ANNALS OF MATHEMATICAL LOGIC 4 (1972) pp. 229–308. North-Holland

THE FINE STRUCTURE OF THE CONSTRUCTIBLE HIERARCHY *

R. Björn JENSEN

Department of Mathematics, University of California, Berkeley, Calif. 94720, U.S.A.

Received 22 November 1971

Introduction

The theory of the "fine structure" of L is essentially the attempt to elucidate the way the constructible hierarchy grows by examining its behavior at arbitrary levels. A typical question would be: At which $\beta \geq \alpha$ does a new L_β-definable subset of α occur (i.e. $\mathfrak{P}(\alpha) \cap L_{\beta+1} \not\subset L_\beta$)? We find such questions both interesting and important in their own right. Admittedly, however, the questions – and the methods used to solve them – are somewhat remote from the normal concerns of the set theorist. One might refer to "micro set theory" in contradistinction to the usual "macro set theory". Happily, micro set theory turns out to have nontrivial applications in macro set theory. These will be treated in some detail in § 5, 6 and in Silver's note at the end of this paper (§ 7).

We have found it convenient to replace the usual L_α hierarchy by a new hierarchy J_α. We define $J_{\alpha+1}$ not as the collection of definable subsets of J_α but as the closure of $J_\alpha \cup \{J_\alpha\}$ under a class of functions which we call "rudimentary". These are just the functions obtained by omitting the recursion schema from the usual list of schemata for primitive recursive set functions. In a sense they form the smallest class of functions $\mathfrak{R}$ such that there is a smooth definability theory for transitive domains closed under $\mathfrak{R}$. The main difference between the two hierarchies is that J_α has rank $\omega\alpha$ rather than α. However, the subsets of J_α

* The typing of the manuscript was supported by Grant GP # – 27964.

which are elements of $J_{\alpha+1}$ are just the definable ones, $J_{\alpha+1}$ is, so to speak, the result of "stretching" the collection of the definable subsets of J_α upwards ω levels in rank without adding new ones. The exact correspondence between the two hierarchies is given by:

$$J_0 = L_0 = \emptyset;\ L_{\omega+\alpha} = V_{\omega+\alpha} \cap J_{1+\alpha}\,.$$

Thus $J_\alpha = L_\alpha$ whenever $\omega\alpha = \alpha$.

§1 develops the theory of rudimentary functions.[1*] §2 defines the hierarchy J_α and develops its elementary properties – including the basic lemmas on admissible ordinals. §3 proves the Σ_n uniformisation lemma: Every $\Sigma_n(J_\alpha)$ relation is uniformisable by a $\Sigma_n(J_\alpha)$ function.

§4 extends the results of §3. §5 uses the results of §4 to prove some combinatorial principles in L. §6 then gives characterisations of weak compactness in L. Specifically, it is shown that if $V = L$ and η is regular, then weak compactness is equivalent to each of the following:

(i) If $A \subset \eta$ is stationary in η, then $A \cap \beta$ is stationary in β for some $\beta < \eta$.

(ii) The η Souslin Hypothesis.

(iii) Any of the partition properties

$$\eta \to (\eta)^2_{\gamma,\tau} \quad (\tau < \gamma < \eta)\,.$$

(The last is proved by showing that any Souslin tree can be partitioned so as to violate the principles (iii); Martin showed this for $\gamma = 3$, $\tau = 2$. The full theorem was proved by Soare). An appendix written by Jack Silver (§7) uses a theorem of §5 to show that the gap-one form of the two cardinals conjecture holds at singular cardinals in L.[2] His proof is shorter and more elegant than my original one.

To my knowledge, the first to study the fine structure of L for its own sake was Hilary Putman who, together with his pupil George Boolas, first proved some of the results in §3. An account of their work can be found in [1]. For a lucid account of the basic properties of L, the reader is referred to [6]. For admissible ordinals (and the related theory of primitive recursive set functions), see [5]. The model theoretic lemmas used in Silver's note can be found in [3].

* For notes see p. 308.

The material in this paper first appeared in a sequence of handwritten notes: "SH = weak compactness in L", "The Σ_n uniformisation lemma", "A note on the two cardinals problem". I am grateful to many people who struggled through these notes and gave me the benefit of their comments. I am particularly grateful to Silver and Solovay for several fruitful discussions. My deepest thanks go to Joseph Rebholz who, in addition to proofreading this paper, read it in manuscript form and made invaluable comments.

§0. Preliminaries

Consider a first order language $\mathcal{L}$ with the predicates $\equiv$ (identity), $\in$ (membership). We add other predicates as necessary. In addition to the usual symbols of first order predicate logic, we suppose $\mathcal{L}$ to contain *bounded quantifiers* $\bigwedge x \in y$, $\bigvee x \in y$ (thus e.g. $\bigwedge x \in y\ \varphi$ means the same as $\bigwedge x (x \in y \to \varphi)$). We call a formula Σ_0 (or Π_0) if it contains no unbounded quantifiers. For $n \geq 1$ we call φ a $\Sigma_n(\Pi_n)$ formula if it has the form: $\bigvee x_1 \bigwedge x_2 \bigvee x_3 \ldots \overset{\vee}{\wedge} x_n\ \psi$ $(\bigwedge x_1 \bigvee x_2 \bigwedge x_3 \ldots \overset{\wedge}{\vee} x_n\ \psi)$, where ψ is Σ_0. We shall deal with structures of the form $M = \langle |M|; =, \in, A_1, \ldots, A_n \rangle$, where $|M|$ is a domain of sets and $A_1, \ldots, A_n$ are relations on $|M|$. Since the first two predicates are fixed, we shall generally write: $M = \langle |M|; A_1, \ldots, A_n \rangle$. Let $n \geq 0$. By $\Sigma_n(M)(\Pi_n(M))$ we mean that set of relations which are M-definable from arbitrary parameters in M by a $\Sigma_n(\Pi_n)$ formula. If we wish to be specific about the parameters, we write: R is $\Sigma_n(M)$ in the parameters $p_1, \ldots, p_m$. [3] We set:

$$\Delta_n = \Sigma_n \cap \Pi_n \ ; \ \Sigma_\omega = \bigcup_{n < \omega} \Sigma_n \ .$$

For $m, n \geq 0$, $\Sigma_n \Sigma_m(M)$ denotes the set of relations R which are $\Sigma_n(\langle |M|; B_1, \ldots, B_q \rangle)$ for $B_1, \ldots, B_q$ which are $\Sigma_m(M)$ (similarly for $\Sigma_n \Pi_m$, $\Sigma_n \Delta_m$, etc.). Obviously we have $\Sigma_n \Sigma_0 = \Sigma_n$. We often write $\Sigma_n(U; A_1, \ldots, A_m)$ as an abbreviation for $\Sigma_n(\langle U; A_1, \ldots, A_m \rangle)$. We call $M = \langle U, A_1, \ldots, A_n \rangle$ *amenable* iff U is transitive and $A_i \cap x \in U$ for $x \in U$. We note the following *absoluteness property* of Σ_0 formulae: If M' is a submodel of M, $|M'|$ is transitive, $x \in |M'|$ and φ is a Σ_0 formula, then

$$\models_{M'} \varphi[x] \leftrightarrow \models_M \varphi[x] \; .$$

We write $M' \prec_{\Sigma_n} M$ $(n \geq 0)$ to mean that M' is a submodel of M and for every Σ_n formula φ and all $x \in M'$ we have

$$\models_{M'} \varphi[x] \leftrightarrow \models_M \varphi[x] \; .$$

Thus $\prec_{\Sigma_\omega}$ is the usual elementary submodel relation. We write $\pi : M' \to_{\Sigma_n} M$ or $M' \xrightarrow{\pi}_{\Sigma_n} M$ to mean that π is an isomorphism of M' onto an $M'' \prec_{\Sigma_n} M$. If $X \subset |M|$, we write $X \prec_{\Sigma_n} M$ to mean $M' \prec_{\Sigma_n} M$, where M' is the result of restricting M to X. For $n \geq 1$, $X \prec_{\Sigma_n} M$ is equivalent to the condition: If $A \subset M$ is Π_{n-1} in parameters from X and $A \neq \emptyset$, then $A \cap X \neq \emptyset$.[4] $\models_M^{\Sigma_n}$ is the satisfaction relation on M for Σ_n formulae. $\mathrm{rn}(x)$ denotes the rank of the set x. $\mathbf{ZF}^-$ consists of all axioms of $\mathbf{ZF}$ set theory except the power set axiom. $\mu\alpha(...)$ means the least ordinal α such that ...

Now let $|M|$ be closed under finite subsets (i.e. $x \subset |M|, \bar{\bar{x}} < \omega \to x \in |M|$). We list some closure properties of $\Sigma_n(M)$ and $\Delta_n(M)$ $(n \geq 1)$.

Property 0.1. If Ryx is Σ_n, so is $\bigvee y Ryx$.

Proof. Let $Ryx \leftrightarrow \bigvee z\, Pzyx$, where P is Π_{n-1}. Then

$$\bigvee y\, Ryx \leftrightarrow \bigvee u\, (Q(u) \wedge \bigvee zy \in u\; Pzyx) \; ,$$

where Q is the Σ_0 condition

$$Q(u) \leftrightarrow \bigvee zy \in u \bigwedge x \in u\, (x = z \vee x = y) \; .$$

If P is Σ_0, we are done. Otherwise we use the equivalences

$$Q(u) \to (\bigwedge x \in u \bigvee y\, S(x,y) \leftrightarrow \bigvee v (Q(v) \wedge$$

$$\wedge \bigwedge x \in u \bigvee y \in v\, S(x,y))$$

to bring the bounded quantifiers successively inward.

An immediate corollary of Property 0.1 is

Property 0.2. If $R_0 x, R_1 x$ are Σ_n, then so are $(R_0 x \vee R_1 x)$ and $(R_0 x \wedge R_1 x)$.

Hence

Property 0.3. Δ_n relations are closed under all sentential operations $(\wedge, \vee, \neg)$.

We call a function $f(x)$ Σ_n iff the relation $y = f(x)$ is Σ_n.

Property 0.4. If $Rz_1 \ldots z_m$ and $f_i(x)$ $(i = 1, \ldots, m)$ are Σ_n, then so is $Rf(x)$.

Proof. $Rf(x) \longleftrightarrow \bigvee \mathbf{z}\, (\bigwedge_{i=1}^{m} z_i = f_i(x) \wedge R\mathbf{z})$.

Property 0.5. If f is Σ_n and $\mathrm{dom}(f)$ is Δ_n, then f is Δ_n.

Proof. $y \neq f(x) \longleftrightarrow (x \notin \mathrm{dom}(f) \vee \bigvee z\, (z = f(x) \wedge y \neq z))$.

§1. Rudimentary functions

Definition. We call a function $f\colon V^n \to V$ *rudimentary* (rud) iff it is finitely generated by the following schemata:

(a) $f(x) = x_i$,
(b) $f(x) = x_i \setminus x_j$,
(c) $f(x) = \{x_i, x_j\}$,
(d) $f(x) = h(g(x))$,
(e) $f(y, x) = \bigcup_{z \in y} g(z, x)$.

Note. This is the usual list of schemata for primitive recursive set functions, minus the recursion schema.

We list some elementary properties of rud functions:

Property 1.1.

(a) $f(x) = \bigcup x_i$ is rud.

(b) $f(x) = x_i \cup x_j = \bigcup \{x_i, x_j\}$ is rud.

(c) $f(x) = \{x\}$ is rud.

(d) $f(x) = \langle x \rangle$ is rud.

(e) If $f(y, x)$ is rud, so is $g(y, x) = \langle f(z, x) | z \in y \rangle$ (since $g(y, x) = \bigcup_{z \in y} \{\langle f(z, x), z \rangle\}$).

Definition. $R \subset \mathrm{V}^n$ is rud iff there is a rud function $r : \mathrm{V}^n \to \mathrm{V}$ such that $R = \{\langle x \rangle | r(x) \neq 0\}$.

Property 1.2.

(a) $\notin$ is rud, since $y \notin x \leftrightarrow \{y\} \setminus x \neq \emptyset$.

(b) If f, R are rud, then so is $g(x) = f(x)$ if Rx, and $g(x) = \emptyset$ if not.
[Proof: Let $Rx \leftrightarrow r(x) \neq 0$. Then $g(x) = \bigcup_{y \in r(x)} f(x)$.]

Let χ_R be the characteristic function of R.

(c) R is rud $\leftrightarrow$ χ_R is rud (proof by 2b).

Hence

(d) R is rud $\leftrightarrow \neg R$ is rud.
[Proof: $\chi_{\neg R}(x) = 1 \setminus \chi_R(x)$.]

(e) Let f_i: $\mathrm{V}^n \to \mathrm{V}$, $R_i \subset \mathrm{V}^n$ be rud ($i = 1, ..., m$). Let $R_i \cap R_j = \emptyset$ for $i \neq j$ and $\bigcup_i R_i = \mathrm{V}^n$. Then f is rud, where $f(x) = f_i(x)$ if $R_i(x)$.
[Proof: Set $\bar{f}_i(x) = f_i(x)$ if $R_i x$ and $\bar{f}_i(x) = \emptyset$ if not. Then $f(x) = \bigcup_{i=1}^m \bar{f}_i(x)$.]

(f) If Ryx is rud, so is $f(y, x) = y \cap \{z | Rzx\}$.
[Proof: $f(y, x) = \bigcup_{z \in y} h(z, x)$, where $h(z, x) = \{z\}$ if Rzx and $h(z, x) = \emptyset$ if not.]

(g) If R is rud and $\bigwedge x \bigvee y\, Ryx$, then so is $f(y, x) =$ that $z \in y$ such that Rzx if $\bigvee z \in y\, Rzx$ and $f(y, x) = \emptyset$ if not.
[Proof: $f(y, x) = \bigcup (y \cap \{z | Rzx\})$.]

(h) If Ryx is rud, then so is $\bigvee z \in y\, Rzx$.

(i) If $R_i x$ is rud ($i = 1, ..., m$), then so are $\bigvee_{i=1}^m R_i x$ and $\bigwedge_{i=1}^m R_i x$.

Property 1.3. The following functions are rud:

(a) $(x)_i^n$ ($i < n < \omega$), where $(\langle z_0, ..., z_{n-1} \rangle)_i^n = z_i$ and $(u)_i^n = \emptyset$ otherwise.

[Proof: $(x)^n_i$ = that $z \in h(x)$ such that $\bigvee u \in h(x)\ (x = \langle u \rangle \wedge u_i = z)$ if such z exists and $(x)^n_i = \emptyset$ if not, where $h(x) = \bigcup x \cup \bigcup^2 x \cup \ldots \cup \bigcup^n x$.]

(b) $x(y)$, where $x(y)$ = the unique $z \in \bigcup^2 x$ such that $\langle z, y \rangle \in x$ if such z exists and $x(y) = \emptyset$ otherwise.

(c) $\mathrm{dom}(x) = \{z \in \bigcup^2 x \mid \bigvee v \in \bigcup^2 x\ \langle v, z \rangle \in x\}$.

(d) $\mathrm{rng}(x) = \{z \in \bigcup^2 x \mid \bigvee v \in \bigcup^2 x\ \langle z, v \rangle \in x\}$.

(e) $x \times y = \bigcup_{u \in x} \bigcup_{v \in y} \{\langle u, v \rangle\}$.

(f) $x \upharpoonright y = x \cap (\mathrm{rng}(x) \times y)$.

(g) $x''y = \mathrm{rng}(x \upharpoonright y)$.

(h) $x^{-1} = h''(x \cap (\mathrm{rng}(x) \times \mathrm{dom}(x)))$, where $h(z) = \langle (z)^2_1, (z)^2_0 \rangle$.

Lemma 1.1. *If f is rud, then there is a $p < \omega$ such that*

$$\bigwedge x\ \mathrm{rn}(f(x)) < \max(\mathrm{rn}(x_1), \ldots, \mathrm{rn}(x_m)) + p\ .$$

Proof. By induction on the defining schemata of f. The induction is straightforward.

By Property 1.2 (h), (i), every Σ_0 relation is rud. We shall now prove the converse; we shall in fact prove a much stronger theorem.

Definition. $f : V^n \to V$ is *simple* iff whenever $\varphi(z, y)$ is a Σ_0 $\in$-formula, then $\varphi(f(x), y)$ is equivalent (in V) to a Σ_0 $\in$-formula [5] (i.e. it has only variables, bounded quantifiers and $\neg, \wedge, \in, \equiv$).

Note that simple functions are closed under composition. The simplicity of a function f is equivalent to the conjunction of the two conditions:

(i) $x \in f(y)$ is Σ_0.

(ii) If Az is Σ_0, then $\bigwedge x \in f(y) Ax$ is Σ_0,

for given these, we can prove by induction on Σ_0 formula φ that $\varphi(f(x))$ is Σ_0.

Lemma 1.2. *All rud functions are simple.*

Proof. We verify by induction on the defining schemata of f that f is simple, using (i), (ii) and the closure of simple functions under composition.

Note. Not all simple functions are rudimentary. For instance f is simple, where $f(\alpha) = \alpha + \omega$ for $\alpha \geq \omega$ and $f(x) = \emptyset$ otherwise. [6]

It is often of interest to consider functions which are rud in a relation A (more precisely: in the characteristic function of A). Not every relation which is rud in A will be Σ_0 in A; for instance, $\{x, y\} \in A$ is not, in general, Σ_0 in A. However, we do have

Lemma 1.3. *If f is (uniformly) rud in A, then f is (uniformly) expressible as a composition of rud functions and the function $a(x) = A \cap x$.*

Proof. Let $\mathfrak{S}$ be the collection of all compositions of rud functions and $a(x)$. It suffices to show

(*) If $g \in \mathfrak{S}$ and $f(y, x) = \bigcup_{z \in y} g(z, x)$, then $f \in \mathfrak{S}$.

Let $\mathfrak{S}_0$ be the collection of all rud functions and $\mathfrak{S}_{n+1}$ the collection of all functions of the form

$$f(x) = h_0(x, A \cap h_1(x), ..., A \cap h_m(x)) ,$$

where $h_0 \in \mathfrak{S}_0$ and $h_1, ..., h_m \in \mathfrak{S}_n$. It is readily checked that $\mathfrak{S} = \bigcup_n \mathfrak{S}_n$ (by induction on $n + m$ prove that $f \in \mathfrak{S}_n$, $g \in \mathfrak{S}_m \to fg(x) \in \mathfrak{S}_{m+n}$).

By induction on n, we prove

(**) If $g \in \mathfrak{S}_n$ and $f(y, x) = \bigcup_{z \in y} g(z, x)$, then $f \in \mathfrak{S}$.

For $n = 0$ this is trivial. Now let $n > 0$ and let (**) hold for $n - 1$. Let $g \in \mathfrak{S}_n$. Then

$$g(z, x) = h_0(z, x, A \cap h_1(z, x), ..., A \cap h_m(z, x)) ,$$

where $h_0 \in \mathfrak{S}_0$, $h_{i+1} \in \mathfrak{S}_{n-1}$. Set

$$\tilde{g}(z, x, u) = h_0(z, x, u \cap h_1(z, x), ..., u \cap h_m(z, x)) ,$$

then $\tilde{g} \in \mathfrak{E}_{n-1}$. Set

$$\tilde{f}(y, x, u) = \bigcup_{z \in y} \tilde{g}(z, x, u),$$

$$\tilde{h}(y, x) = \bigcup_{i=1}^{m} \bigcup_{z \in y} h_i(z, x),$$

then $\tilde{f}, \tilde{h} \in \mathfrak{E}$ by the induction hypothesis. But

$$f(y, x) = \tilde{f}(y, x, A \cap \tilde{h}(y, x)),$$

which proves the lemma.

Definition. X is *rudimentarily closed* (rud closed) iff X is closed under rud functions. $M = \langle U, A \rangle$ is rud closed iff U is closed under functions which are rud in A. The rudimentary closure of X is $X \cup \{f(x) | x \in X \wedge f$ is rudimentary$\}$.

As an immediate corollary of Lemma 1.3, we get

Corollary 1.4.
(a). *$M = \langle |M|, A \rangle$ is rud closed iff $|M|$ is rud closed and M is amenable.*
(b). *If f is rud in A, then f is uniformly $\Sigma_1(U, A \cap U)$ for all transitive rud closed $\langle U, A \cap U \rangle$.*

We now prove

Lemma 1.5. *Let U be transitive. Then the rud closure of U is transitive.*

Proof. Let V = the rud closure of U. Let $Q(x)$ mean: $C(\{x\}) \subset V$ (where $C(z)$ is the transitive closure of z). By induction on the defining schemata of f we show

$$\bigwedge_{i=1}^{n} Q(x_i) \rightarrow Q(f(x)).$$

But $Q(x)$ for $x \in U$ and V is the set of all $f(x)$ such that f is rud and $x \in U$.

An immediate consequence of Lemma 1.2 is

Lemma 1.6. *Let U be transitive and let V be the rud closure of U. Then the restriction of any $\Sigma_0(V)$ relation to U is $\Sigma_0(U)$.*

Definition. Let U be transitive. Set $\mathrm{rud}(U)$ = the rud closure of $U \cup \{U\}$.

Noting that $\mathfrak{P}(U) \cap \Sigma_0(U \cup \{U\}) = \Sigma_\omega$, we get

Corollary 1.7. $\mathfrak{P}(U) \cap \mathrm{rud}(U) = \Sigma_\omega(U)$.

Thus $\mathrm{rud}(U)$, while it has a higher rank than $\Sigma_\omega(U)$, really adds nothing new. It is the result of "stretching" $\Sigma_\omega(U)$, which has the unwieldy rank $\mathrm{rn}(U) + 1$, to length $\mathrm{rn}(U) + \omega$. We shall define the J_α hierarchy, exactly like the L_α hierarchy, except that we take $\mathrm{J}_{\alpha+1}$ to be $\mathrm{rud}(\mathrm{J}_\alpha)$ instead of $\Sigma_\omega(\mathrm{J}_\alpha)$, as in the case of L.

The following characterisation of $\mathrm{rud}(U)$ may be more conceptual, though, since we shall not need it, we do not prove it:

Let $T = T(U)$ be the set of U-definable trees of finite length which have one initial point, and all of whose endpoints have the form $\langle x, 0\rangle$. For $t \in T$, define a function σ_t on the nodes by setting

$$\sigma_t(\langle x, 0\rangle) = x \quad \text{for endpoints;}$$

$$\sigma_t(y) = \{\sigma_t(z) | z >_t y\} \quad \text{otherwise.}$$

Set $\sigma(t) = \sigma_t(x_0)$, where x_0 is the initial point of t. Then

$$\mathrm{rud}(U) = \{\sigma(t) | t \in T(U)\} .$$

It may also be of interest, in this context, to note that a transitive domain ν is rud closed iff it satisfies the following axioms:

A1. $x \setminus y,\ \{x, y\},\ \bigcup x \in \nu$.

A2. $\{u \cap A(x) | x \in w\} \in \nu$.

Here A is Σ_0 and $A(x) = \{y \mid Ayx\}$. Again we omit the proof, since this characterisation is not needed.

Call a family $\mathfrak{F}$ of functions a *basis* iff every rud function can be obtained from $\mathfrak{F}$ by composition alone. We now prove that the rud functions have a finite basis.

Lemma 1.8. *Every rud function is a composition of the following:*

$$\begin{aligned}
&F_0(x, y) = \{x, y\}\,,\\
&F_1(x, y) = x \setminus y\,,\\
&F_2(x, y) = x \times y\,,\\
&F_3(x, y) = \{\langle u, z, v\rangle \mid z \in x \wedge \langle u, v\rangle \in y\}\,,\\
&F_4(x, y) = \{\langle u, v, z\rangle \mid z \in x \wedge \langle u, v\rangle \in y\}\,,\\
&F_5(x, y) = \bigcup x\,,\\
&F_6(x, y) = \mathrm{dom}(x)\,,\\
&F_7(x, y) = \in \cap\, x^2\,,\\
&F_8(x, y) = \{x''\{z\} \mid z \in y\}\,.
\end{aligned}$$

Proof. Let $\mathfrak{C}$ be the class of functions obtainable by composition from $F_0, \ldots, F_8$. For each $\in$-formula $\varphi = \varphi(x_1, \ldots, x_n)$, set

$$t_\varphi(u) = \{\langle x_1, \ldots, x_n\rangle \mid \boldsymbol{x} \in u \wedge \models_{\langle u, \in\rangle} \varphi[\boldsymbol{x}]\}\,.$$

Lemma 1.8.1. $t_\varphi \in \mathfrak{C}$ *for every* $\in$*-formula* φ.

Proof. (a). Let $\varphi(\boldsymbol{x}) \leftrightarrow x_i \in x_j$ $(i < j)$. Then $t_\varphi \in \mathfrak{C}$.

Let $F_x(y) = F_3(x, y)$. Define $X^n(x, y)$ by $X^1(x, y) = x \times y$: $X^n(x, y) = x \times X^{n-1}(x, y)$. Then assuming $\langle x_1, \ldots, x_m\rangle$, $m > 2$, is defined inductively by $\langle x_1, \ldots, x_m\rangle = \langle x_1, \langle x_2, \ldots, x_m\rangle\rangle$, we have

$$\begin{aligned}
&\{\langle x_1, \ldots, x_m\rangle \mid \boldsymbol{x} \in w \wedge \models_{\langle w, \in\rangle} \varphi[\boldsymbol{x}]\}\\
&\qquad = X^{i-1}(w, F_w^{j-i-1}(F_4(w^{m-j}, \in \cap\, w^2)))\,.
\end{aligned}$$

(b). If $\varphi_i(x)$ $(i = 1, ..., p)$ are such that $t_{\varphi_i} \in \mathfrak{S}$, and ψ is any sentential combination of the φ_i's, then $t_\psi \in \mathfrak{S}$.

It suffices to note that $\mathfrak{S}$ contains

$$x \setminus y,\ x \cup y = \bigcup\{x, y\},\ x \cap y = x \setminus (x \setminus y)\,.$$

(c). Consider $\varphi(y, x)$. If $t_\varphi \in \mathfrak{S}$, then $t_{\bigwedge y\varphi}, t_{\bigvee y\varphi} \in \mathfrak{S}$.

This follows from

$$t_{\bigvee y\varphi}(u) = \mathrm{dom}(t_\varphi(u));\ t_{\bigwedge y\varphi}(u) = x^m \setminus \mathrm{dom}(x^m \setminus t_\varphi(u))\,.$$

(d). $t_\varphi \in \mathfrak{S}$, where $\varphi(x) \leftrightarrow x_i = x_j$.

