 & & & \frac{1}{\lambda} & \\ & & 1 & & & 1 \end{array}\right]$ so $A^{-1} = \begin{bmatrix} 1 & & \\ & \frac{1}{\lambda} & \\ & & 1 \end{bmatrix} = I_n + (\frac{1}{\lambda}-1)E_{ii}$

Claim: A has inverse $A' = I_n + (\frac{1}{\lambda}-1)E_{ii}$

Proof: WTS $AA' = I_n \Leftrightarrow (I_n + (\lambda-1)E_{ii})(I_n + (\frac{1}{\lambda}-1)E_{ii}) = I_n$

Let $r,s \in [n]$ be arbitrary. $(I_n)_{rs} = 1$ if $r=s$, 0 if $r \neq s$.

Consider $[(I_n+(\lambda-1)E_{ii})(I_n+(\frac{1}{\lambda}-1)E_{ii})]_{rs}$.

$= \sum_{a=1}^{n} (I_n+(\lambda-1)E_{ii})_{ra}(I_n+(\frac{1}{\lambda}-1)E_{ii})_{as}$

Since $(I_n)_{ra}$ is non-trivial iff $r=a$, and adding $(\lambda-1)E_{ii}$ to I_n only affects the value at row i and col i, the summation is non-trivial only when $a=r$.

4 $= (I_n+(\lambda-1)E_{ii})_{rr}(I_n+(\frac{1}{\lambda}-1)E_{ii})_{rs}$

Case 1: $r \neq s$. Then $(I_n+(\frac{1}{\lambda}-1)E_{ii})_{rs} = 0$ so $(AA')_{rs} = 0$, if $r \neq s$.

Case 2: $r = s$. Then $(AA')_{rs}$ is now $(I_n+(\lambda-1)E_{ii})_{rr}(I_n+(\frac{1}{\lambda}-1)E_{ii})_{rr}$.

Sub-case 1: $r \neq i$. Then $(I_n+(\lambda-1)E_{ii})_{rr}(I_n+(\frac{1}{\lambda}-1)E_{ii}) = (1)(1) = 1$.

Sub-case 2: $r = i$. Then $(I_n+(\lambda-1)E_{ii})_{rr}(I_n+(\frac{1}{\lambda}-1)E_{ii})$ equals

$(1+(\lambda-1))(1+(\frac{1}{\lambda}-1)) = (\lambda)(\frac{1}{\lambda}) = 1$.

So $(AA')_{rs} = 1$ if $r=s$.

Thus, $(AA')_{rs} = 0$ if $r \neq s$, 1 if $r=s$.

So $(AA')_{rs} = (I_n)_{rs} \Rightarrow AA' = I_n \Rightarrow A$ is invertible with inverse A'. $\square$

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4) $$\left[\begin{array}{ccc|ccc} 1 & & & 1 & & \\ & \ddots & & & \ddots & \\ \lambda & & 1 & & & 1 \end{array}\right] \xrightarrow{R_i \leftarrow R_i - \lambda R_j} \left[\begin{array}{ccc|ccc} 1 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & 1 & -\lambda & & 1 \end{array}\right]$$ so $B^{-1} = \begin{bmatrix} 1 & & \\ & \ddots & \\ -\lambda & & 1 \end{bmatrix} = I_n - \lambda E_{ij}$

I claim B has inverse $B' = I_n - \lambda E_{ij}$

$BB' = (I_n + \lambda E_{ij})(I_n - \lambda E_{ij}) = (I_n - \lambda E_{ij}) + \lambda E_{ij}(I_n - \lambda E_{ij})$ by left distributivity

$= (I_n - \lambda E_{ij}) + \lambda E_{ij} - (\lambda E_{ij})^2 = I_n - (\lambda E_{ij})^2 = I_n - \lambda^2 (E_{ij})^2$

$= I_n - \lambda^2 (E_{ij})(E_{ij}) = I_n - \lambda^2(0)$ by formula (1) because $i \neq j$ ✓

Thus $BB' = I_n$ so B is invertible with inverse B'.

$$\left[\begin{array}{ccc|ccc} 1 & & & 1 & & \\ & 0 & 1 & & \ddots & \\ & 1 & 0 & & & \\ & & 1 & & & 1 \end{array}\right] \xrightarrow{R_i \leftrightarrow R_j} \left[\begin{array}{ccc|ccc} 1 & & & 1 & & \\ & \ddots & & & 0 & 1 \\ & & & & 1 & 0 \\ & & 1 & & & 1 \end{array}\right]$$ so $C' = C$.

I claim C has inverse $C' = C$.

$CC' = CC = (I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})(I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji})$

By formula (1) we know $E_{ab}E_{cd}$ has non-trivial value only when $b=c$, so I will write these non-trivial terms only.

$= (I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) + (-E_{ii} + E_{ii}E_{ii} - E_{ij}E_{ji}) + (-E_{jj} + E_{jj}E_{jj} - E_{jj}E_{ji})$

$+ (E_{ij} - E_{ij}E_{jj} + E_{ij}E_{ji}) + (E_{ji} - E_{ji}E_{ii} + E_{ji}E_{ij})$

$= (I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}) + -E_{ii} + -E_{jj} + E_{ij} + E_{ji}$

$= I_n - 2E_{ii} - 2E_{jj} + 2E_{ij} + 2E_{ji}$

is supposed to $= I_n$ ...

something went wrong. :(

right idea