AMERICAN MATHEMATICAL SOCIETY

Lecture Notes Prepared in Connection with the Summer Institute

on

Axiomatic Set Theory

held at

University of California, Los Angeles

Los Angeles, California

July 10 - August 4, 1967

Sponsored by the National Science Foundation under Grant GP-6698

PUSHING UP THE MEASURABLE CARDINAL

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The following work is done in ZF. Classes are to be understood as given by some formula which defines them, and statements concerning them should be taken as abbreviations for schemas.

1. Let $\underset{\sim}{M}_0 = \langle M_0, R_0, c_0 \rangle$ be a relational system with a binary relation R and a distinguished individual c_0. Its language has $\underset{\sim}{R}$ (name of R_0) and $\underset{\sim}{c}$ (name of c_0) as non-logical constants, it has also the equality sign '='.

Let $f : \underset{\sim}{M}_0 \to \underset{\sim}{M}_1$ be an elementary embedding, $\underset{\sim}{M}_1 = \langle M_1, R_1, c_1 \rangle$. $\underset{\sim}{M}_1$ is said to be definable within $\underset{\sim}{M}_0$ if there are formulas $\underline{u}(x)$, $\underline{r}(x,y)$, $\underline{c}(x)$ in the language of $\underset{\sim}{M}_0$ such that:

$$M_1 = \{x : x \in M_0 \ \& \ \underset{\sim}{M}_0 \models \underline{u}(x)\}$$

$$R_1 = \{\langle x,y \rangle : \underset{\sim}{M}_0 \models \underline{r}(x,y)\}$$

$$c_1 = \text{the unique } x \text{ such that } \underset{\sim}{M}_0 \models \underline{c}(x).$$

The mapping f is definable in $\underset{\sim}{M}_0$ if there is a formula $\underline{f}(x,y)$ so that, for every $x \in M_0$,

$$f(x) = \text{the unique } y \text{ such that } M_0 \models \underline{f}(x,y).$$

We will put $O = \langle \underline{u}, \underline{r}, \underline{e}, \underline{f} \rangle$ and refer to it as a self-extending schema.

The property of being a self-extending schema amounts to the satisfaction of a certain set of sentences in M_0. These are the sentences which state that the binary relation which is defined by $\underline{r}$ is over M_1, that $\underline{c}(x)$ holds exactly for one element, that $\underline{f}$ defines a 1-1 mapping of M_0 into M_1 which maps c to the element satisfying $\underline{c}(x)$, and all the sentences of the form

$$(1)\ \forall x_1, \cdots, x_n, y_1, \cdots, y_n \ \{ \bigwedge_{i=1}^{n} f(x_i, y_i) \rightarrow [\varphi(x_1 \cdots x_n) \longleftrightarrow \varphi^0(y_1 \cdots y_n)]\}$$

Where φ is a formula with the free variables $x_1, \cdots, x_n$ and φ^0 is obtained from φ by simultaneously replacing each $\underset{\sim}{R}(u,v)$ by $\underline{r}(u,v)$, each $\underset{\sim}{c}$ by $\iota v \underline{c}(v)$ (= the unique v such that $\underline{c}(v)$) or any other elimination of that term which amounts to the same, and restricting all quantifiers to $\underline{u}(x)$. Changes of bound variables are to be made, if necessary to avoid clash of variables.

Note that, for all $y_1, \cdots, y_n \in M_1$, we have

$$M_1 \models \varphi(y_1, \cdots, y_n) \Longleftrightarrow M_0 \models \varphi^0(y_1, \cdots, y_n).$$

Consequently, one can apply O to any model M of the same similarity type in which all the above mentioned sentences hold, and get another model with an elementary embedding of M into it. In particular one can apply it to every model which is elementary equivalent to M_0.

OM is the model obtained by applying O to M and $O_M : M \to OM$ is the elementary embedding which is defined by $\underline{f}$.

An elementary embedding $g : M \to N$ induces a natural embedding of OM into ON, which turns out to be just the restriction of g to OM. (Remember that the underlying set of OM is a subset of M.) This is denoted by 'Og'. The following diagram is commutative:

$$\begin{array}{ccc} OM & \xrightarrow{Og} & ON \\ {\scriptstyle O_M}\uparrow & & \uparrow{\scriptstyle O_N} \\ M & \xrightarrow[g]{} & N \end{array}$$

For $k < \omega$ we define $O^k M$ to be the result of iterating O k times. That is:

$$O^0 M = M, \qquad O^{k+1} M = O O^k M.$$

Let I be an ordered set of n elements, $n < \omega$, and let J be a subset of I having k elements. For any given

M, the elementary embedding $O_{J,I} : O^k M \to O^n M$ is defined by induction as follows.