By (a), (b); $t_\chi \in \mathfrak{S}$, where

$$\chi(y, x) \leftrightarrow (y \in x_i \leftrightarrow y \in x_j)\,.$$

But then

$$\models_u \varphi[x] \leftrightarrow \wedge\, y \in \bigcup u \models_{u \cup \{u\}} \chi[y, x]\ .$$

Hence

$$t_\varphi(u) = u^m \cap t_{\bigwedge y\chi}(u \cup \bigcup u)\,.$$

(e). $t_\varphi \in \mathfrak{S}$, where $\varphi(x) \leftarrow x_i \in x_j$ $(j \leq i)$.

Let

$$\psi(y, z, x) \leftrightarrow (y \in z \wedge y = x_i \wedge z = x_j)$$

then $t_\psi \in \mathfrak{S}$ by (a), (b), (d). But $\varphi(x) \leftrightarrow \bigvee yz\ \psi(y, z, x)$, hence $t_\varphi \in \mathfrak{S}$ by (c).

By (a), (e), (b), $t_\varphi \in \mathfrak{S}$ for every quantifier free φ. Now let $\varphi(x) \leftrightarrow Q_1 y_1 \ldots Q_n y_n \chi(y, x)$, where χ is quantifier free. $t_\varphi \in \mathfrak{S}$ follows by iterated use of (c), which proves Lemma 1.8.1.

Now set $\mathfrak{S}^*$ = the set of rud functions $f(x)$ such that $F \in \mathfrak{S}$, where $F(u) = f''u^m$ (i.e. $\{z | \bigvee x \in u \; f(x) = z\}$).

Lemma 1.8.2. *If $f \in \mathfrak{S}^*$, then the following functions are in $\mathfrak{S}$:*

$$F(u) = f \upharpoonright u^m \text{ (i.e. } \{\langle z, x\rangle | x \in u \wedge z = f(x)\}) ,$$

$$G(u) = \{\langle z, x\rangle | x \in u \wedge z \in f(x)\} ,$$

$$H(u) = \{\langle z, y, x\rangle | y, x \in u \wedge \bigvee v \in y \; z \in f(v, x)\} .$$

Proof. Set $C_n(u) = u \cup \bigcup u \cup \ldots \cup \bigcup^n u$ $(n < \omega)$. It is a well known fact that if φ is a Σ_0 formula and $n = n(\varphi)$ is the number of quantifiers in φ, then

$$\bigwedge x \in u \, (\models_V \varphi[x] \longleftrightarrow \models_{C_n(u)} \varphi[x]) .$$

We use this to show $F \in \mathfrak{S}$. Let $\varphi(y, x)$ be a Σ_0 formula meaning:

$$y = f(x) .$$

Let $n = n(\varphi)$. Then

$$F(u) = ((f''u^m) \times u^m) \cap t_\varphi C_n(f''u^m \cup u) .$$

The proof that $G, H \in \mathfrak{S}$ is entirely analogous.

Lemma 1.8.3. *Every rud function is in $\mathfrak{S}^*$.*

Proof. We show that $f \in \mathfrak{S}^*$ by induction on the defining schemata of f:

(a). $f(x) = x_i$. Then $f''u^m = u = u \setminus (u \setminus u)$.

(b). $f(x) = x_i \setminus x_j$. Then $f''u^m \{x \setminus y | x, y \in u\}$. Let $\varphi(z, x, y) \longleftrightarrow z \in x \setminus y$. Set

$$F(u) = t_\varphi(u \cup \bigcup u) \cap (\bigcup u \times u^2)$$

$$= \{\langle z, x, y\rangle | x, y \in u \wedge z \in x \setminus y\} .$$

Then $f''u^m = F_8(F(u), u^2)$.

(c). $f(x) = \{x_i, x_j\}$. Then $f''u^m = \{\{x, y\} \mid x, y \in u\} = \bigcup u^2$.
(d). $f(x) = h(g(x))$. Let

$$G_i(u) = g_i''u^m , \qquad H(u) = h''u^m ,$$

$$\overline{G}(u) = \bigcup_{i=1}^{n} G_i(u) , \quad \overline{H}(u) = H(\overline{G}(u)) ,$$

$$\overline{K}(u) = u^m \cup \overline{G}(u) \cup \overline{H}(u) .$$

By hypothesis, $\overline{G}, \overline{H}, \overline{K} \in \mathfrak{S}$. Using Lemma 1.2, let $\varphi(y, x)$ be an $\in$-formula equivalent to the formula

$$\bigvee z_1 \ldots \bigvee z_n (z_1 = g_1(x) \wedge \ldots \wedge z_n = g_n(x) \wedge y$$
$$= h(z_1, \ldots, z_n)) .$$

It is easily seen that

$$f''u^m = F_8(([t_\varphi(\overline{K}(u))] \cap [\overline{H}(u) \times u^m]), u^m) .$$

(e). $f(y, x) = \bigcup_{z \in y} g(z, x)$ Let

$$G(u) = \{\langle z, y, x\rangle \mid \bigvee v \in y \wedge z \in g(v, x) \wedge x \in u \wedge y \in u\} .$$

Then $f''u^{m+1} = F_8(G(u), u^{m+1})$, which proves Lemma 1.8.3.

It remains only to show

Lemma 1.8.4. *Every rud function is in* $\mathfrak{S}$.

Proof. Let $f(x)$ be rud. Define $\bar{f}$ by $\bar{f}(\langle z\rangle) = f(z)$ and $\bar{f}(y) = \emptyset$ otherwise. Then $\bar{f}$ is rud; hence $\bar{f} \in \mathfrak{S}^*$.

Let $F(u) = \bar{f}''u$. Hence $F \in \mathfrak{S}$. Set $P(x) = \{\langle x\rangle\}$. Then $P \in \mathfrak{S}$ since P is gotten by iterating F_0. Then

$$\bigcup F(P(x)) = \bigcup \bar{f}'' \{\langle x\rangle\} = \bigcup \{\bar{f}(\langle x\rangle)\} = f(x) .$$

Combining Lemma 1.8 with Lemma 1.3 we get:

Lemma 1.9. *Every function which is rud in $A \subset V$ is a composition of $F_0, \ldots, F_8$ and F^A, where $F^A(x, y) = A \cap x$.*

These basic lemmas have a number of interesting consequences:

Corollary 1.10. *There is a rud function $s(u)$ such that $u \subset s(u)$ and $\bigcup_n s^n(u)$ is the rud closure of u.*

Proof. Set $s(u) = u \cup \bigcup_{i=0}^{8} F_i''u^2$, which proves the corollary.

Definition. $S(u) = s(u \cup \{u\})$.

Then $u \cup \{u\} \subset S(u)$ and for transitive u we have: $\bigcup_n S^n(u) = \mathrm{rud}(u)$.

Corollary 1.11. *There is a rud function W such that if r is a well ordering of u, then $W(r, u)$ is an end extension of r which well orders $S(u)$.*

The proof is left to the reader.

We can make good use of Lemma 1.9 in proving

Lemma 1.12. *$\models_M^{\Sigma_0}$ is uniformly $\Sigma_1(M)$ over transitive rud closed $M = \langle |M|, A \rangle$.*

Proof. Consider a term language containing just variables and the function symbols f_i $(i = 0, \ldots, 9)$. f_i is interpreted by F_i (where $F_9(x, y) =_{\mathrm{df}} A \cap x$). Let Q be the set of functions each of which maps a finite set of variables into $|M|$. Then Q is rudimentary (given a reasonable arithmetisation). For any term t, let $C(t)$ be the set of its component terms (including variables). We may suppose the function C to be Δ_1. For terms t and $v \in Q$ we define

$$y = t[v] \longleftrightarrow \bigvee g(\varphi(C(t), g, v) \wedge g(t) = y),$$

where

$$\varphi(u, g, v) \longleftrightarrow \mathrm{fun}(g) \wedge \mathrm{dom}(g) = u$$

$\wedge \bigwedge x \in u(\text{x is variable } (x \in \mathrm{dom}(\nu) \wedge g(x) = \nu(x))) \wedge$

$$\bigwedge_{i=0}^{9} (\bigwedge t_0 t_1 \in u \ (x = f_i(t_0 t_1) \to g(x) = F_i(g(t_0), g(t_1)))).$$

Thus φ is rudimentary. Hence $t[\nu]$ is Σ_1. We note now that there is a recursive function σ mapping each Σ_0 formula $\varphi(x)$ onto a $t(x)$ such that $\varphi \longleftrightarrow t = 1$. Hence

$$\models_M^{\Sigma_0} \varphi[\nu] \longleftrightarrow \sigma(\varphi)[\nu] = 1 .$$

Thus $\models_M^{\Sigma_0}$ is Σ_1.

Corollary 1.13. $\models_M^{\Sigma_n}$ *is uniformly* $\Sigma_n(M)$ *over transitive rud closed* $M = \langle |M|, A \rangle$ $(n \geq 1)$.

§2. The hierarchy J_α

Definition. $\mathrm{J}_0 = \emptyset$; $\mathrm{J}_{\alpha+1} = \mathrm{rud}(\mathrm{J}_\alpha)$; $\mathrm{J}_\lambda = \bigcup_{\iota<\lambda} \mathrm{J}_\iota$ for limit λ.

Lemma 2.1.

(a). J_α *is transitive.*

(b). $\alpha \leq \beta \to \mathrm{J}_\alpha \subset \mathrm{J}_\beta$.

(c). $\mathrm{rn}(\mathrm{J}_\alpha) = \mathrm{On} \cap \mathrm{J}_\alpha = \omega \cdot \alpha$.

The proofs are straightforward.

Now define an auxiliary hierarchy S_ν by

$$\mathrm{S}_0 = \emptyset, \quad \mathrm{S}_{\nu+1} = \mathrm{S}(\mathrm{S}_\nu), \quad \mathrm{S}_\lambda = \bigcup_{\iota<\lambda} \mathrm{S}_\iota .$$

It is easily seen that the S_ν hierarchy is cumulative and that

$$\mathrm{J}_\alpha = \bigcup_{\nu<\omega\cdot\alpha} \mathrm{S}_\nu = \mathrm{S}_{\omega\alpha} .$$

Lemma 2.2. *$\langle S_\nu \mid \nu < \omega\alpha \rangle$ is uniformly $\Sigma_1(J_\alpha)$.*

Proof.

$$y = S_\nu \longleftrightarrow \bigvee f(y = f(\nu) \wedge \varphi(f)) ,$$

where

$$\varphi(f) \longleftrightarrow (f \text{ is a function } \wedge \operatorname{dom}(f) \in \mathrm{On} \wedge f(0) = \emptyset$$

$$\wedge \bigwedge (\nu + 1) \in \operatorname{dom}(f)\, (f(\nu + 1) = S(f(\nu)))$$

$$\wedge \bigwedge \lambda \in \operatorname{dom}(f)\, (\operatorname{Lim}(\lambda) \to f(\lambda) = \bigcup_{\nu < \lambda} f(\nu)) .$$

φ is rudimentary, hence Σ_0. Thus it suffices to show that the existence quantifier can be restricted to J_α. That is, we must show:

(*) $\qquad \tilde{s}_\tau \in J_\alpha$ for $\tau < \omega\alpha$,

where $\tilde{s}_\tau = \langle S_\nu \mid \nu < \tau \rangle$. We prove (*) by induction on α. For $\alpha = 0$ it is trivial. For $\operatorname{Lim}(\alpha)$ the induction step is trivial. Now let $\alpha = \beta + 1$. Then $\tilde{s}_{\omega\beta}$ is $\Sigma_1(J_\beta)$ since (*) holds for β; hence $\tilde{s}_{\omega\beta} \in J_\alpha$. But $S_{\omega\beta+n} = S^n(J_\beta) \in J_\alpha$. The conclusion follows easily.

Corollary 2.3. *$\langle J_\nu \mid \nu < \alpha \rangle$ is uniformly $\Sigma_1(J_\alpha)$.*

Proof. It is easily shown that the map $\langle \nu, n \rangle \to \omega\nu + n$ ($\nu < \alpha, n < \omega$) is uniformly $\Sigma_1(J_\alpha)$. But $J_\nu = S_{\omega\nu}$.

Definition. We define well orderings $<_\nu$ of S_ν by

$$<_0 = \emptyset , \quad <_{\nu+1} = W(<_\nu, S_\nu) , \quad <_\lambda = \bigcup_{\nu < \lambda} <_\nu \text{ for limit } \lambda .$$

Then $<_\nu$ well orders S_ν and $<_\tau$ is an end extension of $<_\nu$ for $\nu \leq \tau$.

By repeating the proof of Lemma 2.2, we get

Lemma 2.4. *$\langle <_\nu \mid \nu < \omega\alpha\rangle$ is uniformly $\Sigma_1(\mathrm{J}_\alpha)$.*

Set $<_{\mathrm{J}_\alpha} = <_{\omega\cdot\alpha}$ and $<_\mathrm{J} = \bigcup_{\alpha\in\mathrm{On}} <_{\mathrm{J}_\alpha}$. Then $<_\mathrm{J}$ well orders J and $<_{\mathrm{J}_\alpha}$ well orders J_α.

Corollary 2.5. *$<_{\mathrm{J}_\alpha}$, $\langle <_{\mathrm{J}_\beta} \mid \beta < \alpha\rangle$ and u_α are uniformly $\Sigma_1(\mathrm{J}_\alpha)$, where $u_\alpha(x) = \{y \mid y <_{\mathrm{J}_\alpha} x\}$.*

2.1. *The condensation lemma*

Lemma 2.6. *Let $X \prec_{\Sigma_1} \mathrm{J}_\alpha$. Then $\bigvee \beta X \simeq \mathrm{J}_\beta$.*

Proof. X satisfies extensionality, since $X \prec_{\Sigma_1} \mathrm{J}_\alpha$. Hence there are unique π, M such that $\pi : X \overset{\sim}{\longleftrightarrow} M$, where M is transitive. [7] We claim that $M = \mathrm{J}_\beta$, where $\beta = \pi''\alpha$. We prove this by induction on α.

Let it hold for $\tau < \alpha$. Since $\langle \mathrm{J}_\nu \mid \nu < \alpha\rangle$ is Σ_1, we have $\nu \in X \cap \alpha \leftrightarrow \mathrm{J}_\nu \in X$. But if $\mathrm{J}_\nu \in X$, then $X \cap \mathrm{J}_\nu \prec_{\Sigma_1} \mathrm{J}_\nu$; hence $\pi(\mathrm{J}_\nu) = \pi''(X \cap \mathrm{J}_\nu) = \mathrm{J}_{\pi(\nu)}$ by the induction hypothesis (since $\pi(\nu) = \pi''(X \cap \nu)$). By definition $\mathrm{J}_\alpha = \bigcup_{\nu<\alpha} \mathrm{rud}(\mathrm{J}_\nu)$. Set $\mathrm{rud}_X(\mathrm{J}_\nu)$ = the rud closure of $X \cap (\mathrm{J}_\nu \cup \{\mathrm{J}_\nu\})$. We claim $X = \bigcup_{\nu\in X\cap\alpha} \mathrm{rud}_X(\mathrm{J}_\nu)$. To see this, we note that if $y \in X$, then there is a rud f such that in J_α

$$\bigvee \nu \bigvee x \in \mathrm{J}_\nu\, y = f(\mathrm{J}_\nu, x)\,.$$

Hence $y = f(\mathrm{J}_\nu, x)$ for some $\mathrm{J}_\nu \in X$ and $x \in X \cap \mathrm{J}_\nu$, since $X \prec_{\Sigma_1} \mathrm{J}_\alpha$.

X is rud closed and each rud f has a Σ_0 definition; hence $\pi f(x) = f(\pi(x))$ for rud f. Hence $\pi''\mathrm{rud}_X(\mathrm{J}_\nu) = \mathrm{rud}(\pi(\mathrm{J}_\nu))$ for $\nu \in X \cap \alpha$. Hence

$$M = \pi''X = \bigcup_{\nu\in X\cap\alpha} \mathrm{rud}(\mathrm{J}_{\pi(\nu)}) = \bigcup_{\nu<\pi''\alpha} \mathrm{rud}(\mathrm{J}_\nu) = \mathrm{J}_{\pi''\alpha}$$

Now let $X \prec_{\Sigma_1} \mathrm{J}_\alpha$ and $\pi : X \overset{\sim}{\longleftrightarrow} \mathrm{J}_\beta$. Since $<, <_\mathrm{J}$ are uniformly $\Sigma_1(\mathrm{J}_\alpha)$, we have:

$$\nu < \tau \longleftrightarrow \pi(\nu) < \pi(\tau) \text{ and}$$

$$x <_\mathrm{J} y \longleftrightarrow \pi(x) <_\mathrm{J} \pi(y)\,.$$

By this we can conclude

$$\pi(\nu) \leq \nu \ ; \ \pi(x) \leq_J x \ .$$

To see this, suppose $\pi(\nu) > \nu$. Let $\nu_0 \in X$ be such that $\pi(\nu_0) = \nu$. Then $\nu_0 < \nu$ since $\pi(\nu_0) < \pi(\nu)$. But then $\pi(\nu_0) > \nu_0$ and $\nu_0 < \nu$, so there is $\nu_1 \in X$ such that $\nu_1 < \nu_0$, $\pi(\nu_1) > \nu_1$ ... etc. In this way we generate a decreasing sequence $\nu > \nu_0 > ... > \nu_n >$. Contradiction! The same proof works for $<_J$.

2.2. Σ_n *uniformisation*

Definition. A function r *uniformises* a relation R iff $\mathrm{dom}(r) = \mathrm{dom}(R)$ and

$$\bigwedge x(\bigvee y\, Ryx \leftrightarrow Rr(x)x) \ .$$

Definition. Let $M = \langle |M|, A \rangle$ be amenable. M is Σ_n *uniformisable* iff every Σ_n relation is uniformisable by $\Sigma_n(M)$ function.

Lemma 2.7. *$\langle J_\alpha, A \rangle$ is uniformly Σ_1 uniformisable for amenable $\langle J_\alpha, A \rangle$. (More precisely: Given any Σ_1 formula φ, there is a Σ_1 formula ψ such that $\psi^{\langle J_\alpha, A \rangle}$ is a uniformising function for $\varphi^{\langle J_\alpha, A \rangle}$ whenever $\langle J_\alpha, A \rangle$ is amenable.)*

Proof. We first show that Σ_0 relations can be uniformised.

Let Ryx be Σ_0. Define $r(x)$ by:

$$r(x) \simeq \text{the least } y \text{ (in } <_J) \text{ such that } Ryx \ .$$

Then:

$$y = r(x) \leftrightarrow Ryx \wedge \bigwedge z <_J y \neg Rzx \ .$$

Thus r has a (uniformly) Σ_1 definition since the function $u(x) = \{y \mid y <_J x\}$ is uniformly Σ_1.

Now let Ryx be Σ_1; let

$$Ryx \leftrightarrow \bigvee z\ Pzyx$$

where P is Σ_0. Let $p(x)$ uniformise the Σ_0 relation $\{\langle\langle z, y\rangle, x\rangle | Pzyx\}$. Set: $r(x) \simeq (p(x))^2_1$. Then r uniformises R.

Definition. Let $M = \langle |M|, A\rangle$ be amenable and let $\omega \subset M$. By a Σ_n *Skolem function* for M, we mean a $\Sigma_n(M)$ function h such that $\mathrm{dom}(h) \subset \omega \times M$ and, whenever $A \subset M$ is $\Sigma_n(M)$ in the parameter x, then

$$\bigvee y\ Ay \to \bigvee i\ Ah(i, x)\,.$$

Definition. Let M be as above. We call h a *nice Σ_n Skolem function* iff h is a $\Sigma_n(M)$ function such that $\mathrm{dom}(h) \subset \omega \times M$ and, for some $p \in M$, h is Σ_n in the parameter p and whenever $A \subset M$ is Σ_n in the parameters p, x, then

$$\bigvee y\ Ay \to \bigvee i\ Ah(i, x)\,.$$

The following are easily established:

(1). If h is a Σ_n Skolem function which is Σ_n in no parameters, then h is nice (take $p = 0$).

(2). If h is a Σ_n Skolem function which is Σ_n in p, then $\tilde{h}$ is a nice Σ_n Skolem function, where $\tilde{h}(i, x) \simeq h(i, \langle x, p\rangle)$. Hence the existence of a Skolem function guarantees the existence of a nice Skolem function.

(3). If h is a nice Σ_n Skolem function, then $\bigwedge x \in M\ h''(\omega \times \{x\}) \prec_{\Sigma_n} M$.

Proof. Set $X = h''(\omega \times \{x\})$. Let A be Σ_n in parameters $y_1, \ldots, y_m \in X$. Then $y_i = h(j_i, x)$, where h is Σ_n in p. Hence A is Σ_n in p, x. Hence $\bigvee y\ Ay \to \bigvee y \in X\ Ay$.

(4). If h is a nice Σ_n Skolem function and $X \subset M$ is closed under ordered pairs, then $h''(\omega \times X) \prec_{\Sigma_n} M$.

Proof. Set $Y = h''(\omega \times X)$. Let A be Σ_n in $y_1 \ldots y_m \in Y$. Then $y_i = h(i_i, x_i)$, where h is Σ_n in p. Hence A is Σ_n in $p, \langle x_1, \ldots, x_n \rangle$. Hence

$$\bigvee y Ay \to \bigvee i\, Ah(i, \langle x \rangle) \to \bigvee y \in Y\, Ay.$$

Lemma 2.8. *There is a nice Σ_1 Skolem function $h = h_{\alpha,A}$ which is uniformly $\Sigma_1(J_\alpha, A)$ for amenable $\langle J_\alpha, A\rangle$.*

Proof. $\models^{\Sigma_1}_{\langle J_\alpha, A\rangle}$ is uniformly $\Sigma_1(J_\alpha, A)$ by Corollary 1.13. Let $\langle \varphi_{i<\omega} \rangle$ be a recursive enumeration of the formulae. By Lemma 2.7, there is an h which uniformly uniformises $\{\langle y, i, x\rangle \mid \models^{\Sigma_1}_{\langle J_\alpha, A\rangle} \varphi_i[y, x]\}$.

We shall refer to $h_{\alpha,A}$ as the *canonical* Σ_1 Skolem function for $\langle J_\alpha, A\rangle$.

A similar proof yields

Lemma 2.9. *If $\langle J_\alpha, A\rangle$ is Σ_n uniformisable, then there is a Σ_n Skolem function for $\langle J_\alpha, A\rangle$ ($n \geq 1$).*

Lemma 2.10. *There is a $\Sigma_1(J_\alpha)$ map of $\omega\alpha$ onto J_α.*

We first prove a sublemma.

Lemma 2.10.1. *There is a $\Sigma_1(J_\alpha)$ map of $\omega\alpha$ onto $(\omega\alpha)^2$.*

Proof. Let $<^*$ be Gödel's well ordering of On^2 – i.e. $<^*$ is obtained by ordering the triples $\langle \max(\nu, \tau), \nu, \tau\rangle$ lexicographically. Let $p : \mathrm{On} \to \mathrm{On}^2$ be the monotone enumeration of $<^*$. By induction on α we get

(1) $\qquad (p \restriction \omega\alpha)$ is $\Sigma_1(J_\alpha)$.

Set $Q = \{\alpha \mid p(\alpha) = \langle 0, \alpha\rangle\}$. Then Q is closed, unbounded in On and is the set of α such that $(p|\alpha) : \alpha \leftrightarrow \alpha^2$.

We prove the lemma by induction on α. Let it hold for $\beta<\alpha$.

Case 1. $Q(\omega\alpha)$. Then $p \upharpoonright \omega\alpha$ is Σ_1 and maps $\omega\alpha$ onto $(\omega\alpha)^2$.

Case 2. $\alpha = \beta + 1$; $\neg Q(\omega\alpha)$.

Then $\beta > 0$, since otherswise $Q(\omega\alpha)$. Hence there is a $\Sigma_1(J_\alpha)$ map j: $\alpha \longleftrightarrow \beta$. But there is a $\Sigma_1(J_\beta)$ map of $\omega\beta$ onto $(\omega\beta)^2$ by the induction hypothesis. Hence by Lemma 2.7 there is a $\Sigma_1(J_\beta)$ function g which maps $(\omega\beta)^2$ $1-1$ into $\omega\beta$. Hence $g \in J_\alpha$. Set: $f(\langle\nu, \tau\rangle) = g(\langle j(\nu), j(\tau)\rangle)$. Then f is a $\Sigma_1(J_\alpha)$ function which maps $(\omega\alpha)^2$ $1-1$ into $\omega\beta$. Clearly $\mathrm{rng}(f) \in J_\alpha$, since $\mathrm{rng}(f) = \mathrm{rng}(g)$. Define $h: \omega\alpha \to (\omega\alpha)^2$ by

$$h(\nu) = \begin{cases} f^{-1}(\nu) & \text{if } \nu \in \mathrm{rng}(f) \\ \langle 0, 0\rangle & \text{if not}. \end{cases}$$

Then h has the desired properties.

Case 3. $\mathrm{Lim}(\alpha)$; $\neg Q(\omega\alpha)$.

Let $p(\omega\alpha) = \langle\nu, \tau\rangle$. Then $p \upharpoonright \omega\alpha$ is $\Sigma_1(J_\alpha)$ and maps $\omega\alpha$ $1-1$ onto $v = \{z \mid z <^* \langle\nu, \tau\rangle\} \in J_\alpha$. Let $\gamma < \alpha$ such that $\nu, \tau < \omega\gamma$. Then $p \upharpoonright \omega\alpha$ maps into $(\omega\gamma)^2$. As above, there is a $g \in J_\alpha$ which maps $(\omega\gamma)^2$ $1-1$ into $\omega\gamma$. Set $f(\langle\iota, \kappa\rangle) = g(\langle gp(\iota), gp(\kappa)\rangle)$ for $\iota, \kappa < \omega\alpha$. Then f is $\Sigma_1(J_\alpha)$ and maps $(\omega\alpha)^2$ $1-1$ onto $u = g''(g''v)^2 \in J_\alpha$. Define h by

$$h(\nu) = \begin{cases} f^{-1}(\nu) & \text{if } \nu \in u \\ \langle 0, 0\rangle & \text{if not}. \end{cases}$$

Then h has the desired properties.

Proof of Lemma 2.10. Let $f: \omega\alpha \xrightarrow{\text{onto}} (\omega\alpha)^2$ be $\Sigma_1(J_\alpha)$ in the parameter p. Let p be the least p (in $<_J$) for which such an f exists. Define f^0, f^1 by: $f(\nu) = \langle f^0(\nu), f^1(\nu)\rangle$. We can define maps $f_n: \omega\alpha \xrightarrow{\text{onto}} (\omega\alpha)^n$ by: $f_0 = \mathrm{id} \upharpoonright \omega\alpha$; $f_{n+1}(\nu) = \langle f^0(\nu), f_n f^1(\nu)\rangle$. Then f_n is $\Sigma_1(J_\alpha)$ in p. Let h be the cannonical Σ_1 Skolem function for J_α. Set

$$X = h''(\omega \times (\omega\alpha \times \{p\})).$$

We claim that $X \prec_{\Sigma_1} J_\alpha$. For this it suffices to show that X is closed under ordered pairs. Let $y_1, \ldots, y_n \in X$; $y_i = h(j_i, \langle\nu_i, p\rangle)$. Let $f_n(\tau) = \langle\nu_1, \ldots, \nu_n\rangle$. Then $\{\langle y_1, \ldots, y_n\rangle\}$ is Σ_1 in τ, p. Hence $\langle y_1, \ldots, y_n\rangle \in X$.

We now claim that $X = J_\alpha$. To see this, let $\pi : X \xleftrightarrow{\sim} J_\beta$. Then $\pi(\nu) = \nu$ for $\nu < \omega\alpha$; hence $\beta = \alpha$. Hence $\pi(f) = f$, since $f \subset (\omega\alpha)^2 \times \omega\alpha$. But then $\pi(p) = p$, since p was the least p in which f is Σ_1 (and $\pi(f)$ is Σ_1 in $\pi(p)$, where $\pi(p) \leq p$). Hence

$$\pi h(i, \langle \nu, p \rangle) \simeq h(i, \langle \nu, p \rangle) \text{ for } \nu < \omega\alpha .$$

Hence $\pi \upharpoonright X = \mathrm{id} \upharpoonright X$: $X = J_\alpha$.

It remains only to show that X is the image of a Σ_1 function defined on $\omega\alpha$. Let

$$y = h(i, x) \longleftrightarrow \bigvee z\, Hziyx ,$$

where H is Σ_0. Define $\widetilde{h} : (\omega\alpha)^3 \to J_\alpha$ by

$$\widetilde{h}(\nu, \tau, \kappa) = \begin{cases} y \text{ if } \bigvee z \in S_\tau\, H(z, \nu, y, \langle \kappa, p \rangle) \text{ (hence } \nu < \omega) \\ 0 \text{ if not} . \end{cases}$$

Then $\widetilde{h}''(\omega\alpha)^3 = h''(\omega \times (\omega\alpha \times \{p\})) = X$; hence $\widetilde{h} \cdot f_3 : \omega\alpha \xrightarrow{\text{onto}} J_\alpha$.

2.3. *Admissible ordinals*

Although the concept of admissible ordinal will rarely appear explicitly in the next sections, many of the methods and results are motivated by admissibility theory. Thus, to aid the reader's orientation, we give a brief account of this theory.

Definition. Let $M = \langle |M|, A \rangle$ be amenable. M is *admissible* iff M is a model for the following axioms:

(1). $\emptyset, \{x, y\}, \bigcup x \in V$.

(2). $\bigwedge x \bigvee y\, \varphi(x, y) \to \bigwedge u \bigvee v \bigwedge x \in u \bigvee y \in v\, \varphi(x, y)$, where φ is Σ_0.

(3). $z \cap \{y \mid \varphi(y)\} \in V$ where φ is Σ_0.

It is easily seen that (2) holds when φ is replaced by a Σ_1 formula.

Moreover, we have the Δ_1 Aussonderungs principle: If $B \subset M$ is Δ_1 and $x \in M$, then $x \cap B \in M$. By these two principles, the image of any $x \in M$ under any Σ_1 function which is defined on all of x is an element of M. Using this, we see that M is closed under rudimentary functions and that, in fact, the Σ_1 functions are closed under the schemata for rud functions (translating the last two schemata as

$$f(x) \simeq hg(x)\,,$$

$$f(y, x) \simeq \bigcup_{z \in y} g(z, x))\,.$$

We also have: If Rzx is Σ_1, then so is $\bigwedge z \in y\; Rzx$ (let $Rzx \longleftrightarrow \bigvee w\, Pwzx$; then $\bigwedge z \in y\, \bigvee w\, Pwzx \leftrightarrow \bigvee u \bigwedge z \in y\, \bigvee w \in u\; Pwzx$). The Σ_1 functions are also closed under the following *recursion principle*:

Let R be well founded such that $\{y \mid yRx\} \in M$ for all $x \in M$ and the function $r(x) = \{y \mid yRx\}$ is Σ_1. Let $g(y, x, u)$ be a Σ_1 function. Then there is a unique Σ_1 function f such that $f(y, x) \simeq g(y, x, \langle f(z, x) \mid zRy\rangle)$.

Proof. f has the following Σ_1 definition:

$$u = f(y, x) \longleftrightarrow \bigvee s(y \in \mathrm{dom}(s) \wedge s(y) = u \wedge \varphi(s, x))\,.$$

where

$$\varphi(s, x) \longleftrightarrow s \text{ is a function} \wedge R''\mathrm{dom}(s) \subset \mathrm{dom}(s)$$

$$\underbrace{\wedge \bigwedge z \in \mathrm{dom}(s)\, (s(z) = g(z, x, s \upharpoonright R''\{z\}))}_{\Sigma_1}\,.$$

The adequacy of this definition is shown in the usual way, using the Σ_1 replacement axiom.

Definition. $\omega\alpha$ is an admissible ordinal iff J_α is admissible.

Note. An easy application of the recursion theorem shows that, if α is admissible, then either $\alpha = \omega$ or $\alpha = \omega \cdot \alpha$.

Lemma 2.11. *$\omega\alpha$ is admissible iff there is no $\Sigma_1(J_\alpha)$ map of a $\gamma < \omega\alpha$ onto an unbounded subset of $\omega\alpha$.*

Proof. $(\rightarrow)$ is trivial.

$(\leftarrow)$. Assume that $\omega\alpha$ is not admissible. We show that some $\gamma < \omega\alpha$ is mapped onto an unbounded subset of $\omega\alpha$. For $\alpha = \beta + 1$, this is trivial, for ω maps cofinally into $\omega\beta + \omega$ by the map $n \to \omega\beta + n$. Let α be a limit ordinal. Let R be a Σ_0 relation and let $u \in J_\alpha$ be such that $\bigwedge x \in u \bigvee y Rxy$, but not $\bigwedge x \in u \bigvee y \in z Rxy$ for $z \in J_\alpha$. Let $u \in J_\gamma$, $\gamma < \alpha$. Let $f \in J_\alpha$ such that $f : \omega\gamma \xrightarrow{\text{onto}} u$ (this exists by Lemma 2.10). Define $g : \omega\gamma \to \omega\alpha$ by

$$g(\iota) = \mu\tau \bigvee y \in S_\tau\ Rf(\iota)y\ .$$

Then g is Σ_1 and range of g is unbounded in $\omega\alpha$.

Definition. $M = \langle |M|, A \rangle$ is *strongly admissible* iff M is admissible and $\langle M, B \rangle$ is amenable for all $\Sigma_1(M)$ relations B. It is easily seen that M is strongly admissible iff it satisfies the axioms (1), (3) and

(2') $\bigwedge u \bigvee v \bigwedge x \in u\ (\bigvee y\ \varphi(xy) \to \bigvee y \in v\ \varphi(x, y))$ for $\Sigma_0\varphi$.

Imitating the proof of Lemma 2.11, we get

Lemma 2.12. *$\omega\alpha$ is strongly admissible iff there is no $\Sigma_1(J_\alpha)$ function which maps some subsets of a $\gamma < \omega\alpha$ onto an unbounded subset of $\omega\alpha$.*

The following Lemma (due to Kripke and Platek) is somewhat deeper than the previous two.

Lemma 2.13. *The following conditions are equivalent:*

(i). *$\omega\alpha$ is strongly admissible.*

(ii). *$\langle J_\alpha, A \rangle$ is amenable for every $A \in \Sigma_1(J_\alpha)$.*

(iii). *There is no $\Sigma_1(J_\alpha)$ function which maps a subset of a $\gamma < \omega\alpha$ onto J_α.*

Proof. (i) → (ii) is trivial.

(ii) → (iii) follows by supposing (iii) false and using a diagonal argument to produce a $u \subset \gamma$ such that $u \in \Sigma_1(J_\alpha)\setminus J_\alpha$.

(iii) → (i). Suppose $\omega\alpha$ not to be strongly admissible. We wish to construct a Σ_1 map from a subset of a $\gamma < \omega\alpha$ onto $\omega\alpha$. If $\alpha = \beta + 1$, this follows by the methods of Lemma 2.10. Now let $\mathrm{Lim}(\alpha)$. Let f be $\Sigma_1(J_\alpha)$ such that $f : u \to \omega\alpha$, where $\gamma < \alpha$, $u \subset \omega\gamma$, and range f is unbounded in $\omega\alpha$. Let f be Σ_1 in the parameter p. Suppose $p \in J_\gamma$ (we can insure this by choosing γ sufficiently large). Let $h = h_\alpha$ be the canonical Σ_1 Skolem funciton for J_α. Consider $X = h''(\omega \times J_\gamma)$. Then $X \prec_{\Sigma_1} J_\alpha$. Let $\pi : X \stackrel{\sim}{\longleftrightarrow} J_\beta$. Then $\pi \restriction J_\gamma = \mathrm{id} \restriction J_\gamma$. But then $\pi = \mathrm{id} \restriction X$, since $\pi h(i, x) \simeq h(\pi(i), \pi(x)) = h(i, x)$. Hence $X = J_\beta$. But X is closed under f since f is Σ_1 in $p \in X$. Since range(f) is unbounded in $\omega\alpha$ and X is transitive, we conclude: $\omega\alpha \subset X$. Hence $\beta = \alpha$, $X = J_\alpha$. By Lemma 10 there is a $g \in J_\alpha$ such that, $g : \omega\gamma \xrightarrow{\text{onto}} \omega \times J_\gamma$. Set: $\bar{f}(\nu) \simeq hg(\nu)$. Then $\bar{f}$ is Σ_1 and $\bar{f} : u \xrightarrow{\text{onto}} X = J_\alpha$, where $u \subset \omega\gamma$.

Note. Strongly admissible α are also called *non projectible* since there is no Σ_1 projection of a subset of a $\gamma < \omega\alpha$ onto $\omega\alpha$.

A fairly slight modification of the proof of Lemma 2.13 gives

Lemma 2.14. *The following conditions are equivalent:*

(i). *$\omega\alpha$ is admissible.*

(ii). *$\langle J_\alpha, A\rangle$ is amenable for all $A \in \Delta_1(J_\alpha)$.*

(iii). *There is no $\Sigma_1(J_\alpha)$ map of a $\gamma < \omega\alpha$ onto $\omega\alpha$.*

Proof. (i) → (ii) follows by the Δ_1 Aussonderungs principle.

(ii) → (iii) follows as before.

(iii) → (i). Assume that $\omega\alpha$ is not admissible. We wish to construct a $\Sigma_1 f : \gamma \to \omega\alpha (\gamma < \omega\alpha)$. As before, we may assume $\mathrm{Lim}(\alpha)$. By Lemma 11 let $f : \tau \to \omega\alpha$ be Σ_1 with range unbounded in $\omega\alpha$. Let $\tau < \omega\gamma$, $\gamma < \alpha$. As before, we form $X = h''(\omega \times J_\gamma)$ and show: $X = J_\alpha$. Define a map $\tilde{h} : \omega \times \tau \times J_\gamma \to J_\alpha$ as follows: Let $y = h(i, x) \longleftrightarrow \bigvee z\, H(z, y, i, x)$, where H is Σ_0. Set

$$\tilde{h}(i, \nu, x) = \begin{cases} y \text{ if } \bigvee z \in S_{f(\nu)} H(z, y, i, x) \wedge y \in S_{f(\nu)}\,, \\ 0 \text{ if not}\,. \end{cases}$$

Then $\tilde{h}''(\omega \times \tau \times \{x\}) = h''(\omega \times \{x\})$ since range f is unbounded in $\omega\alpha$. Hence $\tilde{h}''(\omega \times \tau \times J_\gamma) = X = J_\alpha$. Let $g \in J_\alpha$, $g : \omega\gamma \xrightarrow{\text{onto}} \omega \times \tau \times J_\gamma$. Set: $\tilde{f} = \tilde{h} \cdot g$. Then $\tilde{f} : \omega\gamma \xrightarrow{\text{onto}} J_\alpha$.

Some of the results in § 3 can be regarded as a generalization of Lemmas 2.13 and 2.14. (Note that Lemma 14 is also due to Kripke and Platek).

2.4. *The relationship between* J_α *and* L_α

Set $\mathrm{def}(X) = \mathfrak{P}(X) \cap \mathrm{rud}(X)$ for transitive X. As we have seen $\mathrm{def}(X)$ is the set of all $Y \subset X$ which are $\langle X, \in\rangle$-definable from parameters in X. In its usual version, the constructible hierarchy L_α is defined by

$$L_0 = \emptyset, \quad L_{\alpha+1} = \mathrm{def}(L_\alpha), \quad L_\lambda = \bigcup_{\nu<\lambda} L_\nu \text{ for limit } \lambda .$$

We set $L = \bigcup_{\alpha \in On} L_\alpha$. It is obvious that there are many α for which $L_\alpha = J_\alpha$. For our purposes it will suffice to prove

Lemma 2.15. *If* $\omega\alpha$ *is admissible, then* $J_\alpha = L_{\omega\alpha}$.

Proof. For $\alpha = 1$ we have: $J_1 = L_\omega$ = the hereditarily finite sets. Now let $\alpha > 1$ (hence $\alpha = \omega\alpha$). Let M be admissible such that $\alpha \subseteq M$. Since the function $S(x)$ is rud, $\langle S_\nu \mid \nu < \alpha\rangle$ is $\Sigma_1(M)$ by the recursion theorem. Hence $J_\alpha = \bigcup_{\nu<\alpha} S_\nu \subset M$. Since $\omega \in M$, $\mathrm{rud}(x) = \bigcup_{n<\omega} S^n(x)$ is $\Sigma_1(M)$; hence so is $\mathrm{def}(x)$ and $\langle L_\nu \mid \nu < \alpha\rangle$. Hence $L_\alpha = \bigcup_{\nu<\alpha} L_\nu \subset M$. Since J_α is admissible, it follows immediately that $L_\alpha \subset J_\alpha$. To show $J_\alpha \subset L_\alpha$, we must prove that L_α is admissible. Let $x \in L_\alpha$ and let R be $\Sigma_0(L_\alpha)$ such that $\bigwedge y \in x \bigvee z\, Ryz$. We must find $u \in L_\alpha$ such that $\bigwedge y \in x \bigvee z \in u\, Ryz$. Since $\langle L_\nu \mid \nu < \alpha\rangle$ is $\Sigma_1(J_\alpha)$, then so is: $\bar{R}y\nu \leftrightarrow (y \in x \wedge \bigvee z \in L_\nu\, Ryz)$. By the admissibility of J_α there is $\tau < \alpha$ such that $\bigwedge y \in x \bigvee \nu < \tau\, \bar{R}y\nu$. Hence $L_\tau \in L_\alpha$ and $\bigwedge y \in x \bigvee z \in L_\tau\, Ryz$.

Note. If we wished, we could prove the following equations, which establish the precise level-by-level correspondence of L_α and J_α:

(1). $L_{\omega+\eta} = V_{\omega+\eta} \cap J_{1+\eta}$. (Hence $L_\alpha = J_\alpha$ for $\omega\alpha = \alpha$.)

(2). $\Sigma_n(L_{\omega+\eta}) = \mathfrak{P}(L_{\omega+\eta}) \cap \Sigma_n(J_{1+\eta})$ for $n \geq 1$.

§3. The Σ_n uniformisation theorem and the Σ_n projectum

In this section we shall prove

Theorem 3.1. J_α *is* Σ_n *uniformisable* ($\alpha \geq 0, n \geq 1$).

Some of the concepts and lemmas used in proving the theorem turn out to be of independent interest. One concept which is of central importance in the theory of the fine structure is that of the Σ_n projectum:

Definition. The Σ_n *projectum* of α is the largest $\rho \leq \alpha$ such that $\langle J_\rho, A \rangle$ is amenable for all $A \in \Sigma_n(J_\alpha)$($n \geq 0, \alpha \geq 0$). We denote the Σ_n projectum by ρ_α^n.

Note that $\rho_\alpha^0 = \alpha$ and $\rho_\alpha^n \geq 1$ for $\alpha, n \geq 1$. We give some examples of ρ_α^n:

(1). Let J_α be a $\mathbf{ZF}^-$ model. Then $\rho_\alpha^n = \alpha$ for $n < \omega$.

(2). Let J_α be a $\mathbf{ZF}^-$ model, all of whose elements are definable in the parameter α. Then all elements of $J_{\alpha+1}$ are Σ_1 in the parameter α. Let $h = h_{\alpha+1}$ be the canonical Σ_1 Skolem function for $J_{\alpha+1}$. Then $h''(\omega \times \{\alpha\}) = J_{\alpha+1}$. Set : $g(i) \simeq h(i, \alpha)$. Then g is a Σ_1 function which maps a subset of ω onto $J_{\alpha+1}$. Set $a = \{i \in \mathrm{dom}(g) | i \notin g(i)\}$. Then $a \subset \omega$ and $a \in \Sigma_1(J_{\alpha+1}) \setminus J_{\alpha+1}$. It follows that $\rho_{\alpha+1}^1 = 1$.

(3). Let J_α be a $\mathbf{ZF}^-$ model. It follows fairly easily that J_α is Σ_n uniformisable for $n < \omega$. Hence there is a Σ_n nice Skolem function h. Let $X = h''(\omega \times \{0\})$. Let $\pi : X \xrightarrow{\sim} J_\beta$. Then $\rho_\beta^k = \beta$ for $k < n$. However $\rho_\beta^n = 1$ by the above argument, since $h' = \pi h \pi^{-1}$ is a nice Σ_n Skolem function for J_β and $h'{}''\omega \times \{0\} = J_\beta$.

Let us note that $\omega\rho_\alpha^n$ is always strongly admissible by Lemma 2.13. The reason for introducing the Σ_n projectun is this: J_α may be "soft" with respect to predicates in $\Sigma_n(J_\alpha)$. That is, we may find Σ_n subsets of elements of J_α which are not themselves elements of J_α, or even Σ_n functions which project a subset of an element onto the whole of J_α. Thus, we try to isolate the part of J_α which remains "hard" with respect to $\Sigma_n(J_\alpha)$. $J_{\rho_\alpha^n}$ is one explication of this notion. There are at least two others which seem reasonable, however: If we set: γ_α^n = the least $\gamma \leq \alpha$

such that there is a new $\Sigma_n(J_\alpha)$ subset of $\omega\gamma$ (i.e. the least $\gamma \leq \alpha$ such that $\mathfrak{P}(\omega\gamma) \cap \Sigma_n(J_\alpha) \not\subset J_\alpha$.) δ^n_α = the least $\delta \leq \alpha$ such that there is a $\Sigma_n(J_\alpha)$ function which maps a subset of $\omega\delta$ onto J_α. Then either of J_{γ^n}, J_{δ^n} might feasibly be considered the "hard core" of J_α. It is apparent that $\rho^n \leq \gamma^n \leq \delta^n$ for $n \geq 1$. (To see $\gamma^n \leq \delta^n$, use the diagonal argument of example (2)). It turns out that, in fact, equality holds. This is the content of the following theorem.

Theorem 3.2. *There is a $\Sigma_n(J_\alpha)$ function which maps a subset of $\omega \cdot \rho^n_\alpha$ onto J_α ($n \geq 1$).*

We shall prove Theorems 3.1 and 3.2 simultaneously (it seems, in fact, hardly possible to prove one without the other). However, if one assumes Σ_n uniformisability, one can give a direct proof of $\gamma^n = \delta^n$.

Lemma 3.1. *Let $n \geq 1$. Let J_α be Σ_n uniformisable. Let γ be the least $\gamma \leq \alpha$ such that $\mathfrak{P}(\omega\gamma) \cap \Sigma_n(J_\alpha) \not\subset J_\alpha$. Then there is a $\Sigma_n(J_\alpha)$ function which maps a subset of $\omega\gamma$ onto J_α.*

Proof. Since J_α is Σ_n uniformisable, there is a Σ_n Skolem function. Let p be the least p (in $<_{J_\alpha}$) such that some Σ_n Skolem function is Σ_n in the parameter p. Let h be a Σ_n Skolem function which is Σ_n in p. Let $a \subset \omega\gamma$ be a new Σ_n subset of $\omega\gamma$. Let q = the least q (in $<_{J_\alpha}$) such that a is Σ_n in the parameter q. Set $\tilde{h}(i, x) \simeq h(i, \langle x, p, q\rangle)$. Then $\tilde{h}$ is a nice Σ_n Skolem function. Set $X = \tilde{h}''(\omega \times J_\gamma)$. Since there is a $\Sigma_1(J_\alpha)$ map $g: \omega\gamma \xrightarrow{\text{onto}} \omega \times J_\gamma$, then $\tilde{h} \cdot g$ is a $\Sigma_n(J_\alpha)$ function which maps a subset of $\omega\gamma$ onto X. Thus, it suffices to prove $X = J_\alpha$.

Clearly, $X \prec_{\Sigma_n} J_\alpha$. Let $\pi: X \xrightarrow{\sim} J_\beta$. Then $\pi \restriction J_\gamma = \mathrm{id} \restriction J_\gamma$, since J_γ is transitive and $J_\gamma \subset X$. But then $a = \pi''a$ is $\Sigma_n(J_\beta)$ in $\pi(q)$, since a is $\Sigma_n(J_\alpha)$ in q. Hence $\beta = \alpha$, since otherwise $a \in J_{\beta+1} \subset J_\alpha$. Thus a is $\Sigma_n(J_\alpha)$ in $\pi(q)$; but q is the least such and $\pi(q) \leq_{J_\alpha} q$; hence $q = \pi(q)$. $h' = \pi h \pi^{-1}$ is a Σ_n Skolem function for J_α which is Σ_n in $\pi(p)$. But p is the least such and $\pi(p) \leq_{J_\alpha} p$; hence $p = \pi(p)$ and $h' = h$. Hence $\pi\tilde{h}\pi^{-1} = \tilde{h}$, since $\tilde{h}$ is $\Sigma_n(J_\alpha)$ in $\langle p, q\rangle$. But then $\pi\tilde{h}(i, \vec{x}) = \tilde{h}(i, x)$ for $i < \omega$, $x \in J_\gamma$. Hence $\pi \restriction X = \mathrm{id} \restriction X$ and $X = J_\alpha$.[8]

We now introduce a more general notion of Σ_n projectum by

Definition. Let $\langle J_\alpha, A\rangle$ be amenable. By the Σ_n *projectum of* $\langle J_\alpha, A\rangle$ we mean the largest $\rho \le \alpha$ such that $\langle J_\rho, B\rangle$ is amenable for all $B \subset J_\rho$ such that $B \in \Sigma_n(J_\alpha, A)$ ($n \ge 0$). We denote this projectum by $\rho^n_{\alpha,A}$.

Note. In our proofs we shall actually only make use of $\rho^1_{\alpha,A}$.

Note. As before, $\omega\rho^n_{\alpha,A}$ is strongly admissible for $n \ge 1$.

Lemma 3.2. *Let* $\langle J_\alpha, A\rangle$ *be amenable. Let* $\rho = \rho^1_{\alpha,A}$. *If* $B \subset J_\rho$ *is* $\Sigma_1(J_\alpha, A)$, *then* $\Sigma_1(J_\rho, B) \subset \Sigma_2(J_\alpha, A)$.

Proof. We consider two cases.

Case 1. There is a $\Sigma_1(J_\alpha, A)$ map of some $\gamma < \omega\rho$ cofinally into $\omega\alpha$. Let g be the map. Let $Bx \leftrightarrow \bigvee z\, \bar{B}zx$, where $\bar{B}$ is $\Sigma_0(J_\alpha, A)$. Set:

$$B'\langle \nu, x\rangle \leftrightarrow \bigvee z \in S_{g(\nu)}\, \bar{B}zx\,.$$

Then B' is $\Delta_1(J_\alpha, A)$ and B is rudimentary in B' and the parameter γ, since $Bx \leftrightarrow \bigvee \nu < \gamma\, B'\langle \nu, x\rangle$. Hence $\Sigma_1(J_\rho, B) \subset \Sigma_1(J_\rho, B')$ and it suffices to show that $\Sigma_1(J_\rho, B) \subset \Sigma_2(J_\alpha, A)$, where B is $\Delta_1(J_\alpha, A)$.

For this it is enough to show that $\Sigma_0(J_\rho, B) \subset \Sigma_2(J_\alpha, A)$. But if R is $\Sigma_0(J_\rho, B)$, then R is rudimentary in B (and some parameter $p \in J_\rho$). Hence by § 1, Lemma 3, it is enough to show that the function $b(u) = B \cap u$ is $\Sigma_2(J_\alpha, A)$. But this function is in fact $\Pi_1(J_\alpha, A)$, since

$$y = b(u) \leftrightarrow \underbrace{\bigwedge x\, \underbrace{(x \in y \leftrightarrow x \in u \wedge Bx)}_{\Delta_1}}_{\Pi_1}\,.$$

Case 2. Case 1 fails.