$O_{0,0}$ = the identity mapping of M.

If I has n elements, $n > 0$, let i be its maximal element in the given ordering, and put $I' = I - \{i\}$. There are now two cases.

(a) $i \notin J$. Then $J \subseteq I'$ and $O_{J,I}$ is defined as the composition of the mappings

$$O^k M \underset{O_{J,I'}}{\to} O^{n-1} M \underset{O_{O^n M}}{\to} O^n M.$$

(b) $i \in J$. Then put $J' = J - \{i\}$. The mapping $O_{J',I'} : O^{k-1} M \to O^{n-1} M$ being defined, the mapping $O_{J,I} : O^k M \to O^n M$ is the one induced by it; that is, $O_{J,I} = OO_{J',I'}$.

If $K \subseteq J \subseteq I$, where I is finite and ordered, then one gets:

$$O_{K,I} = O_{J,I} \circ O_{K,J}.$$

The fact that $O_M : M \to OM$ is not onto is equivalent to $M \models \exists y\ [\underline{u}(y) \wedge \forall x \sim \underline{f}(x,y)]$. Hence if it holds in M it holds also in every model which is elementary equivalent to it. In this case O is said to be a proper self-extending schema.

If O is proper then one can show that $O_{K,I} \neq O_{J,I}$ whenever I is a finite ordered set, $K,J \subseteq I$, and $K \neq J$.

Now let $I = \langle I,<\rangle$ be any ordered set. Given M, consider the system of all mappings $0_{K,J}$ where $K \subseteq J$, and J is a finite subset of I with the induced ordering. It forms a directed system. Its direct limit is defined to be $0^I M$ (the I^{th} power of 0 applied to M). Up to an isomorphism $0^I M$ depends only on the order type of I. Thus, we get a definition of $0^\xi M$, where ξ is any order-type, which generalizes $0^k M$ for $k < \omega$. If $J \subseteq I$ and we take J with the induced ordering then one has a natural embedding $0_{J,I} : 0^J M \to 0^I M$, which, again, generalizes the finite case.

If 0 is a proper self-extending schema then each element of I will contribute at least one new element to $0^I(M)$ so that the cardinality of $0^I M$ will be at least that of I. The embeddings $0_{J,I}$ will then be proper whenever $J \neq I$.

Note that one can define $0^I M$ also in the case where I is a class which is given by means of some formula. Our claim concerning the cardinality of $0^I M$ is to be interpreted then by saying that one can define a 1-1 mapping of I into $0^I M$.

OR = class of all ordinals.

Every ordinal is to be identified with the set of its predecessors, ordered in the usual way.

If we put $I = \alpha$, for $\alpha \in OR$, then the definition of $0^\alpha M$ and of the mappings $0_{\beta,\alpha}$, $\beta \leq \alpha$, as given here, is equivalent to the following natural definition:

$$O^0 M = M, \quad O_{0,0} = \text{identity mapping of } M$$

$$O^{\alpha+1} M = O O^\alpha M, \quad O_{\beta,\alpha+1} = O_{O^\alpha M} \circ O_{\beta,\alpha}$$

for $\beta \leq \alpha$, $O_{\alpha+1,\alpha+1}$ = identity mapping of $O^{\alpha+1}M$.

For $\alpha = \cup \alpha > 0$, $O^\alpha M$ = direct limit of $\{O_{\beta,\gamma}\}_{\beta \leq \gamma < \alpha}$. If $\beta < \alpha$ then $O_{\beta,\alpha}$ is the natural embedding of $O^\beta M$ into the direct union, and $O_{\alpha,\alpha}$ = identity mapping of $O^\alpha M$.

The definitions are equivalent in the sense that the systems of models and mappings arising out of them are naturally isomorphic. The exact character of the elements of $O^I M$ depends on the way in which one choses representatives for the direct union.

2. Consider now the system $V = \langle V, \epsilon, c \rangle$ where V is, so to speak, the universe of ZF, ϵ is the membership relation, and c is a non-principal ultrafilter in the family of all subsets of a cardinal μ.