By the method of Lemma 2.11, we have

(*) If H is $\Sigma_0(J_\alpha, A)$ and $u \in J_\rho$, then

$$\bigwedge x \in u \bigvee y\, Hxy \leftrightarrow \bigvee v \bigwedge x \in u \bigvee y \in v\, Hxy\,.$$

By (*) it follows that, if Rzx is $\Sigma_1(\mathrm{J}_\alpha, A)$, then so is $R'yx$, where

$$R'yx \longleftrightarrow (y \in \mathrm{J}_\rho \wedge \bigwedge z \in y\, Rzx)\,.$$

We want to show that $\Sigma_1(\mathrm{J}_\rho, B) \subset \Sigma_2(\mathrm{J}_\alpha, A)$. As before, it suffices to show that $\Sigma_0(\mathrm{J}_\rho, B) \subset \Sigma_2(\mathrm{J}_\alpha, A)$. Precisely as before, this reduces to showing that the function $b(u) = B \cap u$ is $\Sigma_2(\mathrm{J}_\alpha, A)$. But this function is in fact $\Sigma_1 \wedge \Pi_1$, since

$$y = b(u) \longleftrightarrow \underbrace{\bigwedge x \in y(x \in u \wedge Bx)}_{-\Sigma_1} \wedge \underbrace{\bigwedge x \in u(Bx \to x \in y)}_{\Pi_1}\,.$$

As an easy corollary of Lemma 3.2, we get

Corollary 3.3. *Let $\langle \mathrm{J}_\alpha, A\rangle$ be amenable, $\rho = \rho^1_{\alpha,A}$. Suppose that there is a $\Sigma_1(\mathrm{J}_\alpha, A)$ function which maps a subset of $\omega\rho$ onto J_α. Then there is a $B \in \Sigma_1(\mathrm{J}_\alpha, A)$ such that $B \subset \mathrm{J}_\rho$ and*

$$\Sigma_n(\mathrm{J}_\rho, B) = \mathfrak{P}(\mathrm{J}_\rho) \cap \Sigma_{n+1}(\mathrm{J}_\alpha, A)$$

for $n \geq 1$.

Proof. Let $f: u \xrightarrow{\text{onto}} \mathrm{J}_\alpha$ be $\Sigma_1(\mathrm{J}_\alpha, A)$ in the parameter p, where $u \subset \omega\rho$. Let $\langle \varphi_{i<\omega}\rangle$ be a recursive enumeration of the formulae. Set

$$B = \{\langle i, x\rangle \mid i < \omega \wedge x \in \mathrm{J}_\rho \wedge \models^{\Sigma_1}_{\langle \mathrm{J}_\alpha, A\rangle} \varphi_i[x, p]\}\;.$$

$\Sigma_n(\mathrm{J}_\rho, B) \subset \Sigma_{n+1}(\mathrm{J}_\alpha, A)$ follows by Lemma 3.2. To see the opposite direction, we note that every $x \in \mathrm{J}_\alpha$ is $\Sigma_1(\mathrm{J}_\alpha, A)$ definable in p and some $\nu < \omega\rho$. Hence if Rx is $\Sigma_1(\mathrm{J}_\alpha, A)$, then the relation $\{\langle x\rangle \mid x \in \mathrm{J}_\rho \wedge Rx\}$ is rudimentary in B and some parameter $\nu < \omega\rho$. Now let Rx be $\Sigma_{n+1}(\mathrm{J}_\alpha, A)$ $(n \geq 1)$. Suppose, for the sake of argument, that n is even (for n odd the proof is entirely similar). We then have

$$Rx \longleftrightarrow \bigvee y_1\; \bigwedge y_2 \ldots \bigwedge y_n Pyx\;,$$

where P is $\Sigma_1(J_\alpha, A)$. Let $\widetilde{P}$ be defined by

$$\widetilde{P}zx \longleftrightarrow (z, x \in J_\rho \wedge Pf(z)x) .$$

Then $\widetilde{P}$ is rudimentary in B and some $\nu < \omega\rho$. Hence $\widetilde{P}$ is $\Delta_1(J_\rho, B)$. Similarly, $D = \mathrm{dom}(f)$ is $\Delta_1(J_\rho, B)$. But if $x \in J_\rho$, then

$$Rx \longleftrightarrow \underbrace{\bigvee z_1 \in D \bigwedge z_2 \in D \dots \bigvee z_{n-1} \in D \bigwedge z_n \in D\, \widetilde{P}zx}_{\Sigma_n(J_\rho, B)} .$$

The following concept will be useful in proving Theorems 3.1 and 3.2 and will also play a large role in §4.

Definition. By a Σ_n *master code* for J_α, we mean a set $A \in \Sigma_n(J_\alpha)$ such that, setting $\rho = \rho_\alpha^n$, $A \subset J_\rho$ and

$$\Sigma_m(J_\rho, A) = \mathfrak{P}(J_\rho) \cap \Sigma_{n+m}(J_\alpha)$$

for $m \geq 1$ $(n, \alpha \geq 0)$.

The following lemma establishes Theorem 3.1 and 3.2, among other things.

Lemma 3.4. *Let α, $n \geq 0$. Let $\rho = \rho_\alpha^n$. Then*

(i). J_α *is* Σ_{n+1} *uniformisable.*

(ii). *There is a $\Sigma_n(J_\alpha)$ function which maps a subset of $\omega\rho$ onto $\omega\alpha$.*

(iii). *If $A \subset J_\rho$ is $\Sigma_n(J_\alpha)$, then $\Sigma_1(J_\rho, A) \subset \Sigma_{n+1}(J_\alpha)$.*

(iv). *α has a Σ_n master code.*

(Theorem 2 follows from (ii) since by Lemma 2.10 there is a $\Sigma_1(J_\alpha)$ map of $\omega\alpha$ onto J_α.)

Proof. Suppose not. Let α be the least α for which the theorem fails. Then $\alpha > 0$. Let n be the least n for which the theorem fails at α. Then $n > 0$, since (ii), (iii) and (iv) are trivial for $n = 0$ and (i) holds by Lemma 2.7. Let $n = m + 1$. Let $\rho = \rho_\alpha^m$ and let A be a Σ_m master code. We first prove (ii). Let δ be the least $\delta \leq \alpha$ such that some $\Sigma_n(J_\alpha)$ function f maps a subset of $\omega\delta$ onto J_α. We claim that

(1). $\delta = \rho_\alpha^n$.

If $\delta < \rho_\alpha^n$, there would be, by the usual diagonal argument, a $\Sigma_n(J_\alpha)$ set $B \subset J_\delta$ such that $\langle J_{\rho_\alpha^n}, B\rangle$ is not amenable. So suppose $\delta > \rho_\alpha^n$. Then $\delta > 1$, since $\alpha > 0$. It follows that δ is a limit ordinal, since if $\delta = \gamma + 1$, there is a Σ_1 map g of $\omega\gamma$ onto $\omega\delta$; but $f \cdot g$ would then be a $\Sigma_n(J_\alpha)$ map of a subset of $\omega\gamma$ onto J_α. Since $\delta > \rho_\alpha^n$ there is a $\Sigma_n(J_\alpha)$ set $B \subset J_\delta$ such that $\langle J_\delta, B\rangle$ is not amenable. Hence there is some $\tau < \delta$ such that $B \cap J_\tau \notin J_\delta$. But then $(B \cap J_\tau) \in J_\alpha \setminus J_\delta$, since $\tau < \delta$ and by Lemma 3.1, δ is the least ordinal such that $\Sigma_n(J_\alpha) \cap \mathfrak{P}(J_\delta) \not\subset J_\alpha$.

But this means that $B \cap J_\tau$ is J_β definable for some β such that $\delta \leq \beta < \alpha$. Let β be the least such and let r be the least r such that $B \cap J_\tau$ is $\Sigma_r(J_\beta)$. Then $\rho_\beta^r \leq \tau < \delta \leq \beta < \alpha$. Hence, by the induction hypothesis, there is a $\Sigma_r(J_\beta)$ function g which maps a subset of $\omega\tau$ onto J_β. But then $f \cdot g$ is a $\Sigma_n(J_\alpha)$ function which maps a subset of $\omega\tau$ onto J_α. Contradiction!

We now prove (iii) and (iv):

(2). $\Sigma_1(J_\delta, B) \subset \Sigma_{n+1}(J_\alpha)$ if $B \subset J_\delta$, $B \in \Sigma_n(J_\alpha)$.

Since $\rho = \rho_\alpha^m$ and A is a Σ_m master code and $B \in \mathfrak{P}(J_\delta) \cap \Sigma_n(J_\alpha)$, we have $B \in \Sigma_1(J_\rho, A)$.
Moreover, $\delta = \rho_{\rho,A}^1$. This follows from the fact that $\delta = \rho_\alpha^n \leq \rho$ and by (iv), for $\xi \leq \rho$, $B \subseteq J_\xi$, we have

$$B \in \Sigma_1(J_\rho, A) \text{ iff } B \in \Sigma_n(J_\alpha).$$

By Lemma 3.2 it follows that

$$\Sigma_1(J_\delta, B) \subset \Sigma_2(J_\rho, A) \subset \Sigma_{n+1}(J_\alpha).$$

(3). J_α has a master code.

Since f is a $\Sigma_n(J_\alpha)$ map of subset of $\omega\delta$ onto J_α then $f' = f \upharpoonright f^{-1}{}''J_\rho$ is a $\Sigma_1(J_\rho, A)$ map of subset of $\omega\delta$ onto J_ρ. Moreover $\delta = \rho_{\rho,A}^1$. Hence the conditions of Corollary 3.3 are fulfilled and we can conclude that there is a $\Sigma_1(J_\rho, A)$ set $B \subset J_\delta$ such that $\Sigma_r(J_\delta, B) = \mathfrak{P}(J_\delta) \cap \Sigma_{r+1}(J_\rho, A)$

for $r \geq 1$. But then $B \in \Sigma_n(J_\alpha)$ and $\Sigma_r(J_\delta, B) = \mathfrak{P}(J_\delta) \cap \Sigma_{n+r}(J_\alpha)$ for $r \geq 1$. Hence B is a Σ_n master code.

We now prove (i). Let B be a Σ_n master code. $\langle J_\delta, B\rangle$ is Σ_1 uniformisable by Lemma 2.7. We use this to prove the Σ_{n+1} uniformisability of J_α. Let Ryx be $\Sigma_{n+1}(J_\alpha)$. Set

$$\widetilde{R}yx \leftrightarrow (y, x \in J_\delta \ \wedge \ Rf(y)f(x)) .$$

Then $\widetilde{R}$ is $\Sigma_1(J_\delta, B)$. Let $\tilde{r}$ be a Σ_1 uniformisation of $\widetilde{R}$. Since f is Σ_n and J_α is Σ_n uniformisable, there is a $\Sigma_n(J_\alpha)$ function $\bar{f}$ which uniformises f^{-1}. Then $r = f\tilde{r}\bar{f}$ uniformises R and is $\Sigma_{n+1}(J_\alpha)$.

Our earlier proof of the Σ_n uniformisation lemma was based on what might be called the "weak projectum" rather than the projectum and was therefore more complicated. However, the earlier proof also yielded more information, which we shall now prove separately.

Definition. The *weak Σ_n projectum* of α is the greatest $\eta \leq \alpha$ such that $\langle J_\eta, A\rangle$ is amenable for every $\Delta_n(J_\alpha)$ set $A \subset J_\eta$. $(n, \alpha \geq 0)$. We denote the weak Σ_n projectum by η_α^n.

Note. As an example of a case in which the weak projectum does not equal the projectum, consider the first admissible $\alpha > \omega$. Then $\eta_\alpha^1 = \alpha$; $\rho_\alpha^1 = \omega$.

We shall prove

Theorem 3.3. *There is a $\Sigma_n(J_\alpha)$ function which maps η_α^n onto $J_\alpha (n \geq 1)$.*

We begin by proving the following analogue of Lemma 3.1.

Lemma 3.5. *Let $n \geq 1$. Let γ be the least $\gamma \leq \alpha$ such that $\mathfrak{P}(J_\gamma) \cap \Delta_n(J_\alpha) \not\subset J_\alpha$. Then there is a $\Sigma_n(J_\alpha)$ function which maps $\omega\gamma$ onto J_α.*

Proof. Let $n = m + 1$. Clearly $\rho^n \leq \gamma \leq \rho^m$. Since a Σ_n function maps a subset of $\omega\rho^m$ onto J_α, then a Σ_n function maps $\omega\rho^m$ onto J_α. Hence it suffices to show that a Σ_n functions maps $\omega\gamma$ onto $\omega\rho^m$. We first show

(*). There is a $\Sigma_n(J_\alpha)$ function g which maps $\omega\gamma$ onto an unbounded subset of $\omega\rho^m$.

Let A be a Σ_m master code for α. Let $b \subset J_\gamma$ such that $b \in \Delta_n(J_\alpha)\setminus J_\alpha$. Then $b \in \Delta_1(J_{\rho^m}, A)$. Since a $\Sigma_1(J_\gamma)$ function maps $\omega\gamma$ onto J_γ, we may assume $b \subset \omega\gamma$. Let b be defined by

$$\nu \in b \leftrightarrow \bigvee y\, B_0 y\nu\,,$$

$$\nu \notin b \leftrightarrow \bigvee y\, B_1 y\nu\,,$$

where B_0, B_1 are $\Sigma_0(J_{\rho^m}, A)$. Then $\bigwedge \nu < \gamma \bigvee y(B_0 y\nu \vee B_1 y\nu)$; however, there is no $\tau < \omega\rho^m$ such that $\bigwedge \nu < \gamma \bigvee y \in S_\tau(B_0 y\nu \vee B_1 y\nu)$, since otherwise $b \in J_{\rho^m}$ by the rudimentary closure of $\langle J_{\rho^m}, A\rangle$. Define g by

$$g(\nu) = \mu\tau \bigvee y \in S_\tau(B_0 y\nu \vee B_1 y\nu)\,.$$

Then g has the desired properties. This proves (*).

Since $\rho^n \leq \gamma$, there is a $\Sigma_n(J_\alpha)$ function f which maps a subset of $\omega\gamma$ onto $\omega\rho^m$. But then f is $\Sigma_1(J_{\rho^m}, A)$. Let f be defined by

$$\tau = f(\nu) \leftrightarrow \bigvee y\, Fy\tau\nu\,,$$

where F is $\Sigma_0(J_{\rho^m}, A)$. Define a map $\bar{f} : (\omega\gamma)^2 \to \omega\rho^m$ by

$$\bar{f}(\nu, \tau) = \begin{cases} \kappa \text{ if } & \bigvee y \in S_{g(\nu)} Fy\kappa\tau\,, \\ 0 \text{ if not}\,. \end{cases}$$

Then $\bar{f}$ is $\Sigma_1(J_{\rho^m}, A)$ and $\bar{f}$ maps $(\omega\gamma)^2$ onto $\omega\rho^m$. Let h be a $\Sigma_1(J_\gamma)$ map of $\omega\gamma$ onto $(\omega\gamma)^2$. Then $f' = \bar{f}h$ is a $\Sigma_h(J_\alpha)$ map of $\omega\gamma$ onto $\omega\gamma^m$.

Theorem 3.3 now follows by

Lemma 3.6. *Let γ be as in Lemma 3.5. Then $\gamma = \eta^n_\alpha$.*

Proof. Suppose not. Then there is an $A \in \Delta_n(J_\alpha)$ such that $\langle J_\gamma, A\rangle$ is not amenable. Hence $\gamma > 1$. But $\gamma \neq \beta + 1$, since otherwise there is a $\Sigma_1(J_\gamma)$ map of $\omega\beta$ onto $\omega\gamma$, hence by Lemma 3.5 there would be a $\Sigma_n(J_\alpha)$ map of $\omega\beta$ onto J_α. Hence γ is a limit ordinal. But then there is some $\tau < \gamma$ such that $A \cap J_\tau \notin J_\gamma$. However, $A \cap J_\tau \in J_\alpha$, since $\tau < \gamma$. Then $A \cap J_\tau$ is J_δ definable for some δ such that $\gamma \leq \delta < \alpha$. By Theorem 3.2 there is a J_δ definable map $f : \omega\tau \xrightarrow{\text{onto}} J_\delta$. Hence $f \in J_\alpha$. But, since $\delta \geq \gamma$, Lemma 3.5 would give us a $\Sigma_n(J_\alpha)$ map of $\omega\tau$ onto J_α. Contradiction!

Note. Theorems 3.2 and 3.3 may be viewed as generalisations of Lemmas 2.13 and 2.14, which are due to Kripke and Platek. They may also be regarded as sharper versions of a still earlier theorem of Putnam, to wit:

If $\mathfrak{P}(\rho) \cap L_{\alpha+1} \not\subset L_\alpha$, then $L_{\alpha+1}$ contains a well ordering of ρ of type α $(\rho \geq \omega)$.

Putnam proved the theorem for the case $\rho = \omega$, but his proof carries over mutatis mutandis.

§4. Standard codes

In §3, we proved that each α has a Σ_n *master code*; i.e. a set $A \subset J_{\rho^n_\alpha}$ such that $A \in \Sigma_n(J_\alpha)$ and $\Sigma_h(J_{\rho^n_\alpha}, A) = \mathfrak{P}(J_{\rho^n_\alpha}) \cap \Sigma_{n+h}(J_\alpha)$ for $h \geq 1$. In this section we pick canonical master codes A^n_α, which we call *standard codes*. We show that the standard codes, in a reasonable sense, are preserved under condensation arguments. This will enable us to do things in a more uniform way than if we had only the results of §3 at our disposal. For instance the Σ_n uniformisation lemma proved in §3 suffers from a serious deficiency vis-à-vis the Σ_1 uniformisation lemma proved in §2 (Lemma 2.7): J_α is not *uniformly* Σ_n uniformisable for $n > 1$. However, the results of this section will enable us, in many contexts, to replace Σ_n uniformisation over J_α by Σ_1 uniformisation over $\langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ – and we know that amenable $\langle J_\rho, A\rangle$ are Σ_1 uniformisable in a uniform way.

Definition. We define *standard codes* A^n_α and standard parameters p^n_α as follows $(n, \alpha \geq 0)$:

(1). $n = 0 : A^n_\alpha = p^n_\alpha = \emptyset$.

(2). $n = m + 1 : p^n_\alpha =$ the least $p \in J_{\rho^m}$ (in $<_{J_{\rho^m}}$) such that every $x \in J_{\rho^m}$ is $\Sigma_1(J_{\rho^m}, A^m_\alpha)$ definable from parameters in $\{p\} \cup J_{\rho^n}$;

$$A^n_\alpha = \{\langle i, x\rangle \mid i < \omega \wedge x \in J_{\rho^n} \wedge \models^{\Sigma_1}_{\langle J^m_\rho, A^m\rangle} \varphi_i[x, p^n_\alpha]\} \quad ,$$

where $\varphi_i (i < \omega)$ is a recursive enumeration of the formulae.

It is easily established that A^n_α is a Σ_n master code for J_α.

We now state our main theorem.

Theorem 4.1. *Let $n, m \geq 0$, $\alpha \geq 1$. Let $\langle J_{\bar\rho}, \bar A\rangle$ be amenable and let*

$$\pi : \langle J_{\bar\rho}, \bar A\rangle \to_{\Sigma_m} \langle J_{\rho^n_\alpha}, A^n_\alpha\rangle .$$

Then

(a). *There is a unique $\bar\alpha \geq \bar\rho$ such that $\bar\rho = \rho^n_{\bar\alpha}$* and $\bar A = A^n_{\bar\alpha}$.

(b). *There is a unique $\tilde\pi \supset \pi$ defined on $J_{\bar\alpha}$ such that for all $i \leq n$,*

$$\tilde\pi(p^i_{\bar\alpha}) = p^i_\alpha$$

and

$$(\tilde\pi \upharpoonright J_{\rho^i_{\bar\alpha}}) : \langle J_{\rho^i_{\bar\alpha}}, A^i_{\bar\alpha}\rangle \to_{\Sigma_{m+(n-i)}} \langle J_{\rho^i_{\bar\alpha}}, A^i_\alpha\rangle .$$

Definition. If $\bar\alpha, \tilde\pi$ are as in (a), (b), we call $J_{\bar\alpha} \xrightarrow{\tilde\pi} J_\alpha$ the *canonical* extension of $\langle J_{\bar\rho}, \bar A\rangle \xrightarrow{\pi} \langle J_{\rho^n}, A^n\rangle$. Before starting the proof of our theorem, we note some facts about the relation $\langle J_{\bar\alpha}, \bar A\rangle \xrightarrow{\pi}_{\Sigma_0} \langle J_\alpha, A\rangle$.

(1). Let $\langle \bar X, \bar A\rangle \xrightarrow{\pi}_{\Sigma_0} \langle X, A\rangle$, where $\bar X, X$ are transitive and $\langle \bar X, \bar A\rangle$ is rud closed. Let $\bar f$ be rud in $\bar A$ and let f be rud in A by the same rud definition. Then $\pi\bar f(x) = f(\pi(x))$.

Proof. Clearly, $\pi(x \cap \bar A) = \pi(x) \cap A$. Moreover, if g is any rud function, then $\pi g(x) = g(\pi(x))$, since the relation $y = g(x)$ is Σ_0. The conclusion follows by Lemma 1.3.

(2). If $J_{\bar{\alpha}} \xrightarrow{\pi}_{\Sigma_0} J_\alpha$, then $\pi(S_\nu) = S_{\pi(\nu)}$ for $\nu < \omega\bar{\alpha}$.
Proof. Let $\nu < \omega\bar{\alpha}$. Then $y = S_\nu \leftrightarrow \bigvee f(y = f(\nu) \wedge \varphi(f))$, where φ is a certain Σ_0 formula (see Lemma 2.2). If $y = S_\nu$ then there is $f \in J_{\bar{\alpha}}$ such that

$$J_{\bar{\alpha}} \models y = f(\nu) \wedge \varphi(f) .$$

But the above formula is Σ_0; so

$$J_\alpha \models \pi(y) = (\pi(f))(\pi(\nu)) \wedge \varphi(\pi(f)) .$$

so there is $f \in J_\alpha$ such that

$$J_\alpha \models \pi(y) = f(\pi(\nu)) \wedge \varphi(f) . \text{ i.e. } \pi(S_\nu) = S_{\pi(\nu)} .$$

(3). If $\langle J_{\bar{\alpha}}, \bar{A}\rangle \xrightarrow{\pi}_{\Sigma_0} \langle J_\alpha, A\rangle$ *cofinally* (i.e. $\sup \pi''\omega\bar{\alpha} = \omega\alpha$), then $\langle J_{\bar{\alpha}}, \bar{A}\rangle \xrightarrow{\pi}_{\Sigma_1} \langle J_\alpha, A\rangle$.
Proof. Let $\bigvee y\, \varphi(y, \pi(x))$ holds in $\langle J_\alpha, A\rangle$, where φ is Σ_0. Then for some $\nu < \omega\bar{\alpha}$,

$$\bigvee y \in S_{\pi(\nu)}\, \varphi(y, \pi(x))$$

holds in $\langle J_\alpha, A\rangle$. But this statement is Σ_0, hence $\bigvee y \in S_\nu \varphi(y, x)$ holds in $\langle J_{\bar{\alpha}}, \bar{A}\rangle$.

(4). If $\langle J_{\bar{\alpha}}, \bar{A}\rangle$ is amenable and $J_{\bar{\alpha}} \xrightarrow{\pi}_{\Sigma_0} J_\alpha$ cofinally, then there is a unique $A \subset J_\alpha$ such that $\langle J_{\bar{\alpha}}, \bar{A}\rangle \xrightarrow{\pi}_{\Sigma_0} \langle J_\alpha, A\rangle$. $\langle J_\alpha, A\rangle$ is then amenable.
Proof. Set $A = \bigcup_{\nu < \omega\bar{\alpha}} \pi(\bar{A} \cap S_\nu)$. Then A is the unique $A \subset J_\alpha$ such that $\pi(\bar{A} \cap S_\nu) = A \cap S_{\pi(\nu)}$ for $\nu < \omega\bar{\alpha}$. To see that $\langle J_\alpha, A\rangle$ is amenable, let $x \in J_\alpha$, $x \subset S_{\pi(\nu)}$ $(\nu < \omega\bar{\alpha})$. Then $x \cap A = x \cap A \cap S_{\pi(\nu)} \in J_\alpha$. By the same argument, if $x \in J_{\bar{\alpha}}$, then $\pi(x \cap \bar{A}) = \pi(x) \cap A$. Now let φ be Σ_0 and let $\models_{\langle J_{\bar{\alpha}}, \bar{A}\rangle} \varphi[x]$. Let $u \in J_{\bar{\alpha}}$ be transitive such that $x \in u$. Then $\models_{\langle u, \bar{A} \cap u\rangle} \varphi[x]$, hence $\models_{\langle \pi(u), A \cap \pi(u)\rangle} \varphi[\pi(x)]$ where $\pi(u)$ is transitive and $\pi(x) \in \pi(u)$). Hence $\models_{\langle J_\alpha, A\rangle} \varphi[\pi(x)]$.

Before beginning the proof of our theorem, we generalise the definitions of $\rho_\alpha^n, A_\alpha^n, p_\alpha^n$:

Definition. Let $\langle J_\beta, B\rangle$ be amenable. Suppose that some $\rho \leq \beta$ satisfies the conditions:

(a). There is a $\Sigma_1(J_\beta, B)$ map of a subset of J_ρ onto J_β.

(b). If $A \in \mathfrak{P}(J_\rho) \cap \Sigma_1(J_\beta, B)$, then $\langle J_\rho, A\rangle$ is amenable.

Then ρ is uniquely determined and we set $\rho_{\beta,B}^1 = \rho$. $p_{\beta,B}^1$ = the least p (in $<_{J_\beta}$) such that every $x \in J_\beta$ is Σ_1 definable in parameters from $J_\rho \cup \{p\}$.

$$A_{\beta,B}^1 = \{\langle i, x\rangle \mid i < \omega \wedge x \in J_\rho \wedge \models^{\Sigma_1}_{\langle J_\beta, B\rangle} \varphi_i[x, p]\} \ .$$

Thus $\rho_\beta^n, p_\beta^n, A_\beta^n$ are definable by

$$\rho^0 = \beta \, , \quad p^0 = A^0 = \emptyset \, ,$$

$$\rho^{n+1} = \rho^1_{\rho^n, A^n} \, , \quad p^{n+1} = p^1_{\rho^n, A^n} \, ,$$

$$A^{n+1} = A^1_{\rho^n, A^n} \, .$$

We prove the theorem by induction on n. For $n = 0$ it is trivial. Now let $n > 0$ and suppose it to hold for $n - 1$. Set $\langle J_\beta, B\rangle = \langle J_{\rho_\alpha^{n-1}}, A_\alpha^{n-1}\rangle$. Set $p = p_\alpha^{n-1}$. Clearly it is enough to prove

(i). There is a unique $\langle J_{\bar\beta}, \bar B\rangle$ such that $\bar\rho = \rho^1_{\bar\beta \bar B}$ and $\bar A = A^1_{\bar\beta \bar B}$.

(ii). There is a unique $\tilde\pi \supset \pi$ such that $\tilde\pi(p^1_{\bar\beta \bar B}) = p$ and $\langle J_{\bar\beta}, \bar B\rangle \xrightarrow{\pi}_{\Sigma_{m+1}} \langle J_\beta, B\rangle$.