Scott has shown that one can construct formulas, $\underline{u}(x)$, $\underline{r}(x,y)$, $\underline{c}(x)$ and $\underline{f}(x,y)$, so that $\underline{u}(x)$ defines in V a class of representatives for all the sequences of length μ modulo c, $\underline{r}(x,y)$ defines the appropriate binary relation between them, $\underline{c}(x)$ defines the representative of the sequence in which c occurs everywhere, and $\underline{f}(x,y)$ the natural embedding of V into its ultrapower.

In this way one gets a proper self-extending schema $\mathcal{O}$ for the whole universe. Each instance of (1) becomes a theorem of ZF.

By iterating this process any finite number of times one can get formulas $\underline{u}_k(x)$, $\underline{r}_k(x,y)$, $\underline{c}_k(x)$, $\underline{f}_{k,k+1}(x,y)$ which will define the k^{th} universe and the elementary embedding of the k^{th} universe into the $k+1^{th}$. These formulas are given by:

$$\underline{u}_1(x) = \underline{u}(x),\ \underline{r}_1(x,y) = \underline{r}(x,y),\ \underline{c}_1(x) = \underline{c}(x)$$

$$\underline{f}_{0,1}(x,y) = \underline{f}(x,y)$$

$$\underline{u}_{n+1}(x) = \underline{u}_n(x) \wedge (\underline{u}(x))^{\mathcal{O}_n},\ \underline{r}_{n+1}(x,y) = \underline{r}(x) \wedge \underline{r}(y) \wedge (\underline{r}(x,y))^{\mathcal{O}_n},$$

$$\underline{c}_{n+1}(x) = \underline{u}_n(x) \wedge (\underline{c}(x))^{\mathcal{O}_n}$$

$$\underline{f}_{n,n+1}(x,y) = \underline{u}_n(x) \wedge \underline{u}_n(y) \wedge (\underline{f}(x,y))^{\mathcal{O}_n}$$

where $\psi^{\mathcal{O}_n}$ is defined like $\psi^{\mathcal{O}}$, except that $\underline{u}_n$, $\underline{r}_n$, $\underline{c}_n$ are used instead of $\underline{u}$, $\underline{r}$, $\underline{c}$, respectively.

One can also give formulas $f_{k,\ell}(x,y)$ defining the embedding of the k^{th} universe into the ℓ^{th}, and, more generally, for every finite ordered set I and every $J \subseteq I$ one can construct a formula $\underline{f}_{J,I}(x,y)$ which defines the mapping $\mathcal{O}_{J,I}$.

For the sake of completeness one might add:

$$\underline{u}_0(x) = (x = x), \qquad \underline{r}_0(x,y) = x \in y, \qquad \underline{c}_0(x) = (x = \underset{\sim}{c}).$$

Note that the indices k,ℓ and J,I are extralinguistical and not variables which occur in the formulas. Our aim is to make them variables. That is, we have to construct formulas $\underline{u}(k,x)$, $\underline{r}(k,x,y)$, $\underline{c}(k,x)$, $\underline{f}(J,I,x,y)$ so that the sentences corresponding to (2) will be provable in ZF. For example, we want to have:

$$ZF \vdash \forall k < \omega \; \forall x \; [\underline{u}(k+1,x) \longleftrightarrow \underline{u}(k,x) \wedge (\underline{u}(x))^{O(k)}]$$

(where $\psi^{O(k)}$ is obtained from ψ in the same way that ψ^{O_k} is obtained, except that here k is a new variable) and similar statements concerning the other formulas as well as the recursive conditions for $O_{J,I}$.

Now, one cannot use here straightforward induction on k because the $k+1^{\text{th}}$ universe is defined in terms of the whole of the k^{th} universe and both are proper classes. Nevertheless our aim can be achieved in the following way.

Let β be any limit ordinal such that $cf(\beta) > \mu$ (i.e., β has cofinality type $> \mu$). Consider the system $R(\beta) = \langle R(\beta), \in, c \rangle$, where $R(\beta)$ = set of all sets of rank $< \beta$.

All the formulas $\underline{u}$, $\underline{r}$, $\underline{c}$, $\underline{f}$ as well as all their components are absolute for all such $R(\beta)$'s. By this we mean that if $\psi(x_1 \cdots x_n)$ is any component of one of the formulas then:

$$ZF \vdash \forall \beta \{\beta = \cup \beta \wedge cf(\beta) > \mu \rightarrow \forall x_1 \cdots x_n \in R(\beta)[\psi(x_1 \cdots x_n) \longleftrightarrow \psi^{R(\beta)}(x_1 \cdots x_n)]\}$$

where $\psi^{R(\beta)}$ is the relativization of ψ to $R(\beta)$. It is assumed here that the representatives of the equivalence classes of the ultrapower are chosen as the sets of equivalent elements of minimal rank, and that $\underline{u}$, $\underline{r}$, $\underline{c}$, $\underline{f}$ are the usual formalizations of these concepts. The fact that $cf(\beta) > \mu$ insures us that $R(\beta)$ is closed under the formation of sequences of length μ.

One can therefore apply the schema 0 to $R(\beta)$ and get the ultrapower of $R(\beta)$ modulo c. Moreover, since $R(\beta)$ is a set one can use the definitions by induction to construct formulas $\underline{u}(\beta,k,x)$, $\underline{r}(\beta,k,x,y)$, $\underline{c}(\beta,k,x)$, $\underline{f}(\beta,J,I,x,y)$ which define the corresponding notions for the case where we start with $R(\beta)$. Note also that β is a variable. Now put

$$\underline{u}(k,x) = \exists \beta \{\beta = \cup \beta \wedge cf(\beta) > \mu \wedge \underline{u}(\beta,k,x)\}$$

$$\underline{r}(k,x,y) = \exists \beta \{\beta = \cup \beta \wedge cf(\beta) > \mu \wedge \underline{r}(\beta,k,x,y)\}$$

$$\underline{c}(k,x) = \exists \beta \{\beta = \cup \beta \wedge cf(\beta) > \mu \wedge \underline{c}(\beta,k,x)\}$$

$$\underline{f}(J,I,x,y) = \exists \beta \{\beta = \cup \beta = J\beta \wedge cf(\beta) > \mu \wedge \underline{f}(\beta,J,I,x,y)\}$$

With these definitions one can prove as theorems of ZF all the recursive conditions corresponding to (2) as well as the recursive conditions for the mappings $O_{J,I}$.

To show this one uses the above-mentioned absoluteness of all the formulas which are employed, as well as the fact that for limit ordinals γ,β having cofinality types $>\mu$, if $\gamma > \beta$ then $O^kR(\beta)$ is the subsystem of $O^kR(\gamma)$ which is defined in $O^kR(\gamma)$ by: rank $(x) < \beta_k$, where β_k is the image of β under the elementary mapping of $R(\gamma)$ into $O^kR(\gamma)$. The mappings $O_{J,I}$ for the systems obtained from $R(\beta)$ are the restrictions of those obtained by starting from $R(\gamma)$.

Now having defined the k^{th} universe and the mappings $O_{J,I}$ where k, J, I are variables one can go on and define in the usual way the universe O^IV, where I is any ordered class. By this we mean that, given any formula $\pi(x,y)$, which might contain additional free variables as parameters, one can construct formulas $\underline{u}_\pi(x)$, $\underline{r}_\pi(x,y)$, $\underline{c}_\pi(x)$, $\underline{f}_\pi(x,y)$, having as additional variables the parameters of π, and prove in ZF the schema which amounts to the following statement:

If $\pi(x,y)$ is an ordering of its field then $\underline{f}_\pi(u,v)$ is an elementary embedding of V into the universe which is defined by $\underline{u}_\pi$, $\underline{r}_\pi$, and $\underline{c}_\pi$.

Moreover for any two formulas $\pi_1(x,y)$ $\pi_2(x,y)$ and any formula $\rho(u,v)$, one can construct a formula $\underline{f}_{\pi_1,\pi_2,\rho}(u,v)$ and prove in ZF the schema which amounts to the following statement:

If π_i, $i = 1,2$ are orderings of their respective fields and ρ is an isomorphism of the first into the second then $\underline{f}_{\pi_1 \pi_2,\rho}$ is an elementary embedding of the universe V_{π_1} into V_{π_2}.

Similarly, all the other properties of the mappings $0_{J,I}$ where $J \subseteq I$, and $I = \langle I,<\rangle$, can be expressed by means of schemas which are provable in ZF.

In particular, given a formula $\pi(x,y)$ which is assumed to define some ordering, and letting p,q, range over the field of this ordering, one can construct formulas $\underline{u}_\pi(p,x)$, $\underline{r}_\pi(p,x,y)$, $\underline{c}_\pi(p,x)$ and $\underline{f}_\pi(p,q,x,y)$, which define the universes of the form $0^{I_p}V$, where I_p is the class $\{x : x < p\}$, and the embeddings $0^{I_p}V \to 0^{I_q}V$ for $p \leq q$.