We begin by proving the existence part of (i) and (ii). Set $\rho = \rho_\alpha^n$, $A = A_\alpha^n$. Define $\tilde\rho \leq \rho$ by $\omega\tilde\rho = \sup_{\nu < \omega\bar\rho} \pi(\nu)$. Set $\tilde A = A \cap J_{\tilde\rho}$. Then $\langle J_{\bar\rho}, \bar A\rangle \xrightarrow{\pi}_{\Sigma_0} \langle J_{\tilde\rho}, \tilde A\rangle$ cofinally. Set X = the set of all $x \in J_\beta$ which are $\Sigma_1(J_\beta, B)$ in parameters from $\mathrm{rng}(\pi) \cup \{p\}$.

Lemma 4.1. $X \cap J_{\tilde\rho} = \mathrm{rng}(\pi)$.

Proof. Let $y \in X \cap J_{\tilde{\rho}}$. Then for some $x \in \mathrm{rng}(\pi)$ and some Σ_1 formula φ_i, y is the unique y such that $\models_{\langle J_\beta, B\rangle} \varphi_i[\langle y, x\rangle, p]$. Hence y is the unique $y \in J_{\tilde{\rho}}$ such that $\tilde{A}(i, \langle y, x\rangle)$. But $\langle J_{\bar{\rho}}, \bar{A}\rangle \xrightarrow{\pi}_{\Sigma_1} \langle J_{\tilde{\rho}}, \tilde{A}\rangle$; hence $y \in \mathrm{rng}(\pi)$.

Now let $J_{\bar{\beta}}$ be the transitivisation of X and set $\tilde{\pi} : \langle J_{\bar{\beta}}, \bar{B}\rangle \stackrel{\sim}{\longleftrightarrow} \langle X, X \cap B\rangle$. Then $\langle J_{\bar{\beta}}, \bar{B}\rangle \xrightarrow{\tilde{\pi}}_{\Sigma_1} \langle J_\beta, B\rangle$. Since $(\tilde{\pi} \restriction J_{\bar{\rho}}) : J_{\bar{\rho}} \stackrel{\sim}{\longleftrightarrow} X \cap J_{\tilde{\rho}} = \mathrm{rng}(\pi)$, we have $\tilde{\pi} \restriction J_{\bar{\rho}} = \pi$.

Lemma 4.2. $\langle J_{\bar{\beta}}, \bar{B}\rangle \xrightarrow{\tilde{\pi}}_{\Sigma_{m+1}} \langle J_\beta, B\rangle$.

Proof. For $m = 0$ the assertion is proven. Assume $m > 0$. We must show:

(*). If $y \in J_\beta$ is $\Sigma_{m+1}(J_\beta, B)$ in parameters from $\mathrm{rng}(\pi) \cup \{p\}$, then $y \in X$.

Let y be defined by the condition

$$\models_{\langle J_\beta, B\rangle} \varphi(y, x, p)\,, \tag{1}$$

where φ is Σ_{m+1}. Let h be the canonical Skolem function for $\langle J_\beta, B\rangle$ and set $\tilde{h}(\langle i, x\rangle) \simeq h(i, \langle x, p\rangle)$. Then $\tilde{h}''J_\rho = J_\beta$ and $\tilde{h}''\mathrm{rng}(\pi) = X$. Hence it suffices to show that the condition

$$\models_{\langle J_\beta, B\rangle} \varphi(\tilde{h}(z), x, p) \tag{2}$$

is satisfied by some $z \in X \cap J_\rho$. Let $\varphi = \bigvee z_1 \bigwedge z_2 \ldots \bigvee z_m\, \psi$, where ψ is Σ_1 if m is even and Π_1 if m is odd. Then we must show that the condition

$$\bigvee_{z_1 \in J_\rho} \bigwedge_{z_2 \in J_\rho} \ldots \bigvee_{z_n \in J_\rho} \models_{\langle J_\beta, B\rangle} \psi[\tilde{h}(z), \tilde{h}(z), x, p] \tag{3}$$

is satisfied by some $z \in J_\rho \cap X$. But (3) is clearly $\Sigma_m(J_\rho, A)$ in the parameters x. Since $x \in X \cap J_\rho \prec_{\Sigma_m} \langle J_\rho, A\rangle$, we conclude that (3) is satisfied by some $z \in X \cap J_\rho$.

Obviously $p \in X$. Set $\bar{p} = \tilde{\pi}^{-1}(p)$.

Lemma 4.3. $\bar{A} = \{\langle i, x\rangle \mid i < \omega \wedge x \in J_{\bar{\rho}} \wedge \models^{\Sigma_1}_{\langle J_{\bar{\beta}}, \bar{B}\rangle} \varphi_i[x, \bar{p}]\}$.

Proof. We have

(1) $\qquad \bar{A}\langle i, x\rangle \longleftrightarrow A\langle i, \pi(x)\rangle$

and

(2) $\qquad \models_{\langle J_{\bar{\beta}}, \bar{B}\rangle} \varphi_i[x, \bar{p}] \longleftrightarrow \models_{\langle J_\beta, B\rangle} \varphi_i[\pi(x), p]$.

Since the right-hand sides of (1) and (2) are equivalent, so are the left.

Lemma 4.4. $\bar{\rho} = \rho^1_{\bar{\beta}\bar{B}}$.

Proof. ($\geq$). By the construction of $J_{\bar{\beta}}$, every $x \in J_{\bar{\beta}}$ is $\Sigma_1(J_{\bar{\beta}}, \bar{B})$ in parameters from $J_{\bar{\rho}} \cup \{\bar{p}\}$. Hence if $\bar{h}$ is the canonical Σ_1 Skolem function for $\langle J_{\bar{\beta}}, \bar{B}\rangle$, we have

$$J_{\bar{\beta}} = \bar{h}''(\omega \times J_{\bar{\rho}} \times \{\bar{p}\}).$$

($\leq$). Let $C \in \mathfrak{P}(J_{\bar{\rho}}) \cap \Sigma_1(J_{\bar{\beta}}, \bar{B})$. We must show that $\langle J_{\bar{\rho}}, C\rangle$ is amenable. Every $x \in J_{\bar{\beta}}$ is $\Sigma_1(J_{\bar{\beta}}, \bar{B})$ in parameters from $J_{\bar{\rho}} \cup \{\bar{p}\}$, hense so is C. By Lemma 4.3 it follows that C is rud in $\bar{A}$. Hence $\langle J_{\bar{\rho}}, C\rangle$ is amenable.

Lemma 4.5. $\bar{p} = p^1_{\bar{\beta}\bar{B}}$; $\bar{A} = A^1_{\bar{\beta}\bar{B}}$.

Proof. By Lemma 4.3 it suffices to show $\bar{p} = p^1_{\bar{\beta}\bar{B}}$. Now $\bar{p}$ satisfies the condition:

(*) Each $x \in J_{\bar{\beta}}$ is $\Sigma_1(J_{\bar{\beta}}, \bar{B})$ in parameters from $J_{\bar{\rho}} \cup \{\bar{p}\}$.

We must show that $\bar{p}$ is the least such in $<_J$. Suppose not. Let $p' <_J \bar{p}$

satisfy (*). Then $\bar{p} = \bar{h}(i, \langle x, p' \rangle)$ for some $x \in J_{\bar{\rho}}$. But then $p = h(i, \langle \pi(x), \tilde{\pi}(p') \rangle)$, where $\pi(x) \in J_\rho$ and $\tilde{\pi}(p') <_J p$. But then every $x \in J_\beta$ would be $\Sigma_1(J_\beta, B)$ in parameters from $J_\rho \cup \{\tilde{\pi}(p')\}$ and we would have $p >_J \tilde{\pi}(p') \geq_J p^1_{\beta,B}$. Contradiction!

This establishes the existence part of (i) and (ii). It remains only to prove uniqueness.

Lemma 4.6. *There is at most one* $\langle J_{\bar{\beta}}, \bar{B} \rangle$ *such that* $\bar{p} = \rho^1_{\bar{\beta},\bar{B}}$ *and* $\bar{A} = A^1_{\bar{\beta},\bar{B}}$.

Proof. Let $\langle J_{\beta_i}, B_i \rangle$ have the property ($i = 0, 1$). Set $p_i = p^1_{\beta_i B_i}$. Then

$$(1) \qquad \models_{\langle J_{\beta_0}, B_0 \rangle} \varphi[x, p_0] \leftrightarrow \models_{\langle J_{\beta_1}, B_1 \rangle} \varphi[x, p_1]$$

for Σ_1 formulae φ and $x \in J_{\bar{\rho}}$, since

$$\models^{\Sigma_1}_{\langle J_{\beta_i}, B_i \rangle} \varphi_j[\langle x \rangle, p_i] \leftrightarrow \bar{A}\langle j, x \rangle .$$

Let h_i be the canonical Skolem function for $\langle J_{\beta_i}, B_i \rangle$ and set $\tilde{h}_i(\langle j, x \rangle) \simeq h_i(j, \langle x, p_i \rangle)$. Then $J_{\beta_i} = \tilde{h}_i{}'' J_{\bar{\rho}}$. By (1) we have

$$(2) \qquad \tilde{h}_0(x) \in \tilde{h}_0(y) \leftrightarrow \tilde{h}_1(x) \in \tilde{h}_1(y) ,$$

$$\tilde{h}_0(x) = \tilde{h}_0(y) \leftrightarrow \tilde{h}_1(x) = \tilde{h}_1(y) ,$$

$$B_0 \tilde{h}_0(x) \leftrightarrow B_1 \tilde{h}_1(x) .$$

for $x, y \in J_{\bar{\rho}}$. Thus we may define an isomorphism $\sigma : \langle J_{\beta_0}, B_0 \rangle \stackrel{\sim}{\longleftrightarrow} \langle J_{\beta_1}, B_1 \rangle$ by: $\sigma \tilde{h}_0(x) = \tilde{h}_1(x)$. But σ is an $\in$-isomorphism; hence $\sigma = \mathrm{id} \restriction J_{\beta_0}$.

Lemma 4.7. *There is at most one* $\tilde{\pi} \supset \pi$ *such that* $\langle J_{\bar{\beta}}, \bar{B} \rangle \xrightarrow{\tilde{\pi}}_{\Sigma_1} \langle J_\beta, B \rangle$ *and* $\tilde{\pi}(\bar{p}) = p$.

Proof. Let $\tilde{\pi}_i$ have the property ($i = 0, 1$). Let $\bar{h}$ be the canonical Σ_1 Skolem function for $\langle J_{\bar{\beta}}, \bar{B}\rangle$. Then

$$\tilde{\pi}_0 \bar{h}(j, x, \bar{p}) = \tilde{\pi}_1 \bar{h}(j, x, \bar{p}) = h(j, \pi(x), p)$$

for $x \in J_{\bar{\rho}}$. Hence $\tilde{\pi}_0 = \tilde{\pi}_1$.

§5. Combinatorial principles in L

In this section we use the results of §4 to derive some combinatorial principles from the assumption V = L. These principles enables us to carry out inductions which would otherwise break down. In §6 and 7 we shall make use of them to settle some classical problems of set theory and model theory on the assumption V = L.

Definition. Let α be a limit ordinal. $A \subset \alpha$ is *Mahlo* (*stationary*) in α iff $A \cap C \neq \emptyset$ for every $C \subset \alpha$ which is closed and unbounded in α.

Theorem 5.1. *Assume* V = L. *Then there is a class E of limit ordinals and a sequence C_λ defined on singular limit ordinals λ such that*

(i) *$E \cap \kappa$ is Mahlo in κ for all regular $\kappa > \omega$;*

(ii) *C_λ is closed, unbounded in λ;*

(iii) *if $\gamma < \lambda$ is a limit point of C_λ, then γ is singular, $\gamma \notin E$ and $C_\gamma = \gamma \cap C_\lambda$.*

(Hence, in particular, there is a class E such that $E \cap \kappa$ is Mahlo in all regular κ but no singular κ.)

We begin the proof of Theorem 5.1 by defining the set E.

Definition. E is the set of limit ordinals α such that for some $\beta > \alpha$

(i) J_β is a **ZF**$^-$ model,

(ii) α is the largest cardinal in J_β,

(iii) α is regular in J_β,

(iv) for some $p \in J_\beta$, J_β is the smallest $X \prec J_\beta$ such that $p \in X$ and $\alpha \cap X$ is transitive.

Note: $\mathbf{ZF}^- = \mathbf{ZF}$ without the power set axiom.

Lemma 5.1. *If κ is regular, then $E \cap \kappa$ is Mahlo in κ.*

Proof. Let $C \subset \kappa$ be closed and unbounded in κ. We claim that $C \cap E \neq \emptyset$. Let $U =$ the smallest $U \prec J_{\kappa^+}$ such that $C \in U$ and $\kappa \cap U$ is transitive. Let $\pi^{-1} : U \overset{\sim}{\longleftrightarrow} J_\beta$. Then $J_\beta \xrightarrow{\pi}_{\Sigma_\omega} J_{\kappa^+}$. Let $\alpha = \kappa \cap U$. It is clear that $\pi \restriction J_\alpha = \mathrm{id} \restriction J_\alpha$, since $J_\alpha \subset U$. Moreover, if $X \in U$ and $X \in J_\kappa$, then $\pi^{-1}(X) = X \cap J_\alpha$, since $J_\alpha = U \cap J_\kappa$. In particular, $\pi^{-1}(\kappa) = \alpha$: $\pi^{-1}(C) = C \cap \alpha$. Since $E \cap \kappa$ is J_{κ^+}-definable, we have

$$E \cap \kappa \in U \quad \text{and} \quad \pi^{-1}(E \cap \kappa) = E \cap \alpha .$$

By the definition of U, J_β is the smallest $U' \prec J_\beta$ such that $C \cap \alpha \in U'$ and $\alpha \cap U'$ is transitive. Hence $\alpha \in E$. But $C \cap \alpha$ is unbounded in α, since $\pi(C \cap \alpha) = C$ is unbounded in κ. Hence $\alpha \in C$, since C is closed.

We now define the sequence C_α. We consider several cases, all but one of which are trivial.

Case 1. $\alpha < \omega_1$. Let C_α be any unbounded subset of order type ω. Let $s : \mathrm{On}^2 \leftrightarrow \mathrm{On}$ be Gödel's pairing function and let $Q = \{\nu | (s \restriction \nu^2) : \nu^2 \leftrightarrow \nu\}$.

Case 2a. $\alpha > \omega_1$ and $s''\alpha^2 \not\subseteq \alpha$. Let γ be the maximal $\gamma < \alpha$ such that $s''\gamma^2 \subset \gamma$. Then $E \cap (\alpha \setminus \gamma) = \emptyset$, since $\beta \in E \to s''\beta^2 \subset \beta$. Set $C_\alpha = \alpha \setminus \gamma$.

Case 2b. $\alpha > \omega_1$ and $s''\alpha^2 \subseteq \alpha$ and $Q \cap \alpha$ is bounded in α. Let γ be the maximal $\gamma < \alpha$ such that $s''\gamma^2 \subseteq \gamma$. Then there is a $\Sigma_1(J_\alpha)$ map f of ω onto an unbounded subset of α (E.g. define f by $f(0) = \gamma$, $f(n+1) = \sup s''f(n)^2$. Since s is uniformly definable in terms of $<_J$, f is easily seen to be $\Sigma_1(J_\alpha)$ in the parameter γ.) Set $C_\alpha = \mathrm{rng}(f)$.

Note. If cases (1) and (2) fail, then $\omega\alpha = \alpha$.

Before considering the next case, we have the definitions:

Definition. α is *regular* in β iff $\beta \geq \alpha$ and there is no $\Sigma_\omega(J_\beta)$ mapping of a $\gamma < \alpha$ cofinally into α (i.e. onto an unbounded subset of α.)

Definition. α is Σ_n *regular in* β iff $\beta \geq \alpha$ and there is no $\Sigma_n(J_\beta)$ function mapping a subset of some $\gamma < \alpha$ cofinally into α.

We set (for singular α) $\beta = \beta(\alpha) = \mu\beta \geq \alpha$ such that α is not regular in β, $n = n(\alpha) = \mu n \geq 1$ such that α is not Σ_n regular in $\beta(\alpha)$.

Case 3. $n = 1$ and β is a successor ordinal. Then α is ω-cofinal and we again take C_ε as being of order type ω. We show that α is ω-cofinal as follows:

Let $f : u \to \alpha$ be $\Sigma_1(J_\beta)$ where $\gamma < \alpha$, $u \subset \gamma$ and $f''\gamma$ is unbounded in α. Let

$$\tau = f(\nu) \longleftrightarrow \bigvee z\, Fz\tau\nu\ ,$$

where F is Σ_0. Let $\beta = \delta + 1$. Define f_i $(i < \omega)$ by

$$\tau = f_i(\nu) \longleftrightarrow \bigvee z \in S_{\omega\delta+i}\, Fz\tau\nu\ .$$

Then f_i is J_δ definable, since $f_i \in J_\beta$ and $f_i \subset J_\delta$. Set $\alpha_i = \sup f_i''\gamma$. Then $\alpha_i < \alpha$, since α is regular in δ, but $\sup\limits_i \alpha_i = \sup f''\gamma = \alpha$.

Before proceeding to the last case, we note that each $\alpha \in E$ falls under case 3. This follows from

Lemma 5.2. *If $\alpha \in E$ and β is as in the definition of E. Then α is not Σ_1 regular in $\beta + 1$*

Proof. Let $p \in J_\beta$ be such that J_β is the smallest $X \prec J_\beta$ such that $p \in S$ and $\alpha \cap X$ is transitive. Let h be the canonical Σ_1 Skolem function for $J_{\beta+1}$. Let

$$y = h(i, x) \longleftrightarrow \bigvee z\, Hzyix \quad \text{where } H \text{ is } \Sigma_0 .$$

Define h_j $(j < \omega)$ by

$$y = h_j(i, x) \longleftrightarrow y, x \in S_{\omega\beta+j} \wedge \bigvee z \in S_{\omega\beta+j}\, Hzyix\ .$$

Then $h_j \in J_{\beta+1}$; moreover $\langle h_j | j < \omega\rangle$ is $\Sigma_1(J_{\beta+1})$. Define α_i, X_i $(i < \omega)$ by

$$\alpha_0 = \omega\,, \quad X_i = h_i''(\omega \times (J_{\alpha_i} \times \{p\}))\,,$$

$$\alpha_{i+1} = \alpha_{X_i} \quad (\alpha_X =_{\mathrm{df}} \sup(\alpha \cap X))\,.$$

By induction on i, we get $\alpha_i < \alpha$, using the facts: $h_i \cap J_\beta$ is J_β definable; α is regular in J_β; there is a function in J_β which maps α_i onto J_{α_i}. Clearly, $\langle \alpha_i | i < \omega\rangle$ is $\Sigma_1(J_{\beta+1})$. Thus it suffices to show that $\bar\alpha = \alpha$ where $\bar\alpha = \sup_i \alpha_i$.

Now $\bar\alpha = \alpha_X$, where $X = \bigcup_i X_i$. But $X = h''(\omega \times (J_{\bar\alpha} \times \{p\}))$; hence $X \prec_{\Sigma_1} J_{\beta+1}$. Set $Y = X \cap J_\beta$. Then $Y \prec J_\beta$. $p \in Y$ and $\bar\alpha = \alpha \cap Y$ is transitive. Hence $Y = J_\beta$. Hence $\bar\alpha = \alpha$.

We turn now to the most difficult case.

Case 4. Cases 1–3 fail. Let $\beta = \beta(\alpha)$, $n = n(\alpha)$. Set $\rho = \rho(\alpha) = \rho_\beta^{n-1}$, $A = A(\alpha) = A_\beta^{n-1}$.
Then $\rho_\beta^n \le \alpha \le \rho$, since α is Σ_{n-1} regular but not Σ_n regular in β.
Set

$$p = p(\alpha) = \text{ the least } p \text{ (in } <_{J_\rho}) \text{ such that every } x \in J_\rho \text{ is } \Sigma_1(J_\rho, A) \text{ in parameters from } \alpha \cup \{p\}\,.$$

(Note that $p \le_J p_\beta^n$ but *not* necessarily $p_\beta^n \le p$.)

Let h be the canonical Σ_1 Skolem function for $\langle J_\rho, A\rangle$. Set

$$\tilde h(i, x) \simeq h(i, \langle x, p\rangle)\,.$$

Then $\tilde h$ is a nice Σ_1 Skolem function for $\langle J_\rho, A\rangle$ and $\tilde h''(\omega \times \alpha) = J_\rho$.

Lemma 5.3. *There is a* $\gamma < \alpha$ *such that* $\alpha \cap \tilde h''(\omega \times \gamma)$ *is unbounded in* α.

Proof. There is a $\tau < \alpha$ and a $\Sigma_1(J_\rho, A)$ function f such that $f''\tau$ is unbounded in α. We may assume that $\tau \le \rho_\beta^n$ (since there is a $\Sigma_1(J_\rho, A)$ function mapping a subset of ρ_β^n onto α). Let f be Σ_1 in the parameter q. Then $q = \tilde h(i_0, \nu_0)$ for some $i_0 < \omega$, $\nu_0 < \alpha$. Let $s : \mathrm{On} \leftrightarrow \mathrm{On}^2$ be

Gödels pair enumeration. Let

$$Q = \{\nu \mid (s \upharpoonright \nu) : \nu \longleftrightarrow \nu^2\} .$$

Since cases 1 and 2 fail, we have $\alpha > \omega_1$ and $Q \cap \alpha$ is unbounded in α. Pick $\gamma \in Q$ such that $\gamma_0 < \gamma, \tau \leq \gamma, \gamma < \alpha$. It suffices to show that $f''\tau \subset \widetilde{h}''(\omega \times \gamma)$. Let $X = \widetilde{h}''(\omega \times \gamma)$. Since $q \in X$ and $\tau \subset X$, it suffices to show $X \prec_{\Sigma_1} \langle J_\rho, A\rangle$. Since $\widetilde{h}''(\omega \times X) \subset X$, it suffices to show that X is closed under ordered pairs. This follows by $\gamma \in Q$. Let $x_0, x_1 \in X$, $x_i = \widetilde{h}(j_i, \eta_i)$ $(i = 0, 1; j_i < \omega, \eta_i < \gamma)$. Let $s_\eta = \langle \eta_0, \eta_1\rangle$. Since $s \upharpoonright \rho$ is $\Sigma_1(J_\rho, A)$ in no parameters, η_0, η_1 are Σ_1 in η and x_0, x_1 are Σ_1 in p, η. Hence $\langle x_0, x_1\rangle = \widetilde{h}(j, \eta)$ for some $j < \omega$.

Now let h have the (uniform) definition

$$y = h(i, x) \longleftrightarrow \bigvee z\ Hzyix ,$$

where H is Σ_0. For $\tau < \rho$, set

$$y = h_\tau(i, x) \longleftrightarrow y, x \in J_\tau \wedge \bigvee z \in J_\tau\ Hzyix .$$

Then h_τ is the canonical Σ_1 Skolem function for amenable $\langle J_\tau, A \cap J_\tau\rangle$. We define a map $g : u \to \alpha$ where $u \subset \alpha$ by

$$g(\omega\nu + i) \simeq \begin{cases} \widetilde{h}(i, \nu) & \text{if } \widetilde{h}(i, \nu) < \alpha , \\ \text{undefined otherwise} . \end{cases}$$

Then g is $\Sigma_1(J_\rho, A)$ in p, α (if $\alpha < \rho$) and

$$\tau = g(\nu) \longleftrightarrow \bigvee z\ G(z, \tau, \nu) ,$$

where

$$G(z, \tau, \nu) \longleftrightarrow \nu < \alpha \wedge \tau < \omega\alpha \wedge H(z, \tau(\nu/\omega), \langle [\nu/\omega], p\rangle) .$$

G is uniformly $\Sigma_1(J_\tau, A \cap J_\tau)$ in p and α (if $\alpha < \rho$) for $\tau < \rho$ such that $\langle J_\tau, A \cap J_\tau\rangle$ is amenable and $p \in J_\tau$, $\alpha < \tau$ (if $\alpha < \rho$).

Note that $g''\omega\tau = \alpha \cap \widetilde{h}''(\omega \times \tau)$ for $\tau \leq \rho$. Set γ = the least γ such that $\sup g''\gamma = \alpha$. Then $\gamma < \alpha$ by Lemma 5.3. Obviously γ is a limit ordinal. Since $\sup g''\tau > \tau$ for $\gamma \leq \tau < \alpha$, there is a maximal $\kappa < \alpha$ such that $\sup g''\kappa \leq \kappa$. But then $\kappa < \gamma$. This κ is fixed for the rest of the proof. We have $\kappa < \gamma < \alpha$ and

$$(*) \qquad \sup g''\tau > \tau \text{ for } \kappa < \tau < \alpha .$$

Lemma 5.4. *If $X \prec_{\Sigma_1} \langle \mathrm{J}_\rho, A\rangle$, $\kappa \in X, p \in X$ and $\alpha \cap X$ is transitive, then $\alpha \cap X = \alpha$.*

Proof. Since $\alpha \cap X$ is transitive we have $\alpha_X =_{\mathrm{df}} \sup(\alpha \cap X) = \alpha \cap X$. $g \subset \alpha^2$ is $\Sigma_1(\mathrm{J}_\rho, A)$. So $g''\alpha_X \subset \alpha \cap X$ and $\sup g''\alpha_X \leq \alpha_X$. Suppose $\alpha \cap X \neq \alpha$. Then $\alpha \cap X \subsetneqq \alpha$ and since $\alpha \cap X$ is transitive, $\alpha_X < \alpha$. $\kappa \in \alpha \cap X$; so $\kappa < \alpha_X$. Then by (*), $\sup g''\alpha_X > \alpha_X$. Contradiction!

We turn now to the definition of C_α. We shall first define three functions k, l, m from a limit ordinal $\theta \leq \gamma$ into γ, α, ρ, respectively, such that

$$\sup_{\nu<\theta} k(\nu) = \gamma\,, \quad \sup_{\nu<\theta} l(\nu) = \alpha\,, \quad \sup_{\nu<\theta} m(\nu) = \rho\,.$$

k will be monotone and l, m will be normal. k maps into $\mathrm{dom}(g)$ in such a way that gk is monotone and $gk(\nu) > k(\nu)$. l will be defined in such a way that $l(\nu) < gk(\nu) < l(\nu+1)$. C_α will then be defined as a closed cofinal subset of $\{l(\nu) | \nu < \theta\}$.

We define k, l, m by the following simultaneous recursion:

(a). $k(\nu)$ = the least $\tau \in \mathrm{dom}(g)\setminus\kappa$ such that

(i) $\tau > k(\iota)$ for $\iota < \nu$,

(ii) $g(\tau) > l(\nu)$,

(iii) $m(\nu) \in \widetilde{h}''(\omega \times g(\tau))$.