All the properties of this system can be proved in ZF. By this, we mean, again, that certain sentences which depend on π are provable. For instance the sentence expressing the following statement:

If π is an ordering of its field, then, for every p in its field which has a successor $p + 1$, and for every x,
$\underline{u}_\pi(p + 1,x) \longleftrightarrow \underline{u}_\pi(p,x) \wedge (\underline{u}(x))^{0_\pi(p)}$,
(where $\psi^{0_\pi(p)}$ is obtained from ψ in the usual way, namely,

replacing every $x \in y$ by $\underline{r}_\pi(p,x,y)$ etc.)

or the statement:

If π is an ordering of its field, then for every q in its field which is a limit element and every y, $\underline{u}_\pi(q,y) \leftrightarrow \exists\, p,x\, [\pi(p,q) \wedge \underline{u}_\pi(p,x) \wedge \underline{f}_\pi(p,q,x,y)]$.

(Actually, the universe $0^{I^{p+1}}V$ is not really the one obtained from $0^{I^p}V$ by an application of 0. Both are defined as direct limits and their elements depend on the way in which the representatives are chosen. However, natural isomorphisms exist here, and they can be obviously defined by a formula having p as a parameter. Thus, we prefer to overlook this point of pedantry).

3. From now on assume that μ is a measurable cardinal and that c is μ-additive. In this case, as it is well known, $0V$ is well-founded.

The schema which corresponds to the following theorem is provable in ZF.

THEOREM. If I is any well-ordered class then $0^I V$ is well-founded.

(Here, by a well-founded relation, we mean a relation which has no descending chain of power $\aleph_0$. Since we assume the axiom of regularity it can be easily proved that in a well-founded class every given non-empty subclass has a minimal element.)

Note that it is enough to prove the theorem for countable I's. If there exists a descending infinite chain in $0^I V$, then, since $0^I V$ is a direct limit of systems $0^k V$, $k < \omega$, every element will be contributed by some $0^J V$ where $J \subseteq I$ and J is finite. Hence a countable descending chain in $0^I V$ would imply an existence of an isomorphic copy of it in some $0^J V$ where $J \subseteq I$ and J is countable.

Note also that if any universe U which is elementary equivalent to V is assumed to be well-founded one can easily deduce the same fact about OU. This is so because the order type of the ordinals in OV can be easily seen to be OR. Hence, one has a formula which defines in V an isomorphism of OV into V. The same formula will define in U an isomorphism of OU into U implying that OU must be well-founded as well.

The only difficult point in the proof (which we will not present here) is to show that if $0^\beta V$ is well-founded for all $\beta < \alpha$, where α is some limit countable ordinal, so is $0^\alpha V$.

From now on assume that $I = \langle I, < \rangle$ is a well-ordered class (that is, we are given a formula, which might also have parameters, and which is assumed to define a well-ordered relation). With no loss of generality we might assume that if I is a set then $I = \alpha$ for some $\alpha \in OR$, and that otherwise OR is an initial segment of I.

Let $V_p = \langle V_p, \epsilon_p, c_p \rangle$ be $_0{}^{I_p}V$ where $I_p = \{x : x < p\}$, $(p \in I)$. In particular, $V_0 = V$.

Let $f_{p,q} : V_p \to V_q$ be the elementary embedding of V_p into V_q, $(p < q)$.

Let $\mu_p = f_{0,p}(\mu)$, that is, μ_p is the cardinal which corresponds to μ in the p^{th} universe, and let OR_p be the class of the ordinals of V_p.

The following results hold:

(I) If $p < q$ then $\langle \mu_p, \epsilon_p \rangle$ is isomorphic to a proper initial segement of $\langle \mu_q, \epsilon_q \rangle$.

(That is a formula with parameters p, q can be constructed, which will define the required isomorphism).

(II) $\{x: x \epsilon_1 \mu_1\}$ has higher power than μ.

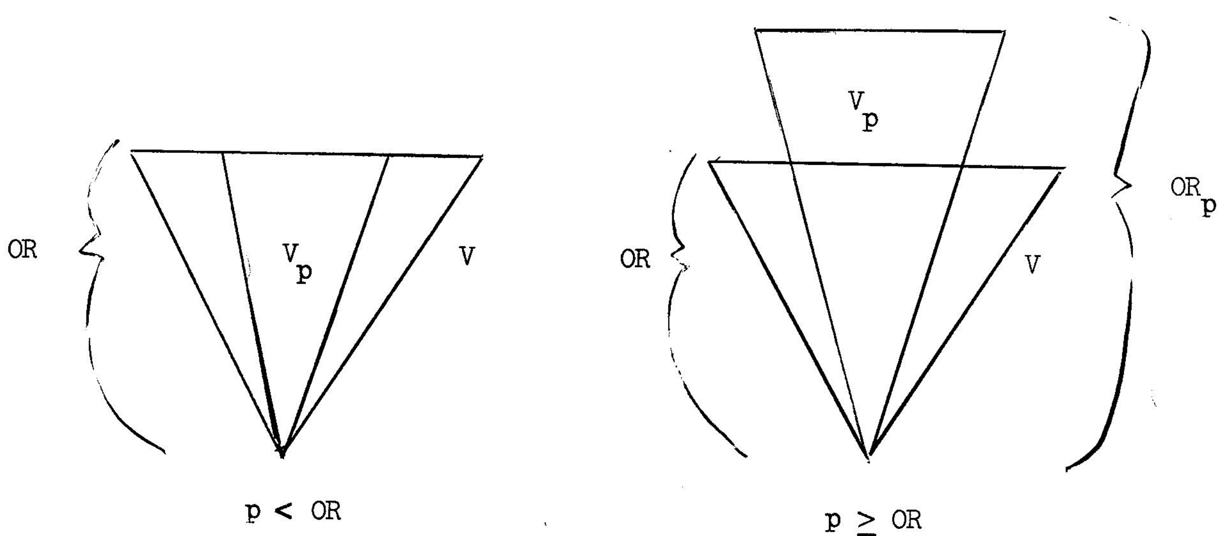
(III) For $0 < p$ the classes $\{x : x \epsilon_p \mu_p\}$ and $\{x : x \epsilon_{p+1} \mu_{p+1}\}$ are equinumerous (that is, a formula having p as a parameter and defining the 1-1 correspondence can be constructed).

(IV) If p is a limit element of I then $\langle \mu_p, \epsilon_p \rangle$ is the union of all its initial segments which are the isomorphic copies of $\langle \mu_q, \epsilon_q \rangle$ where $q < p$.

(V) $\langle OR_\alpha, \epsilon_\alpha \rangle$ is isomorphic to $\langle OR, \epsilon \rangle$ for all $\alpha \in OR$.

Note that if $\{x : x < p\} = OR$ then $\langle \mu_p, \epsilon_p \rangle$ is isomorphic to $\langle OR, \epsilon \rangle$. (This follows from (I), (III), and (IV)). Hence in the OR^{th} universe the measurable cardinal μ is actually equal to OR.

The p^{th} universe in relation to V can be pictured as follows:



However, the mapping $f_{0,p}$ maps OR onto a cofinal subclass of OR_p.

One gets the following theorem:

THEOREM. Given any formula $\pi(x,y)$ one can construct formulas defining a universe V_π and an elementary embedding of V into it, and a formula $\sigma(u,v)$, so that in ZF one can prove:

If $\pi(x,y)$ defines a well-ordering then V_π is well-founded and $\sigma(x,y)$ is an isomorphic embedding of the well-ordered class defined by π into the ordinals of V_π.

Further results can be gotten by considering the systems $R_p(\mu_p) = \langle R_p(\mu_p), \epsilon_p \rangle$ where $R_p(x) = \{y : V_p \models y \in R(x)\}$. These are all models of ZF, and the mappings $f_{p,q}$ map $R_p(\mu_p)$ elementarily into $R_q(\mu_q)$.

For these systems one can give a truth definition in ZF. By considering the constructible part of $R_p(\mu_p)$ one arrives at a theorem which parallels the last one except that now the universe, say, V_π^* satisfies also $V = L$ and a truth definition can be given for it.

Still further results can be obtained by applying O^{I_p} not to V but to some countable elementary submodel of $R(\beta)$ where $\beta = \bigcup \beta$ and $cf(\beta) > \mu$. One still gets systems, say $V_p^\#$, in which $OR_p^\#$ (the class of ordinals) is arbitrary high. The cardinals $\mu_p^\#$ which correspond to μ are now spaced in a much finer way. It turns out that $\{x : x \in_p^\# \mu_p^\#\}$ has the same cardinality as I_p.

Note that such a countable model amounts actually to a certain set of natural numbers. One can start the whole process of expansion and keep it going, just by throwing this set of natural numbers into the constructible universe.