Using (*) it is easily seen that $g(k(\nu)) > k(\nu)$.

(b). $m(0) = \max(\kappa+1, \mu\tau\, p \in \mathrm{J}_\tau)$; $m(\nu+1)$ = the least $\eta < \rho$ such that

(i) $m(\nu), k(\nu), gk(\nu) < \eta$,

(ii) $\bigvee z \in \mathrm{J}_\eta\ G(z, gk(\nu), k(\nu))$,

(iii) $m(\nu) \in h''_\eta(\omega \times gk(\nu) \times \{p\})$,

(iv) $A \cap \mathrm{J}_{m(\nu)} \in \mathrm{J}_\eta$.

$m(\lambda) = \sup_{\nu<\lambda} m(\nu)$ if $\sup_{\nu<\lambda} m(\nu) < \rho$ for limit λ.

(c). $l(\nu) = \alpha_{X_\nu}$ if $\alpha_{X_\nu} < \alpha$, where

$$X_\nu = h_{m(\nu)}{}''(\omega \times J_\eta \times \{p\}) ,$$

$$\eta = \max(\kappa + 1, \sup_{\iota<\nu} gk(\iota)) .$$

Thus k, l, m are all defined on the ordinal $\theta \leq \gamma$, where $\theta = \mathrm{dom}(\kappa) \cap \mathrm{dom}(l) \cap \mathrm{dom}(m)$. It is clear that $l(\nu) < gk(\nu) < l(\nu + 1)$ and that $m(\iota) \in X_\nu$ for $\iota < \nu$. It is also easily seen that l is normal. To see that θ is a limit ordinal, we must show that $l(\nu + 1)$ is defined, where $l(\nu)$ is. But

$$l(\nu + 1) = \sup (\alpha \cap h_{m(\nu+1)}{}''(\omega \times J_{\eta(\nu+1)} \times \{p\}) ,$$

where $\eta(\nu + 1) = g(k(\nu))$. $h_{m(\nu+1)} \in J_\rho$ and there is an $f \in J_\rho$ which maps $\eta(\nu + 1)$ onto $\omega \times J_{\eta(\nu+1)} \times \{p\}$. Hence there is an $f \in J_\rho$ mapping $\eta(\nu + 1)$ unboundedly into $l(\nu + 1)$. But $\eta(\nu + 1) < \alpha$ and α is regular in all $\tau < \rho$. Hence $l(\nu + 1) < \alpha$.

Lemma 5.5.

(i). $\sup_{\nu<\theta} k(\nu) = \gamma$.

(ii). $\sup_{\nu<\theta} m(\nu) = \rho$.

(iii). $\sup_{\nu<\theta} l(\nu) = \alpha$.

Proof. It suffices to prove (iii). (i) then follows since $\sup g''\tau = \alpha$, where $\tau = \sup_{\nu<\theta} k(\nu)$, hence $\tau = \gamma$. (ii) follows since otherwise, letting $\tau = \sup_{\nu<\theta} m(\nu)$, where $\tau < \rho$, $\alpha \cap h_\tau{}''(\omega \times J_\gamma)$ is unbounded in α; but $h_\tau \in J_\rho$ and there is an $f \in J_\rho$ mapping γ onto J_γ; hence α would fail to be regular in some $\eta < \rho$.

Proof of (iii): Suppose not. Let $\bar{\alpha} = \sup_{\nu<\theta} l(\nu) < \alpha$. Then $\sup_{\nu<\theta} m(\nu) < \rho$, since otherwise $\bar{\alpha} = \bigcup_\nu \alpha_{X_\nu} = (\alpha \cap \bigcup_\nu X_\nu)$, and letting $X = \bigcup_\nu X_\nu$ we have

$$X = h''(\omega \times (J_{\bar{\alpha}} \times \{p\})) = \tilde{h}(\omega \times J_{\bar{\alpha}}) .$$

Then by Lemma 5.4, we have $\bar{\alpha} = \alpha$, since $\bar{\alpha} = \alpha \cap X$, $\kappa < \bar{\alpha}$, $p \in X$ and $X \prec_{\Sigma_1} \langle J_\rho, A \rangle$. Contradiction!

But then $\sup_{\nu<\theta} k(\nu) \geq \gamma$ since otherwise $k(\theta)$, $m(\theta)$, $l(\theta)$ would be defined. Let τ = the least $\tau \in \mathrm{dom}(g) \setminus \kappa$ such that $g(\tau) > \bar{\alpha}$. Then there is a least $\nu < \theta$ such that $k(\nu) > \tau$. But $k(\nu)$ = the least $\tau' \in \mathrm{dom}(g) \setminus \kappa$ such that

$$(**)\ \tau' > k(\iota) \text{ for } \iota < \nu \text{ and } g(\tau') > l(\nu), \text{ and } m(\nu) \in \tilde{h}''(\omega \times g(\tau')).$$

But then $\tau \geq k(\nu)$, since τ satisfies (**). Contradiction!

As a corollary of the proof of Lemma 5.5 we obtain

Lemma 5.6. *Let $\lambda < \theta$ be a limit ordinal. Let $\tau = \sup_{\nu<\lambda} k(\nu)$. Then $\tau > \kappa$ and $l(\lambda) = \sup g''\tau$ (hence $l(\lambda) > \tau \geq \lambda$).*

Proof. Suppose not. Let $\eta < \tau$ such that $g(\eta) > l(\lambda)$. Let ν be the least $\nu < \tau$ such that $k(\nu) > \eta$. As before we get $k(\nu) \leq \eta$. Contradiction!

For $\nu < \theta$ let g_ν be related to $h_{m(\nu)}$ as g is related to h; i.e.

$$\tau = g_\nu(\iota) \leftrightarrow \tau, \iota < l(\nu) \wedge \bigvee z \in J_{m(\nu)} G(z, \tau, \iota)\,.$$

Then g_ν is uniformly $\Sigma_1(J_{m(\nu)}, A \cap J_{m(\nu)})$ in p, $l(\nu)$ (if $l(\nu) < m(\nu)$). Let κ_ν be defined from g_ν as κ is defined from g; i.e.

$$\kappa_\nu = \max\{\kappa \mid \kappa \leq l(\nu) \wedge \sup g_\nu''\kappa \leq \kappa\}\,.$$

Preparatory to defining C_α, we prove

Lemma 5.7. *$\kappa_\nu = \kappa$ for sufficiently large ν.*

Proof. $\nu < \tau \to g_\nu \subseteq g_\tau \subseteq \bigcup_{\nu<\theta} g_\nu = g$. Then $\sup g_\nu''\nu \leq \sup g''\kappa \leq \kappa$. So $\kappa \leq \kappa_\nu$. Similarly $\nu < \tau \to \kappa \leq \kappa_\tau \leq \kappa_\nu$. Let $\kappa < \xi < \alpha$. Then $\bigcup_{\nu<\theta} \sup g_\nu''\xi = \sup g''\xi > \xi$. So there is $\nu_\xi < \theta$ such that for all $\nu > \nu_\xi$, $\sup g_\nu''\xi > \xi$; i.e. $\kappa_\nu < \xi$. Since there is no infinite desending sequence of ordinals, it follows that $\kappa_\nu = \kappa$ for all $\nu >$ some ν_0.

We are now ready to define C_α. We define a normal function $t:\widetilde{\theta} \to \theta$ ($\widetilde{\theta} < \theta$) such that $\sup t''\widetilde{\theta} = \theta$ and then set $C_\alpha = l''t''\widetilde{\theta}$. $t(\iota)$ is defined recursively as follows:

Case 1. $\iota = 0$. $t(0) =$ the least ν such that

(i) $\kappa_\tau = \kappa$ for $\tau \geq \nu$.

(ii) $\alpha \in X_\nu$ if $\alpha < \rho$.

(iii) $l(\nu) > \omega_1$.

For $\iota > 0$ we consider two cases.

Case 2. $n = 1$. Let $s : \mathrm{On}^2 \leftrightarrow \mathrm{On}$ be Gödel's pairing function. Set

$$t(\iota + 1) = \text{the least } \nu > t(\iota) \text{ such that } s''(lt(\iota))^2 \subset l(\nu)\ ,$$

$$t(\lambda) = \sup_{\iota<\lambda} t(\iota) \ \text{ if } \sup_{\iota<\lambda} t(\iota) < \theta \ \text{ for limit } \lambda\ .$$

$t(\iota + 1)$ is always defined, since $s''\alpha^2 \subset \alpha$.

Case 3. $n > 1$. Set $\rho^* = \rho_\beta^{n-2}$, $A^* = A_\beta^{n-2}$, $p^* = p_\beta^{n-1}$. Let h^* be the canonical Σ_1 Skolem function for $\langle \mathrm{J}_\rho *, A^* \rangle$. Set

$$t(\iota + 1) = \text{the least } \nu > t(\iota) \text{ such that}$$

$$s''(lt(\iota))^2 \subset l(\nu) \wedge \mathrm{J}_\alpha \cap h^{*''}(\omega \times X_{t(\iota)} \times \{p^*\}) \subset X_\nu\ ,$$

$$t(\lambda) = \sup_{\iota<\lambda} t(\iota) \ \text{ if } \sup_{\iota<\lambda} t(\iota) < \theta \text{ for limit } \lambda\ .$$

We must show that $t(\iota + 1)$ is defined. Let $\nu = t(\iota)$. Let $Y = \mathrm{J}_\alpha \cap h^{*''}(\omega \times X_\nu \times \{p^*\})$. We must show that $Y \subset Y_\xi$ for some $\xi < \theta$. Since $Y \subset \mathrm{J}_\alpha$ it suffices to show that $Y \subset \mathrm{J}_\tau$ for some $\tau < \alpha$, for if $l(\xi) \geq \tau$, we then have $X \subset \mathrm{J}_\tau \subset X_{\xi+1}$.

$X_\nu = h_{m(\nu)}''(\omega \times \mathrm{J}_\eta \times \{p\})$ for some $\eta < \alpha$. $h_{m(\nu)} \in \mathrm{J}_\beta$ and J_β contains a function mapping η onto $(\omega \times \mathrm{J}_\eta \times \{p\})$. Hence J_β contains a function mapping η onto $\omega \times X_\nu \times \{p^*\}$. Hence there is a $\Sigma_{n-1}(\mathrm{J}_\beta)$ function f mapping a subset of η onto Y. Since α is Σ_{n-1} regular, the function $f'(\nu) \simeq \mu\tau f(\nu) \in \mathrm{J}_\tau$ is bounded in α. Hence $Y \subset \mathrm{J}_\tau$ for some $\tau < \alpha$. Set: $C_\alpha = \{lt(\nu) | \nu < \widetilde{\theta}\}$. C_α is obviously closed and unbounded in α. As an immediate corollary of the definition of C_α we have

Lemma 5.8. *Let $\bar{\alpha} < \alpha$ be a limit point of C_α. Then $\bar{\alpha} > \omega_1$ and $s''\bar{\alpha}^2 \subset \bar{\alpha}$, where s is Gödels pairing function. Moreover, suppose that $n > 1$ and $\bar{\alpha} = l(\lambda)$. Let f be a function $\Sigma_1(J_{\rho^*}, A^*)$ in parameters from $X_\lambda \cup \{p^*\}$ which maps a bounded subset of $\bar{\alpha}$ into $\bar{\alpha}$. Then f is bounded in $\bar{\alpha}$.*

For the rest of the proof let $\bar{\alpha} < \alpha$ be a fixed limit point of C_α. We must show $\bar{\alpha} \notin E \wedge C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$. Let $\bar{\alpha} = l(\lambda)$. Set

$$\pi^{-1} : \langle X_\lambda, A \cap X_\lambda \rangle \overset{\sim}{\longleftrightarrow} \langle J_{\bar{\rho}}, \bar{A} \rangle .$$

Then $\langle J_{\bar{\rho}}, \bar{A} \rangle \xrightarrow{\pi}_{\Sigma_0} \langle J_\rho, A \rangle$. Let $J_{\bar{\beta}} \xrightarrow{\tilde{\pi}} J_\beta$ be the canonical extension of $\langle J_{\bar{\rho}}, \bar{A} \rangle \xrightarrow{\pi} \langle J_\rho, A \rangle$. Then $\bar{\rho} = \rho_{\bar{\beta}}^{n-1}$, $\bar{A} = A_{\bar{\beta}}^{n-1}$ and $\pi(p_{\bar{\beta}}^{n-1}) = p_\beta^{n-1}$. Let $\bar{h}$ be the canonical Skolem function on $\langle J_{\bar{\rho}}, \bar{A} \rangle$. Then $\bar{h} = \pi^{-1} h_{m(\lambda)} \pi$ and has the canonical Σ_1 definition:

$$y = \bar{h}(i, x) \longleftrightarrow \bigvee z \in J_{\bar{\rho}} \; \bar{H}(y, i, x) ,$$

where $\bar{H} = \pi^{-1}(H)$. Set $\bar{p} = \pi^{-1}(p)$.

Lemma 5.9. $\bar{\beta} = \beta(\bar{\alpha})$, $n = n(\bar{\alpha})$, $\bar{p} = p(\bar{\alpha})$; *moreover, if $\bar{g}$, $\bar{\kappa}$ are defined from $\bar{\alpha}$ as g, κ were defined from α, then $\bar{g} = g_\lambda$, $\bar{\kappa} = \kappa$.*

Proof. Set p' = the least p' (in $<_J$) such that every $x \in J_{\bar{\rho}}$ is $\Sigma_1(J_{\bar{\rho}}, \bar{A})$ in parameters from $\bar{\alpha} \cup \{p'\}$.

(a). $p' = \bar{p}$.

($\leq$). $\bar{h}''(\omega \times J_{\bar{\alpha}} \times \{\bar{p}\}) = J_{\bar{\rho}}$. But $\bar{\alpha}$ is closed under Gödels pairing function, whence it easily follows that $J_{\bar{\rho}} = \bar{h}''(\omega \times \bar{\alpha} \times \{\bar{p}\})$ (cf. the proof of Lemma 5.3). Thus each $x \in J_{\bar{\rho}}$ is Σ_1 in parameters from $\bar{\alpha} \cup \{\bar{p}\}$.

($\geq$). Since $\bar{p} \in J_{\bar{\rho}}$, there is a $\nu < \bar{\alpha}$ and $i < \omega$ such that $\bar{p} = \bar{h}(i, \langle \nu, p' \rangle)$. Hence $p = h(i, \langle \nu, \pi(p') \rangle)$. Hence each $x \in J_\rho$ is $\Sigma_1(J_\rho, A)$ in parameters from $\alpha \cup \{\pi(p')\}$. Hence $\pi(p') \geq p$ and $p' \geq \bar{p}$.

Define g' in terms of $\bar{h}, \bar{\alpha}, \bar{p}$ as g was defined from h, α, p. It is immediate that

(b). $g' = g_\lambda$.

Defining κ' from g' as κ was defined from g we then get (c). $\kappa' = \kappa_\lambda = \kappa$.

Thus it remains only to show that $\bar{\beta} = \beta(\bar{\alpha})$, $n = n(\bar{\alpha})$, for we then have $p' = p(\bar{\alpha})$, $g' = \bar{g}$, $\kappa' = \bar{\kappa}$.

(d). $\bar{\beta} = \beta(\bar{\alpha})$.

($\geq$). Set $\eta = \sup_{\nu<\lambda} k(\nu)$. Then $\eta < \bar{\alpha}$ since $\bar{\alpha} = \sup_{\nu<\lambda} gk(\nu) = \sup g''\eta$ and $\eta > \kappa$. Hence g' maps a subset of η unboundedly into $\bar{\alpha}$. But g' is $\Sigma_1(J_{\bar{\rho}}, \bar{A})$, hence $\Sigma_n(J_{\bar{\beta}})$.

($\leq$). If not, i.e., if $\bar{\beta} > \beta(\bar{\alpha})$, then $\bar{\beta} > \bar{\alpha}$ and there is an $f \in J_{\bar{\beta}}$ which maps some $\tau < \bar{\alpha}$ unboundedly into $\bar{\alpha}$. $\tilde{\pi} \upharpoonright \bar{\alpha} = \mathrm{id} \upharpoonright \bar{\alpha}$ and dom (f) is bounded in $\bar{\alpha}$. So $\tilde{\pi}(f) = f$. But the statement "f is a function, dom $(f) = \tau < \bar{\alpha}$ and range f is unbounded in $\bar{\alpha}$," is Σ_0. But then the same statement must hold with only $\bar{\alpha}$ replaced by $\tilde{\pi}(\bar{\alpha})$. Contradiction since $\tilde{\pi}(\bar{\alpha}) \geq \alpha > \bar{\alpha}$. (Actually $\tilde{\pi}(\bar{\alpha}) = \alpha$).

(e). $n = n(\bar{\alpha})$.

($\geq$). The proof of the ($\geq$) part of (d) showed that there is a $\Sigma_n(J_{\bar{\beta}})$ function mapping a bounded subset of $\bar{\alpha}$ unboundedly into $\bar{\alpha}$.

($\leq$). For $n = 1$ this is trivial. Let $n > 1$. We must show that $\bar{\alpha}$ is Σ_{n-1} regular in $\bar{\beta}$. Let f be a $\Sigma_{n-1}(J_{\bar{\beta}})$ function mapping a bounded subset of $\bar{\alpha}$ into $\bar{\alpha}$. Set

$$\bar{\rho}^* = \rho_{\bar{\beta}}^{n-2},\ \bar{A}^* = A_{\bar{\beta}}^{n-2},\ \bar{p}^* = p_{\bar{\beta}}^{n-1},\ \pi^* = \pi \upharpoonright J_{\bar{\rho}^*}.$$

Then $\langle J_{\bar{\rho}^*}, \bar{A}^*\rangle \xrightarrow{\pi^*}_{\Sigma_1} \langle J_{\rho^*}, A^*\rangle$ and $\pi^*(\bar{p}^*) = p^*$. f is $\Sigma_1(J_{\bar{\rho}^*}, \bar{A}^*)$ in $\bar{p}^*$ and some $x \in J_{\bar{\rho}}$. Let f' have the same Σ_1 definition over $\langle J_{\rho^*}, A^*\rangle$ in the parameters p^*, $\pi(x)$. Since $f \subset J_{\bar{\alpha}}$ and $\tilde{\pi} \upharpoonright J_{\bar{\alpha}} = \mathrm{id} \upharpoonright J_{\bar{\alpha}}$, we have: $f \subset f'$. Let $u = \mathrm{dom}(f)$. Since u is $\Sigma_{n-1}(J_{\bar{\beta}})$ and bounded in $\bar{\rho}$ we have: $u \in J_{\bar{\rho}}$. Since u is bounded in $\bar{\alpha}$, we have $\pi(u) = u$. The statements "f is a function" and "$\mathrm{dom}(f) \subset u$" are $\Pi_1(J_{\bar{\rho}^*}, \bar{A}^*)$ in $\bar{p}^*, x, u$. Hence f' is a function and $\mathrm{dom}(f') \subset u$. Hence $f = f'$. Thus f is $\Sigma_1(J_{\rho^*}, A^*)$ in p^*, $\pi(x)$. Since $\pi(x) \in X_\lambda$, and the domain of f is bounded in α, we conclude by Lemma 5.8 that f is bounded in $\bar{\alpha}$. This proves Lemma 5.9.

Lemma 5.10. *$\bar{\alpha}$ falls under case 4 in the definition of $C_{\bar{\alpha}}$.*

Proof. We must show that cases 1–3 fail. Cases 1 and 2 fail by Lemma 5.8 and the fact that $\tilde{\pi}(\bar{\alpha} \cap Q) = \alpha \cap Q$. In case 3, $n = 1$ and $\bar{\beta}$ is a successor ordinal. But if $n = 1$, then $\bar{\beta} = \bar{\rho}$. $\bar{\rho} = \sup_{\iota<\lambda} \pi^{-1}(m(\iota))$ is a limit ordinal.

Since each $\alpha \in E$ falls under case 3, we conclude

Corollary 5.11. $\bar{\alpha} \notin E$.

It remains only to prove

Lemma 5.12. $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$.

Proof. Define $\bar{k}, \bar{l}, \bar{m}$ from $\bar{\alpha}$ as k, l, m are defined from α. Using Lemma 5.9, we prove by induction on $\nu < \lambda$ that $\bar{k}(\nu) = k(\nu)$, $\bar{l}(\nu) = l(\nu)$, $\pi\bar{m}(\nu) = m(\nu)$. The induction is straightforward. For limit $\tau < \lambda$, we must use $m(\tau) \in X_\lambda$ to show that

$$\pi\bar{m}(\tau) = \pi(\sup_{\nu<\tau} \bar{m}(\nu))$$

$$= \text{the least } \eta \in X_\lambda \text{ such that } \eta > m(\nu) \text{ for } \nu < \tau$$

$$= m(\tau)\,.$$

Now define $\bar{t}$ from $\bar{\alpha}$ as t was defined from α. Let $\lambda = t(\bar{\lambda})$. By induction on $\nu < \bar{\lambda}$ we prove that $\bar{t}(\nu) = t(\nu)$. For $\nu = 0$, this follows by the fact that $\bar{\kappa} = \kappa$ and $\pi(\bar{\alpha}) = \alpha$. For $n = 1$ the rest of the induction is trivial. For $n > 1$ we use the facts $\langle J_{\bar{\rho}^*}, \bar{A}^* \rangle \xrightarrow{\pi^*}_{\Sigma_1} \langle J_{\rho^*}, A^* \rangle$ and $\pi^*(\bar{p}^*) = p^*$ to show that $\bar{t}(\nu+1) = t(\nu+1)$ if $\bar{t}(\nu) = t(\nu)$. Thus $\bar{l} = l \restriction \lambda$ and $\bar{t} = t \restriction \bar{\lambda}$. Hence

$$C_{\bar{\alpha}} = \bar{l}\bar{t}''\bar{\lambda} = lt''\bar{\lambda} = \bar{\alpha} \cap C_\alpha\,.$$

5.1. *The principle* $\Box_\kappa$

Let κ be any infinite cardinal. Consider the statement:

($\Box_\kappa$) There is a sequence C_λ defined on limit ordinals $< \kappa^+$ such that

(i) C_λ is closed, unbounded in λ;

(ii) if $\mathrm{cf}(\lambda) < \kappa$, then $\overline{\overline{C}}_\lambda < \kappa$;

(iii) if γ is a limit point of C_λ, then $C_\gamma = \gamma \cap C_\lambda$.

Note. It follows that if $\mathrm{cf}(\lambda) = \kappa$, then C_λ has order type κ.

$\square_\kappa$ is not provable in **ZF** + **GCH**, for Solovay has shown that **ZF** + **GCH** + $\neg \square_{\omega_1}$ is consistent relative to **ZF** + "there is a Mahlo cardinal".

A somewhat weaker version of $\square_\kappa$ is:

($\square_\kappa^*$) There is a sequence C_λ defined on limit ordinals $< \kappa^+$ such that

(i) C_λ is closed, unbounded in λ;

(ii) C_λ has order type $\mathrm{cf}(\lambda)$;

(iii) if $\tau < \kappa^+$ then $\{C_\lambda \cap \tau \mid \lambda < \kappa^+\}$ has cardinality $< \kappa^+$.

If $2^\kappa = \kappa^+$, then $\square_\kappa^*$ is equivalent to the proposition: There is a special Aronszajn tree on κ^+. Hence $\square_\kappa^*$ follows from **ZF** + **GCH** for regular κ. For singular κ the problem is still open.

Theorem 5.2. *Assume* V = L. *Let κ be any infinite cardinal. Then $\square_\kappa$ holds. In fact, there is a set $E \subset \kappa^+$ and a sequence C_λ* ($\mathrm{Lim}(\lambda)$, $\lambda < \kappa^+$) *such that*

(i) *E is Mahlo in κ^+;*

(ii) *C_λ is closed, unbounded in λ;*

(iii) *if* $\mathrm{cf}(\lambda) < \kappa$, *then* $\overline{\overline{C}}_\lambda < \kappa$;

(iv) *if γ is a limit point of C_λ then $\gamma \notin E$ and $C_\gamma = \gamma \cap C_\lambda$.*

Proof. Let S = the set of all limit ordinals α such that

(a) $\kappa < \alpha < \kappa^+$;

(b) α is closed under Gödel's pairing function;

(c) each $\nu < \alpha$ has cardinality $\leq \kappa$ in J_α (i.e. some $f \in J_\alpha$ maps κ onto ν).

Then S is closed, unbounded in κ^+.

Lemma 5.13. *There is a set $\bar{E} \subset S$ and a sequence $\bar{C}_\lambda (\lambda \in S)$ such that*

(i) *$\bar{E}$ is Mahlo in κ^+;*

(ii) *$\bar{C}_\lambda$ is closed, unbounded in λ;*

(iii) *$\bar{C}_\lambda$ has order type $\leq \kappa$;*

(iv) *if γ is a limit point of $\bar{C}_\lambda$, then $\gamma \in S$, $\gamma \notin \bar{E}$, $\bar{C}_\gamma = \gamma \cap \bar{C}_\lambda$.*

Proof. Set $\bar{E} = E \cap S$, where E is the class defined in the proof of Theorem 5.1. Since $E \cap \kappa^+$ is Mahlo in κ^+, $\bar{E}$ is also Mahlo in κ^+. Now let

$\alpha \in S$ and let C_α be defined as in the proof of Theorem 5.1. If cases 1, 2b or 3 in the definition of C_α apply, set $\overline{C}_\alpha = C_\alpha$. (Case 2a does not apply since $\alpha \in S$.) It remains to consider case 4. Let $\beta = \beta(\alpha)$ and $n = n(\alpha)$. Then $\rho^n_\beta = \kappa$ for since α is not Σ_n regular in β and each $\nu < \alpha$ has cardinality $\leq \kappa$ in J_α, there must be a $\Sigma_n(J_\beta)$ subset of κ not in J_β. Now the C_α constructed in Theorem 5.1 has order type γ for some $\gamma < \alpha$, whereas in the present situation we wish to have the order type of $\overline{C}_\alpha \leq \kappa$. We construct $\overline{C}_\alpha$ by modifying the construction of C_α in Theorem 5.1. Proceed as in Theorem 5.1, case 4, but redefine $p, \tilde{h}, g, k, l, m$ as follows (but first replace the "free variable" κ occurring in the proof of Theorem 5.1 by some other symbol, say μ, to avoid confusion):

Let $p = p(\alpha) =$ the $<_J$-least $p \in J_\rho$ such that $h''(\omega \times (\kappa \times \{p\})) = J_\rho$. Then $p = p^n_\beta$. Define $\tilde{h}$ by $\tilde{h}(i,x) = h(i, \langle x, p\rangle)$. Define g from a subset of κ onto α by

$$g(\omega\nu + i) = \begin{cases} \tilde{h}(i,\nu) & \text{if } \tilde{h}(i,\nu) \in \alpha\,, \\ \text{undefined otherwise}\,. \end{cases}$$

Define k, l, m on $\theta \leq \kappa$ as follows:

$k(\nu) =$ the least $\tau \in \mathrm{dom}(g)$ such that $g(\tau) > l(\nu)$ and $l(\nu)$ has cardinality $\leq \kappa$ in $J_{g(\tau)}$;
$m(0) = \max(\kappa + 1, \mu\tau(p \in J_\tau))$;
$m(\nu + 1) =$ the least $\eta < \rho$ such that
  (i) $m(\nu), g(k(\nu)) < \eta$,
  (ii) $\bigvee z \in J_\eta\, G(z, g(k(\nu)), k(\nu))$,
  (iii) $l(\nu), m(\nu) \in h_\tau{}''(\omega \times (\kappa \times \{p\}))$,
  (iv) $A \cap J_{m(\nu)} \in J_\eta$;
$m(\lambda) = \sup_{\nu<\lambda} m(\nu)$ if $\sup_{\nu<\gamma} m(\nu) < \rho$ for $\lim(\lambda)$;
$l(\nu) = \alpha_{X_\nu} = \sup(\alpha \cap X_\nu)$ where

$$X_\nu = h_{m(\nu)}{}''(\omega \times (\kappa \times \{p\}))\,.$$

The rest of the proof of Theorem 5.1 may be followed almost verbatim, although some of the apparatus developed there is not needed in the present situation, and the remaining lemmas are somewhat more easily proved.

Lemma 5.14. *Let $\overline{E}$ be as in Lemma 5.13. There is a sequence $C_\lambda (\lambda \in S)$ such that*
(i) *C_λ is closed, unbounded in λ;*
(ii) *$\overline{\overline{C}}_\lambda < \kappa$ if $\mathrm{cf}(\lambda) < \kappa$;*
(iii) *if $\gamma < \lambda$ is a limit point of C_λ, then $\gamma \notin \overline{E}$, $\gamma \in S$, $C_\gamma = \gamma \cap C_\lambda$.*

Proof. If κ is regular, we may set: $C_\lambda = \overline{C}_\lambda$. Now let κ be singular and let $\mathrm{cf}(\kappa) = \delta$. Let $\langle \delta_\nu \mid \nu < \delta \rangle$ be a normal function such that $\sup_{\nu<\delta} \delta_\nu = \kappa$. Let $\langle \gamma_\nu \mid \nu < \theta \rangle$ be the monotone enumeration of $\overline{C}_\alpha$. Define C_α as follows.

Case 1. $\delta_\nu < \theta \le \delta_{\nu+1}$ for some ν. Set $C_\alpha = \{\gamma_\iota \mid \iota > \delta_\nu\}$

Case 2. $\theta = \sup\{\delta_\nu \mid \delta_\nu < \theta\}$. Set $C_\alpha = \{\gamma_{\delta_\nu} \mid \delta_\nu < \theta\}$.

The C_α's clearly have the desired properties.

Now let $\mathfrak{J}$ be the set of half open intervals $I = [\tau_0, \tau_1)$ such that $\tau_1 \in S$ and τ_0 is the least τ such that $[\tau, \tau_1) \cap S = \emptyset$. Then $(\kappa^+ \setminus S) = \bigcup_{I \in \mathfrak{J}} I$. Note that, since S is closed, τ_0 is never a limit ordinal.

Lemma 5.15. *Let $I \in \mathfrak{J}$. Then there is a sequence $C^I_\lambda (\mathrm{Lim}(\lambda), \lambda \in I)$ such that*
(i) *C^I_λ is closed, unbounded in λ;*
(ii) *if $\mathrm{cf}(\lambda) < \kappa$, then $\overline{\overline{C^I_\lambda}} < \kappa$;*
(iii) *if $\gamma < \lambda$ is a limit point of C^I_λ, then $\gamma \in I$ and $C^I_\gamma = \gamma \cap C^I_\lambda$.*

Proof. Let λ_ι $(\iota \le \delta)$ enumerate monotonically the limit ordinals of $I \cup \{\sup(I)\}$. Let $\mathfrak{S}_\iota$ be the set of functions $C = \langle C_\nu \mid \nu \le \lambda_\iota \rangle$ satisfying (i)-(iii). By induction on ι we prove

(*) $\mathfrak{S}_\iota \neq \emptyset$ and for each $\tau < \iota$, if $C \in \mathfrak{S}_\tau$, then there is $C' \in \mathfrak{S}_\iota$ such that $C \subset C'$.

For $\iota = 0$ the assertion is trivial. Let it hold for ι. We can then extend $C \in \mathfrak{S}_\iota$ to $C' \in \mathfrak{S}_{\iota+1}$ by setting $C'_{\lambda_{\iota+1}} = \lambda_{\iota+1} \setminus \lambda_\iota$. Now assume $\mathrm{Lim}(\eta)$, $\iota < \eta \le \delta$, $C \in \mathfrak{S}_\iota$. Let $\rho = \mathrm{cf}(\eta)$ and let $\langle \eta_\nu \mid \nu \le \rho \rangle$ be a normal function such that $\eta_0 = \iota$ and $\eta_\rho = \sup_{\nu<\rho} \eta_\nu = \eta$. Define a sequence $C^\nu \in \mathfrak{S}_{\eta_\nu}$ such that $C^0 \subset C^1 \subset \ldots \subset C^\nu \ldots$ as follows:

$C^0 = C$; $C^{\nu+1} \in \mathfrak{S}_{\eta_{\nu+1}}$ such that $C^{\nu+1} \supset C^\nu$.
For limit τ let $C^* = \bigcup_{\nu<\tau} C^\nu$ and extend C^* to C^τ by setting

$$C^\tau_{\lambda_{\eta_\tau}} = \{\lambda_{\eta_\nu} \mid \nu < \tau\} .$$

Then $C^\rho \in \mathfrak{S}_\eta$ and $C^\rho \supset C$. This proves (*).

By (*) there is a $C \in \mathfrak{S}_\delta$. But then $C^I = C \restriction \lambda_\delta$ has the desired properties. This proves Lemma 5.15.

The theorem follows by Lemma 5.14, Lemma 5.15 and the fact that $\bar{E} \subset S$. Let $\langle C_\lambda \mid \lambda \in S\rangle$ be the sequence given by Lemma 5.14. We can extend this to a sequence defined on all limit ordinals $< \kappa^+$ by setting

$$C_\lambda = C^I_\lambda \ \text{ for } \lambda \in I \in \mathfrak{J} \ .$$

This sequence has the desired properties. This completes the proof of Theorem 5.2.

Remarks. (1). By combining the proof of Theorem 5.1 with the methods of Theorem 6.1, we could prove: There is a class E and a sequence C_λ defined on accessible λ such that

(i) E is Mahlo in inaccessible κ;

(ii) C_λ is closed, unbounded in λ;

(iii) if $\gamma < \lambda$ is a limit point of C_λ, then $\gamma \notin E$ and $C_\gamma = \gamma \cap C_\lambda$.

Similarly for "inaccessible limit of inaccessibles", "Mahlo", "hyper Mahlo" etc. However, there is a limit of this process: there is no E such that $E \cap \kappa$ is Mahlo in κ iff κ is weakly compact.

(2). We can prove a version of Theorem 5.1 under the assumption $V = L[A]$, $A \subset \mathrm{On}$. In this version C_λ would be defined on all λ such that λ is singular in $L[A \cap \lambda]$ and whenever κ is regular in $L[A \cap \kappa]$, $E \cap \kappa$ would be Mahlo in κ in the model $L[A \cap \kappa]$. The proof is virtually the same, but some reworking of §3 and §4 is required.

(3). Similarly, we can weaken the premiss of Theorem 5.2 to: $V = L[A]$ for an $A \subset \kappa^+$ such that $\alpha < \kappa^+$ has cardinality $\leq \kappa$ in $L[A \cap \alpha]$. In particular, if κ^+ not Mahlo in L, then $\square_\kappa$ holds. Hence Solovay's relative consistency result is the best possible.

§6. Weakly compact cardinals in L

Each of the following conditions is known to characterize weakly compact cardinals κ:

(a) κ is Π^1_1 indescribable;

(b) κ is strongly inaccessible and there is no κ-Aronszajn tree;

(c) $\kappa \to (\kappa)^2_2$.

In this section we show that, if V = L, then apparently weaker forms of each of these conditions suffice to characterize weak compactness.

We start with (a). A consequence of Π^1_1 indescribability is:

(*) If $E \subset \kappa$ is Mahlo in κ, then $E \cap \beta$ is Mahlo in β for some $\beta < \kappa$.

The assertion (*) characterises weak compactness for regular κ in L. In fact we shall prove

Theorem 6.1. *Assume* V = L. *Let* $\kappa > \omega$ *be regular but not weakly compact. There is an* $E \subset \kappa$ *and a sequence* C_λ (Lim(λ), $\lambda < \kappa$) *such that*

(i) *E is Mahlo in κ*,

(ii) *C_λ is closed, unbounded in λ*,

(iii) *if $\gamma < \lambda$ is a limit point of C_λ, then $\gamma \notin E$ and $C_\gamma = \gamma \cap C_\lambda$.*

Proof. We may assume that κ is inaccessible since the theorem has been proved for successor cardinals (Theorem 5.2). Since κ is not weakly compact, it is Π^1_1 describable. Hence there is a set $B \subset \kappa$ and a first order formula φ (with predicates $\in$, B, D) such that

$$\bigwedge D \subset \kappa \models_{J_\kappa} \varphi[D, B] ,$$

but

$$\bigvee D \subset \beta \models_{J_\beta} \neg\varphi[D, B \cap \beta] \quad \text{for } \beta < \kappa .$$

We make use of B, φ in defining a Mahlo set $\bar{E} \subset \kappa$.

Definition. $\bar{E}$ = the set of limit cardinals $\alpha < \kappa$ such that for some $\beta > \alpha$:

(i) α is regular in β;
(ii) α is the largest cardinal in J_β;
(iii) J_β is a model of ZF^-;
(iv) For some $p \in J_\beta$, J_β = the smallest $X \prec J_\beta$ such that $p \in X$ and $\alpha \cap X$ is transitive;
(v) $B \cap \alpha \in J_\beta$ and $\bigwedge D \in \mathfrak{P}(\alpha) \cap J_\beta \models_{J_\alpha} \varphi[D, B \cap \alpha]$.

Note that $\overline{E} \subset E$, where E is the class defined in Theorem 5.1.

Lemma 6.1. *$\overline{E}$ is Mahlo in κ.*

Proof. Exactly like Lemma 5.1.

We wish to define $\overline{C}_\lambda$ (Lim(λ), $\lambda < \kappa$) such that $\overline{E}$, $\overline{C}_\lambda$ satisfy (i)–(iii) of Theorem 6.1. Since each $\alpha \in \overline{E}$ is a limit cardinal, we can dispose quickly of the case that λ is not a limit cardinal. There is then a maximal $\tau < \lambda$ such that $\tau = 0$ or τ is a limit cardinal. Set $\overline{C}_\lambda = \lambda \setminus \tau$.

We now define a set Q of limit cardinals $< \kappa$ (containing all regular ones) on which $\overline{C}_\lambda$ can be defined in a fairly simple fashion. We will have $Q \cap \overline{E} = \emptyset$. The definition will give us: If $\lambda \in Q$ and γ is a limit point of $\overline{C}_\lambda$, then $\gamma \in Q$ and $\overline{C}_\gamma = \gamma \cap \overline{C}_\lambda$. Afterwards we shall make use of §5 in defining $\overline{C}_\lambda$ on the remaining limit cardinals $\lambda \notin Q$. We begin with

Definition. Q' is the set of limit cardinals α such that for some $\beta > \alpha$:
(i) α is regular in β;
(ii) $B \cap \alpha \in J_\beta$;
(iii) there is a $D \in \mathfrak{P}(\alpha) \cap J_\beta$ such that $\models_{J_\alpha} \neg\varphi[D, B \cap \alpha]$.

Lemma 6.2. $Q' \cap \overline{E} = \emptyset$.

Proof. Let $\alpha \in \overline{E}$ and let $\beta > \alpha$ be as in the definition of $\overline{E}$. Then no $\beta' \le \beta$ satisfies (iii). But α is not regular in $\beta + 1$ by Lemma 5.2. Hence no $\beta' > \beta$ satisfies (i).

We define Q as a subset of Q :

Definition. Q is the set of $\alpha \in Q'$ such that, letting β be the least $\beta > \alpha$ to satisfy (i)-(iii), we have

(iv) if $p \in J_\beta$, there is an $X \prec J_\beta$ such that $p \in X$, $\alpha \cap X$ is transitive and $\alpha \cap X < \alpha$.

We note that, if $\alpha \in Q' \setminus Q$, then there is precisely one $\beta > \alpha$ satisfying (i)-(iii). This follows from

Lemma 6.3. *Let $\alpha \in Q' \setminus Q$ and let $\beta > \alpha$ be the least β to satisfy* (i)-(iii) *in the definition of Q'. Then α is not Σ_1-regular in $\beta + 1$.*

Proof. Let p be the least counterexample to (iv) such that $B \cap \alpha, \alpha$ are J_β-definable in the parameter p. Let X be the smallest $X \prec J_\beta$ such that $p \in X$ and $\alpha \cap X$ is transitive, then $\alpha \subset X$. It suffices to show $X = J_\beta$, for we may then repeat the proof of Lemma 5.2. Let $\pi : X \widetilde{\longleftrightarrow} J_{\bar\beta}$, then $\pi \upharpoonright \alpha = \mathrm{id} \upharpoonright \alpha$, $\pi(\alpha) = \alpha$; hence $\pi(B \cap \alpha) = (B \cap \alpha)$. Hence $\bar\beta$ satisfies (i)-(iii). Hence $\bar\beta = \beta$. But then $\pi(p) = p$ by the minimality of p. Since every $x \in X$ is J_β-definable in parameters from $\alpha \cup \{p\}$, we conclude: $\pi \upharpoonright X = \mathrm{id} \upharpoonright X$.

We now define $\bar C_\lambda$ for $\lambda \in Q$.

Definition. Let $\alpha \in Q$. Let β be the least β to satisfy (i)-(iii) in the definition of Q. Define a sequence $X_\nu \prec J_\beta$ by

X_0 = the smallest $X \prec J_\beta$ such that $\alpha \cap X$ is transitive and $\alpha, B \cap \alpha \in X$;

$X_{\nu+1}$ = the smallest $X \prec J_\beta$ such that $\alpha \cap X$ is transitive and $\alpha_\nu, \alpha, B \cap \alpha \in X$ where $\alpha_\nu = \sup(\alpha \cap X_\nu)$,

$X_\lambda = \bigcup_{\nu<\lambda} X_\nu$ if $\alpha \cap \bigcup_{\nu<\lambda} X_\nu < \alpha$ for limit λ.

Then X_ν is defined for $\nu < \eta = \eta_\alpha$, where η is a limit ordinal, and $X_\nu \prec X_\tau \prec J_\beta$ where $\nu < \tau < \eta$. Set $\alpha_\nu = \alpha_{X_\nu}$ and $\bar C_\alpha = \{\alpha_\nu \mid \nu < \eta_\alpha\}$. Clearly $\bar C_\alpha$ is closed and unbounded in α. We must prove

Lemma 6.4. *Let $\alpha \in Q$. Let $\bar\alpha < \alpha$ be a limit point of $\bar C_\alpha$. Then $\bar\alpha \in Q$ and $\bar C_{\bar\alpha} = \bar\alpha \cap \bar C_\alpha$.*

Proof. We first prove $\bar{\alpha} \in Q$. Let $\bar{\alpha} = \alpha_\lambda$. Let $\pi^{-1} : X_\lambda \xrightarrow{\sim} J_{\bar{\beta}}$. Then

$$\pi : J_{\bar{\beta}} \to_{\Sigma_\omega} J_\beta, \quad \pi(\bar{\alpha}) = \alpha, \quad \pi(B \cap \bar{\alpha}) = B \cap \alpha .$$

Hence $\bar{\alpha}, \bar{\beta}$ satisfy (i)-(iii) and $\bar{\beta}$ is the least one to do so. We must show that $\bar{\alpha}$ satisfies (iv). Let $p \in J_{\bar{\beta}}$. Then $\pi(p) \in X_\nu$ for some $\nu < \lambda$. Set $\bar{X} = \pi^{-1}(X_\nu)$. Then $p \in \bar{X}, \bar{X} \prec J_{\bar{\beta}}, \bar{\alpha} \cap \bar{X} = \alpha_\nu < \alpha$.

We now prove $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$. Define $\bar{X}_\nu, \bar{\alpha}_\nu$ $(\nu < \bar{\eta})$ from $\bar{\alpha}, \bar{\beta}$ as X_ν, α_ν were defined from α, β. It is easily seen that $\bar{\eta} = \lambda$ and $X_\nu = \pi(\bar{X}_\nu)$ for $\nu < \lambda$. Hence $\bar{\alpha}_\nu = \bar{\alpha}_{\bar{X}_\nu} = \alpha_{X_\nu} = \alpha_\nu$.

We turn now to the definition of $\bar{C}_\lambda$ for $\lambda \notin Q$. Let $\alpha \in \kappa\setminus Q$ be a limit cardinal. Then α is singular. (If α were regular, (i)-(iv) in the definition of Q would be satisfied with $\beta = \alpha^+$). Let C_α be as in Theorem 5.1. Let $\tilde{C}_\alpha$ be the set of limit cardinals $\eta < \alpha$ such that η is a limit point of C_α. Then $\tilde{C}_\alpha$ is closed, but may be bounded in α.
We consider four cases.

Case 1. $\tilde{C}_\alpha$ is bounded in α. Then α is ω-cofinal and we let $\bar{C}_\alpha$ be an unbounded set of order type ω.

α cannot satisfy cases 1, 2 in the definition of C_α since α is a limit cardinal. If case 3 in the definition of C_α applies, then α falls under case 1 above, since C_α has no limit points. Thus, in particular, case 1 takes care of $\alpha \in \bar{E}$ (by Lemma 5.2) and $\alpha \in Q'\setminus Q$ (by Lemma 6.3).

Now let $\tilde{C}_\alpha$ be unbounded in α. We shall define $\bar{C}_\alpha$ as a closed cofinal subset of $\tilde{C}_\alpha$. We note that α satisfies case 4 in the definition of C_α. Hence each $\eta \in \tilde{C}_\alpha$ satisfies case 4 and we have: $\bar{C}_\alpha \cap E = \tilde{C}_\alpha \cap E = \emptyset$.

Let $\beta = \beta(\alpha)$, $n = n(\alpha)$ be as in §5. Let $\langle \alpha_\nu \mid \nu < \theta \rangle$ be the monotone enumeration of $\tilde{C}_\alpha$. Set $\beta_\nu = \beta(\alpha_\nu)$ (the $\bar{\beta}$ of §5 for $\bar{\alpha} = \alpha_\nu$). By §5 we have $n(\alpha_\nu) = n$. Moreover, there are maps $\pi_\nu (\tilde{\pi}$ of §5) such that

$$J_{\beta_\nu} \xrightarrow{\pi_\nu}{}_{\Sigma_{n-1}} J_\beta \quad \text{and} \quad \pi_\nu \restriction \alpha_\nu = \mathrm{id} \restriction \alpha_\nu .$$

If $\alpha = \beta$, then $\alpha_\nu = \beta_\nu$; if $\alpha < \beta$, then $\alpha_\nu < \beta_\nu$ and $\pi_\nu(\alpha_\nu) = \alpha$. Set $\pi_{\nu\tau} = \pi_\tau^{-1} \cdot \pi_\nu$ [9] for $\nu \leq \tau$. Then

$$J_{\beta_\nu} \xrightarrow{\pi_{\nu\tau}}{}_{\Sigma_{n-1}} J_{\beta_\tau}$$

and $J_{\beta_\nu}, \pi_{\nu\tau}$ $(\nu \leq \tau < \theta)$ is a directed system whose direct limit is J_β, $\pi_\nu(\nu < \theta)$. $J_{\beta_\lambda}, \pi_{\nu\lambda}(\nu < \lambda)$ is the direct limit of $J_{\beta_\nu}, \pi_{\nu\tau}(\nu \leq \tau < \lambda)$ for limit λ. Since $\alpha \notin Q'$, we have three more cases to consider.

(2) $\alpha = \beta$;

(3) $\alpha < \beta$ and $B \cap \alpha \in J_\delta$ for some $\delta < \beta$, but for all $\delta < \beta$, if $D \in \mathfrak{P}(\alpha) \cap J_\delta$, then $\models_{J_\alpha} \varphi[D, B \cap \alpha]$;

(4) $\alpha < \beta$ and $B \cap \alpha \notin J_\delta$ for $\delta < \beta$.

Case 2. $\alpha = \beta$. Set $\overline{C}_\alpha = \widetilde{C}_\alpha$. If $\overline{\alpha} = \alpha_\lambda$ is a limit point, then $\alpha_\lambda = \beta_\lambda$ and case 2 applies. Hence $\overline{C}_{\overline{\alpha}} = \widetilde{C}_{\overline{\alpha}} = \overline{\alpha} \cap \overline{C}_\alpha$.

Case 3. $\alpha < \beta$; $B \cap \alpha \in J_\delta$ for some $\delta < \beta$; if $D \in \mathfrak{P}(\alpha) \cap J_\delta$ for $\delta < \beta$, then $\models_{J_\alpha} \varphi[D, B \cap \alpha]$.

Let δ be the least δ such that $B \cap \alpha \in J_\delta$. Let ν_0 be the least ν such that $\delta, B \cap \alpha \in \pi_\nu{}''J_{\beta_\nu}$.[10] Set $\overline{C}_\alpha = \{\alpha_\nu \mid \nu_0 \leq \nu < \theta\}$. If $\overline{\alpha} = \alpha_\lambda$ is a limit point of $\overline{C}_\alpha$, then

$$B \cap \overline{\alpha} = \pi_\lambda^{-1}(B \cap \alpha) \in J_{\pi_\lambda^{-1}(\delta)}$$

where $\pi_\lambda^{-1}(\delta) < \beta_\lambda$. We use $J_{\beta_\lambda} \xrightarrow{\pi_\lambda}{}_{\Sigma_{n-1}} J_\beta$ to conclude that case 3 applies to $\overline{\alpha}$. Hence $\overline{C}_{\overline{\alpha}} = \{\alpha_\nu \mid \nu_0 \leq \nu < \lambda\} = \overline{\alpha} \cap \overline{C}_\alpha$.

Case 4. $\alpha < \beta$; $B \cap \alpha \notin J_\delta$ for $\delta < \beta$. If $\delta < \beta_\nu$ and $B \cap \alpha_\nu \in J_\delta$, then $\pi_{\nu\tau}(B \cap \alpha_\nu) \neq B \cap \alpha_\tau$ for some $\tau > \nu$; since otherwise we should have $B \cap \alpha = \bigcup_{\tau > \nu} \pi_{\nu\tau}(B \cap \alpha_\nu) = \pi_\nu(B \cap \alpha_\nu) \in J_{\pi_\nu(\delta)}$ where $\pi_\nu(\delta) < \beta$. Define a normal function $\langle \nu_\iota \mid \iota < \overline{\theta} \rangle$ by

$$\begin{aligned} \nu_0 &= 0\,; \\ \nu_{\iota+1} &= \text{the least } \nu > \nu_\iota \text{ such that if } \bigvee \delta < \beta_{\nu_\iota}\, B \cap \alpha_{\nu_\iota} \in J_\delta\,, \\ &\qquad \text{then } \pi_{\nu_\iota \nu}(B \cap \alpha_{\nu_\iota}) \neq B \cap \alpha_\nu\,, \\ \nu_\lambda &= \sup_{\iota < \lambda} \nu_\iota \quad \text{if} \quad \sup_{\iota < \lambda} \nu_\iota < \theta \quad \text{for limit } \lambda\,. \end{aligned}$$

Set $\overline{C}_\alpha = \{\alpha_{\nu_\iota} \mid \iota < \overline{\theta}\}$. Let $\overline{\alpha} = \alpha_{\nu_\lambda}$ be a limit point of $\overline{C}_\alpha$. Then case 4 holds for $\overline{\alpha}$, since otherwise $B \cap \overline{\alpha} \in J_\delta$ for some $\delta < \beta_{\nu_\lambda}$. But then $B \cap \overline{\alpha} = \pi_{\nu_\iota \nu_\lambda}(B \cap \alpha_{\nu_\iota})$ for some $\iota < \lambda$. Hence $\pi_{\nu_\iota \nu_{\iota+1}}(B \cap \alpha_{\nu_\iota}) = B \cap \alpha_{\nu_{\iota+1}}$. Contradiction!

It follows readily that $\overline{C}_{\overline{\alpha}} = \{\alpha_{\nu_\iota} \mid \iota < \lambda\} = \overline{\alpha} \cap \overline{C}_\alpha$.

6.1. *Souslin's hypothesis in* L

The usual characterisation of weakly compact cardinals in terms of trees can be sharpened considerably if we assume V = L.

First the relevant definitions. By a *tree* we mean a partially ordered set $T = \langle T, < \rangle$ such that for any point $x \in T$, the set of predecessors $\{y \mid y < x\}$ is well ordered by $<$. Thus every $x \in T$ has a *rank* $|x|$ defined as the order type of $\{y \mid y < x\}$. The *length* $|X|$ of a set $X \subset T$ is defined by $|X| = \text{lub } \{|x| \mid x \in X\}$. By a *branch* we mean a $b \subset T$ which is closed under $<$ and well ordered by $<$. By an *antichain* we mean a set of mutually incomparable points in T.

Definition. Let κ be a regular cardinal. We call a tree T *κ-normal* iff
(i) T has just one initial point;
(ii) every non-maximal point has ≥ 2 immediate successors;
(iii) each $x \in T$ has successors at arbitrarily high levels $\alpha < |T|$;
(iv) a branch of limit length has at most one immediate successor;
(v) for all α, $\{y \mid |y| = \alpha\}$ has cardinality $< \kappa$.

It follows easily that, if T is κ-normal, then $|T| \leq \kappa$.

By a *κ-Aronszajn tree* we mean a normal tree of length κ which has no branch of length κ. By the "κ-Aronszajn hypothesis" ($\mathbf{AH}_\kappa$) let us mean the statement: There is no κ-Aronszajn tree. It is provable in **ZFC** that κ is weakly compact iff κ is strongly inaccessible and $\mathbf{AH}_\kappa$. If we assume **GCH**, this can be improved to: κ is regular and $\mathbf{AH}_\kappa$.

By a *Souslin tree* we mean a κ-normal tree of length κ which has no antichain of cardinality κ.

The *κ-Souslin hypothesis* ($\mathbf{SH}_\kappa$) says that there is no κ-Souslin tree. (Note. $\mathbf{SH}_\kappa$ is equivalent to: Every linear ordering whose intervals satisfy the κ-antichain condition has a dense subset of cardinality $< \kappa$).

Clearly, every Souslin tree is Aronszajn and hence $\mathbf{AH}_\kappa \to \mathbf{SH}_\kappa$. The converse is known not to be provable for $\kappa = \omega_1$, even with **GCH**. However, if V = L, we get: $\mathbf{AH}_\kappa \leftrightarrow \mathbf{SH}_\kappa \leftrightarrow \kappa$ is weakly compact for regular κ, as the following theorem shows:

Theorem 6.2. *Assume* V = L. *Let $\kappa > \omega$ be regular but not weakly compact. Then there is a κ-Souslin tree.*

In the proof of Theorem 6.2 we will make use of a further combinatorial property of L:
Let $A \subset \kappa$. Consider the following principle.

($\Diamond_\kappa(A)$) There is a sequence S_α ($\alpha \in A$) such that $S_\alpha \subset \alpha$ and for each $X \subset \kappa$ the set $\{\alpha \mid X \cap \alpha = S_\alpha\}$ is Mahlo in κ.

$\Diamond_\kappa(A)$ clearly implies that A is Mahlo in κ.

Lemma 6.5. *Assume* $V = L$. *Let* κ *be regular. Then* $\Diamond_\kappa(A)$ *holds for every Mahlo set* $A \subset \kappa$.

Proof. Assume (w.l.o.g.) that A contains only limit ordinals. Define a sequence $\langle S_\alpha, C_\alpha\rangle$ ($\alpha \in A$) by induction on α as follows.

$\langle S_\alpha, C_\alpha\rangle =$ the least pair $\langle S, C\rangle$ (in $<_J$) such that $S \subset \alpha$, C is closed, unbounded in α and $\bigwedge \tau \in C \;\; S \cap \tau \neq S_\tau$.

If no such pair exists, set

$$\langle S_\alpha, C_\alpha\rangle = \langle \emptyset, \emptyset\rangle .$$

We claim that the sequence $\langle S_\alpha \mid \alpha \in A\rangle$ fulfills $\Diamond_\kappa(A)$. Suppose not. Then there is an $S \subset \kappa$ and a closed, unbounded $C \subset \kappa$ such that $\bigwedge \alpha \in C \;\; S \cap \alpha \neq S_\alpha$. Let $\langle S, C\rangle$ be the least such pair (in $<_J$). Define a sequence of elementary submodels $X_\nu \prec J_{\kappa^+}$ ($\nu < \kappa$) as follows.

$X_0 =$ the smallest $X \prec J_{\kappa^+}$ such that $A \in X$ and $\kappa \cap X$ is transitive;

$X_{\nu+1} =$ the smallest $X \prec J_{\kappa^+}$ such that $X_\nu \cup \{X_\nu\} \subset X$ and $\kappa \cap X$ is transitive;

$X_\lambda = \bigcup_{\nu<\lambda} X_\nu$ for limit λ.

Set $\alpha_\nu = \kappa \cap X_\nu$. Then $\langle \alpha_\nu \mid \nu < \kappa\rangle$ is a normal function. Since A is Mahlo, there is an $\alpha = \alpha_\alpha$ such that $\alpha \in A$. Now let $\pi : X_\alpha \overset{\sim}{\longleftrightarrow} J_\beta$. Then $\pi \upharpoonright J_\alpha = \mathrm{id} \upharpoonright J_\alpha$ and $\kappa(\pi) = \alpha$. Now

$$\langle\langle S_\nu, C_\nu\rangle \mid \nu < \kappa\rangle\ ,\ \langle S, C\rangle \in X_\alpha\ ,$$

since these are J_{κ^+} definable from A. It is easily seen that

$$\pi(\langle\langle S_\nu, C_\nu\rangle \mid \nu < \kappa\rangle) = (\langle S_\nu, C_\nu\rangle \mid \nu < \alpha\rangle\ ,$$

$$\pi(\langle S, C\rangle) = \langle S \cap \alpha, C \cap \alpha\rangle\ .$$

Since $\pi^{-1} : J_\beta \to_{\Sigma_\omega} J_{\kappa^+}$, we conclude that $\langle S \cap \alpha, C \cap \alpha\rangle$ = the least pair $\langle S', C'\rangle$ (in J_β) such that $S' \subset \alpha$, C' is closed and unbounded in α and $\bigwedge \tau \in C'\ S' \cap \tau \neq S_\tau$. Hence $\langle S \cap \alpha, C \cap \alpha\rangle = \langle S_\alpha, C_\alpha\rangle$. But $\alpha \in C$, since $C \cap \alpha$ is unbounded in α. Hence $\alpha \in C$ and $S \cap \alpha = S_\alpha$. Contradiction! This completes the proof of Lemma 6.5.

We are now ready to prove Theorem 6.2. Let E, $C_\lambda(\mathrm{Lim}(\lambda), \lambda < \kappa)$ be as in Theorem 6.1. Let S_α $(\alpha \in E)$ be the sequence given by $\diamondsuit_\kappa(E)$. We wish to construct a Souslin tree T. The points of T will be ordinals $< \kappa$. We shall construct T in stages $T_\alpha (1 \leq \alpha < \kappa)$. T_α is to be the restriction of T to points of rank $< \alpha$. Hence T_α will be a normal tree of length α and T_β will be an end extension of T_α for $\beta > \alpha$. We define T_α by induction on α as follows.

Case 1. $\alpha = 1$. $T_1 = \{0\}$.

Case 2. $T_{\alpha+1}$ is defined. Define $T_{\alpha+2}$ by appointing two immediate successors for each maximal point of $T_{\alpha+1}$.

Case 3. $\mathrm{Lim}(\alpha)$ and T_ν is defined for $\nu < \alpha$. Set $T_\alpha = \bigcup_{\nu<\alpha} T_\nu$.

The remaining case is the crucial one:

Case 4. $\mathrm{Lim}(\alpha)$ and T_α is defined. We must define $T_{\alpha+1}$. For each $x \in T_\alpha$ we first select a branch b_x of length α through T_α. b_x is defined as follows:

Let $\gamma_\nu(\nu < \lambda)$ be the monotone enumeration of C_α. Let $\bar{\nu} = \bar{\nu}_x$ be the least ν such that $|x| \leq \gamma_\nu$. We define a sequence $p_\nu = p_\nu^x (\bar{\nu} \leq \nu < \lambda)$ of points in T_α as follows.

$p_{\bar{\nu}}$ = the least ordinal y such that $|y| = \gamma_{\bar{\nu}}$ and $y \geq x$ in T_α ;

$p_{\nu+1}$ = the least ordinal y such that $|y| = \gamma_{\nu+1}$ and $y > p_\nu$ in T_α ;

for limit η:

$$p_\eta = \text{the unique } y \text{ such that } |y| = \gamma_\eta \text{ and } y > p_\nu (\nu < \eta) \text{ in } T_\alpha$$
$$\text{if such } y \text{ exists; otherwise undefined.}$$

If any p^x_η is undefined, then $T_{\alpha+1}$ is undefined. Otherwise we set $b_x = \{y \mid \bigvee \nu\, y \leq p^x_\nu \text{ in } T_\alpha\}$.

If $\alpha \notin E$, we form $T_{\alpha+1}$ by appointing an immediate successor to each b_x. If $\alpha \in E$ but S_α is not a maximal antichain in T_α, we do the same. If $\alpha \in E$ and S_α is a maximal antichain in T_α, we appoint an immediate successor only to b_x such that $\bigvee z \in S_\alpha\; x \leq z$ in T_α.

It is clear that $T_{\alpha+1}$ is a normal tree of length α. We must prove that T_α is defined for $\alpha < \kappa$. In cases 1–3 this is trivial. In case 4 we must show that $p_\nu = p^x_\nu$ is defined for $\bar{\nu} \leq \nu < \lambda$. The nontrivial case is p_η $(\mathrm{Lim}(\eta))$. Since γ_η is a limit point of C_α, then $\gamma_\eta \notin E$ and $C_{\gamma_\eta} = \gamma_\eta \cap C_\alpha = \{\gamma_\nu \mid \nu < \eta\}$. It follows that if we define p'^x_ν $(\nu < \eta)$ from γ_η as p^x_ν was defined from α, then $p'^x_\nu = p^x_\nu$. But $b'_x = \{y \mid \bigvee \nu < \eta\; y \leq p'_\nu$ in $T_{\gamma_\eta}\}$ has a successor in T_α by case 4, since $\gamma_\eta \notin E$. Hence p_η is defined.

Set $T = \bigcup_{\alpha<\kappa} T_\alpha$. T is clearly a normal tree of length κ. We must prove that T is Souslin. Let $X \subset T$ be a maximal antichain in T. Let A be the set of limit $\alpha < \kappa$ such that $\alpha \cap X$ is a maximal antichain in T_α. A is easily seen to be closed and unbounded in κ. Hence there is $\alpha \in A \cap E$ such that $S_\alpha = X \cap \alpha$ by $\diamondsuit_\kappa(E)$. By the construction of $T_{\alpha+1}$, we then have:
Every $x \in T$ of level α lies above an element of $X \cap \alpha$. Hence $X \cap \alpha$ is a maximal antichain in T. Hence $X = X \cap \alpha$ has cardinality $< \kappa$. This proves Theorem 6.2.

6.2. *Partition properties in* L

Definition. Let $[X]^n$ denote the collection of all n element subsets of $X (n < \omega)$. Let a partition $\Delta = \langle \Delta_\iota \mid \iota \in I \rangle$ of $[X]^n$ be given. Let τ be a cardinal. We call $Y \subset X$ *τ-homogeneous* with respect to Δ iff $[Y]^n \subset \bigcup_{\iota \in s} \Delta_\iota$ for some s such that $\bar{\bar{s}} \leq \tau$.

Definition. Let $\kappa, \delta, \gamma, \tau$ be cardinals such that $\tau < \gamma < \kappa$. We write

$$\kappa \to (\delta)^n_{\gamma\tau}$$

to mean that every partition of $[\kappa]^n$ γ parts has a τ homogeneous set of cardinality δ. Clearly $\kappa \to (\delta)^n_{\gamma\tau}$ implies $\kappa' \to (\delta')^{n'}_{\gamma'\tau'}$ for $n' \leq n$, $\kappa \leq \kappa'$, $\delta' \leq \delta$, $\gamma' \leq \gamma$, $\tau \leq \tau'$. It is known that, for $\kappa > \omega$, $\kappa \to (\kappa)^2_{21}$ implies weak compactness and weak compactness implies $\kappa \to (\kappa)^n_{\gamma 1}$ for $n < \omega$, $\gamma < \kappa$. If we assume V = L we can sharpen this result by showing that, for regular κ, each of the principles $\kappa \to (\kappa)^2_{\gamma\tau}$ ($\tau < \gamma < \kappa$) implies weak compactness.

Theorem 6.3. *Assume* V = L. *Let κ be regular but not weakly compact. Then $\kappa \not\to (\kappa)^2_{\gamma\tau}$ for $\tau < \gamma < \kappa$.*

Theorem 6.3 is an immediate corollary of Theorem 6.2 and the following lemma.

Lemma 6.6. *Assume* ZFC. *Let* SH_κ *fail. Then $\kappa \not\to (\kappa)^2_{\gamma\tau}$ for $\tau < \gamma < \kappa$.*

Proof. Let T be a Souslin tree. We may suppose (without loss of generality) that each point of T has $\geq \gamma$ many immediate successors. Let $S(x)$ be the set of immediate successors of x. For each $x \in T$ partition $[S(x)]^2$ into disjoint nonempty sets Δ^x_i ($1 \leq i < \gamma$). We now define a partition Δ_i ($i < \gamma$) of T.

If $y_0, y_1 \in T$ are comparable, put $\{y_0, y_1\} \in \Delta_0$. Otherwise let x be the greatest common predecessor of y_0, y_1. Then there are unique $x_i \in S(x)$ such that $x_i \leq y_i$ in T. Put $\{y_0, y_1\} \in \Delta_i$ if $\{x_0, x_1\} \in \Delta^x_i$. If $\kappa \to (\kappa)^2_{\gamma\tau}$ held, there would be a set $X \subset T$ of cardinality κ and an $s \subset \gamma$ of cardinality τ such that $[X]^2 \subset \bigcup_{i \in s} \Delta_i$. We derive a contradiction as follows.

Case 1. $0 \notin s$. Then X is an antichain of cardinality κ.

Case 2. $0 \in s$. Set $Y = \{y \mid \bigvee x \in X\ y \leq x \text{ in } T\}$. Then $[Y]^2 \subset \bigcup_{i \in s} \Delta_i$. But for each $x \in Y$, some immediate successor of x is not in Y. Let Z be the set of $z \notin Y$ such that z immediately succeeds an element of Y. Then Z is an antichain of cardinality κ.

Note. Lemma 6.6 was first proved by Tony Martin for the case $\gamma = 3$, $\tau = 2$. The general case is due to Soare.

Remarks. (1). Theorem 6.1 can be proved under the following weaker assumption. $\mathrm{V} = \mathrm{L}[A]$ for an $A \subset \kappa$ such that for some Π^1_1 statement φ

$$\bigwedge D \subset \kappa \models_{\mathrm{L}_\kappa[A]} \varphi[D, A] ,$$

but for $\beta < \kappa$

$$\bigvee D \in \mathfrak{P}(\beta) \cap \mathrm{L}_\beta[A \cap \beta] \models_{\mathrm{L}_\beta[A \cap \beta]} \neg \varphi[D, A \cap \beta] .$$

(2). Lemma 6.5 can be proved under the assumption

$$\bigvee A \subset \kappa \ \ \mathrm{V} = \mathrm{L}[A] .$$

(3). A weaker form of Theorem 6.1 can be used in the proof of Theorem 6.2; (iii) can be replaced by (iii)': $\{\beta \cap C_\lambda \mid \lambda < \kappa\}$ has cardinality $< \kappa$ for $\beta < \kappa$.

This form of Theorem 1 holds trivially for successors of regular cardinals γ such that $2^\gamma = \gamma^+$. It also holds trivially for e.g. the first Mahlo cardinal.

(4). With a slight modification of the proof of Lemma 6.6 one can sharpen the conclusion to: There is a partition $\Delta_\nu (\nu < \kappa)$ of $[\kappa]^2$ such that if $X \subset \kappa$ has cardinality κ, then $\bigwedge \nu [X]^2 \cap \Delta_\nu \neq \emptyset$. This is the version proved by Soare.

§7. The one-gap two-cardinal conjecture holds in L

by Jack SILVER

Let $\square_\kappa$ be this combinatorial proposition: There is a sequence $\langle C_\alpha : \alpha$ is a limit ordinal $< \kappa^+\rangle$ such that each C_α is a closed, cofinal subset of α; if $\mathrm{cf}(\alpha) < \kappa$, then C_α has cardinality less than κ; and finally, if β is a limit point of C_α, then $C_\beta = \beta \cap C_\alpha$.

In §5, Jensen has established that $\square_\kappa$ holds in L for all cardinals κ (and indeed holds under somewhat weaker hypotheses). It is the burden of this note to show that:

($\#$) If $(\forall\lambda < \kappa)\,(2^\lambda = \lambda^+)$ and $\square_\kappa$ holds, κ singular, then $(\omega_1, \omega_0) \to (\kappa^+, \kappa)$.[11]

The expression $(\lambda^+, \lambda) \to (\kappa^+, \kappa)$ means that any countable first-order theory having a model of type (λ^+, λ), i.e. a model whose universe has cardinality λ^+ and in which the unary predicate U denotes a set of cardinality λ, also has a model of type (κ^+, κ). Vaught [7] has shown that, for any infinite cardinal λ, $(\lambda^+, \lambda) \to (\omega_1, \omega_0)$, and Chang [2], assuming the **GCH**, has shown that whenever τ is a regular cardinal, $(\omega_1, \omega_0) \to (\tau^+, \tau)$. The **GCH** being a consequence of the axiom of constructibility, the above considerations reveal that ($\#$) together with Jensen's proof of the combinatorial principles from the axion of constructibility fill in what is needed to see that the full one-gap conjecture, ($\forall$ infinite κ, λ) $((\lambda^+, \lambda) \to (\kappa^+, \kappa))$, is indeed a consequence of the axiom of constructibility, only the singular case having been problematical. It is still not known whether the full one-gap conjecture may fail in a model where the **GCH** holds, the solution very possibly awaiting further progress on the singular cardinals problem of set theory. On the other hand, Mitchell (Ph.D. dissertation, Univ. of Calif., Berkeley, 1970) has found a non-**GCH** model in which the one-gap conjecture fails for very low regular cardinals. It should also be noted that, by a more difficult argument, Jensen has shown that $(\forall\lambda < \kappa)(2^\lambda = \lambda^+)$ can be weakened to $(\forall\lambda < \kappa)(2^\lambda \leq \kappa)$ in ($\#$).

Before getting down to business it is instructive and perhaps even useful to see that $\square_\kappa$ can be reformulated in the following manner: There is a sequence $\langle S_\alpha : \alpha$ is a limit ordinal $< \kappa^+\rangle$ such that (i) each S_α is a closed subset of α, and if $\mathrm{cf}(\alpha)$ exceeds ω, a closed cofinal subset of α; (ii) if $\mathrm{cf}(\alpha) < \kappa$, then the cardinality of S_α is less than κ; (iii) if $\beta \in S_\alpha$, then $\beta \cap S_\alpha = S_\beta$. To derive this new formulation from the original one, simply take S_α to consist of all limit points of C_α other than α itself. As for the other direction, define C_α by induction on α, taking C_α to be the union of all C_β as β ranges over S_α together with, if $\sup S_\alpha < \alpha$, the greatest element of S_α and an ω-sequence above it converging to α. Noting that '$\beta \in S_\alpha$' is a partial ordering, we can obtain a still more refined formulation.

The model-theoretic arguments here are modelled on those of C.C. Chang which appear in his well-known paper on the two cardinal problem [2]. Let us digress for a moment to summarize that work briefly. Assume the **GCH**, or at least enough of it to make the following arguments work. Let L be a countable first-order language with equality having a unary predicate symbol U. If $\mathfrak{A}$ is an L-structure, the type of $\mathfrak{A}$ is defined to be that ordered pair of cardinals whose first component is the cardinality of the universe of $\mathfrak{A}$ and whose second component is the cardinality of the set denoted by U in $\mathfrak{A}$. Suppose that $\mathfrak{P}$ is an L-structure having type (ω_1, ω_0), and that κ is a regular infinite cardinal. There is no loss of generality in supposing that L has a binary predicate symbol E which denotes in $\mathfrak{B}$ an extensional relation such that, whenever H is a finite subset of $U^{\mathfrak{B}}$, there is an element of $U^{\mathfrak{B}}$ whose 'E-members' are precisely the elements of H. (Let T_0 be the first-order theory involving U, E, and equality which expresses the properties in the last sentence.) To incorporate the finite set structure into the original structure is one of the key devices due to Chang. A key lemma of Chang states that, if $\langle \mathfrak{A}_\alpha : \alpha < \beta \rangle$ is an elementary tower of U-saturated structures, each a model of T_0 and each having cardinality κ, and $\beta \leq \kappa$, and $U^{\mathfrak{A}_\alpha}$ is the same for all α, then the union of the structures is itself U-saturated (the notion of U-saturadness is to be defined later).

We seek a structure of type (κ^+, κ) elementarily equivalent to $\mathfrak{B}$. This is to be obtained by forming an elementary tower of height κ^+ of saturated structures, each having power κ, each elementarily equivalent to $\mathfrak{B}$, and all having the same U. (It being understood that inclusion in the tower is proper, the union must have cardinality κ^+.) To form this tower inductively, one need only show that any U-saturated structure elementarily equivalent to $\mathfrak{B}$ has a proper U-saturated elementary extension with the same U. (Chang's lemma mentioned at the end of the last paragraph takes care of the limit stage in the construction.) Let $\mathfrak{B}$ be such a structure. If $\mathfrak{A}$ is not itself (fully) saturated, then $\mathfrak{A}$ can be extended (properly) to a saturated elementary extension without changing U. On the other hand, if $\mathfrak{A}$ is itself saturated, then, as had been known for some time, $\mathfrak{A}$, being elementarily equivalent to a 'two-cardinal' model $\mathfrak{B}$, has a proper saturated elementary extension with the same U. This will be seen below for special models, using the same proof.

Where, then, does the above proof break down if κ is assumed to be singular instead of regular? There will not in general be a saturated model of power κ elementarily equivalent to $\mathfrak{B}$ (the number of subsets of cardinality cf(κ) being greater than κ), in view of which the above construction cannot be started. Suppose we replace 'saturated' everywhere by 'special' (a special model being the union of a certain kind of elementary tower of saturated models, to be defined below). All of the above steps go through except the needed analogue of Chang's lemma concerning unions of U-saturated structures. To remedy this difficulty, we need to associate to each special model in our tower a ranking of its elements (which simply tells us how to write the model as a union of saturated elementary submodels) as we go along, the choice of the ranking being critical, and to make use of Jensen's combinatorial principles $\square_\kappa$ in assigning the ranking at limit stages. Thus it is not known whether, in the absence of that combinatorial principles, an argument of this kind (or indeed any other proof of the desired two-cardinal result) can be carried out.

We commence the proof of (#). Suppose κ is singular, the **GCH** holds beneath κ, and $\square_\kappa$ holds. Let τ be the cofinality of κ, and suppose that $G(\alpha)$, $\alpha < \tau$, is a strictly increasing sequence of regular cardinals converging from below to κ, $G(0) = 0$, $G(1) > \omega$. As before, let L be a countable first-order language with equality having a unary predicate symbol U.

Definition 7.1. *Saturated and special models. Rankings.*

(7.1.1) If card $A = \lambda$ and $\mathfrak{A} = (A, U^{\mathfrak{A}}, \ldots)$ is an L structure, then $\mathfrak{A}$ is said to be *U-saturated* if the following condition holds: whenever S is a set of unary formulas (i.e. having only the free variable x) with parameters from A such that card $S < \lambda$ and S is finitely satisfiable in $U^{\mathfrak{A}}$ (i.e. any finite subset of S is simultaneously satisfied by some element of $U^{\mathfrak{A}}$ in the structure $\mathfrak{A}$), then there is some element of $U^{\mathfrak{A}}$ which simultaneously satisfies all formulas of S in $\mathfrak{A}$.

(7.1.2) For the definition of '$\mathfrak{A}$ is saturated' simply replace $U^{\mathfrak{A}}$ everywhere in 7.1.1 by A.

(7.1.3) An L-structure $\mathfrak{A}$ (of cardinality κ) is said to be *U-special* if it is the union of some ascending elementary tower $\langle \mathfrak{A}_\alpha : \alpha < \tau \rangle$ where each $\mathfrak{A}_\alpha$ is a U-saturated structure of power $G(\alpha)$. A mapping $r : |\mathfrak{A}| \to \tau$ is said to be a *U-ranking* of $\mathfrak{A}$ iff there exists such an elementary tower for which $(\forall x \in |\mathfrak{A}|)(r(x) =$ the least α such that $x \in |\mathfrak{A}_{\alpha+1}|)$.

(7.1.4) 'Special' and 'ranking are defined in an analogous way (one simply omits all references to U).

Definition 7.2. If $\mathfrak{A} = (A, U^{\mathfrak{A}}, ...)$ is an L-structure, *type* $\mathfrak{A}$ is defined to be (card A, card $U^{\mathfrak{A}}$).

Lemma 7.3. Saturated and special models.

(7.3.1) *If λ is a regular uncountable cardinal less than κ, then any L-structure of cardinality $\leq \lambda$ has a saturated elementary extension of cardinality λ.*

(7.3.2) *If T is a theory in L having infinite models, then T has a saturated model in each uncountable regular cardinality less than κ, and has a special model (of cardinality κ).*

(7.3.3) *If $\mathfrak{A}$ and $\mathfrak{A}'$ are special models with rankings r and r′ respectively, then there exists an isomorphism f of $\mathfrak{A}$ onto $\mathfrak{A}'$ which sends r into r′, i.e. $(\forall x \in |\mathfrak{A}|)(r'(f(x)) = r(x))$. Also, if $\mathfrak{A}'$ is special with ranking r′ and $\mathfrak{A}$ is U-special with U-ranking r, then there exists an elementary monomorphism f of $\mathfrak{A}$ into $\mathfrak{A}'$ such that the range of f includes $U^{\mathfrak{A}'}$ and $(\forall x)(r'(f(x)) = r(x))$.*

Remarks on the proof of Lemma 7.3. The first part of (7.3.1), enabling us inductively to form a tower of the required type, directly gives (7.3.2). To do the first part of (7.3.3), let $\mathfrak{A}_\alpha$ and $\mathfrak{A}'_\alpha$ be the representations of $\mathfrak{A}$ and $\mathfrak{A}'$ respectively given by r and r'. Define inductively an ascending chain of isomorphisms f_α, f_α being an isomorphism between $\mathfrak{A}_\alpha$ and $\mathfrak{A}'_\alpha$ in each case. This is possible owing to a basic property of saturated models, that an elementary map of cardinality $< \lambda$ between subsets of two elementarily equivalent saturated models of cardinality λ can be extended to an isomorphism between the models. Finally let f be the union of all the f_α. To do the second part of (7.3.3), imitate the argument just completed, making use instead of the following principle: If $\mathfrak{S}$ is U-saturated, $\mathfrak{S}'$ is fully saturated, $\mathfrak{S}$ and $\mathfrak{S}'$ are elementarily equivalent and have the same cardinality, and h is an elementary (i.e. satisfaction preserving) map of a substructure of $\mathfrak{S}$ having cardinality less than that of $\mathfrak{S}$ onto a substructure of $\mathfrak{S}'$, then h can be extended to an elementary monomorphism of $\mathfrak{S}$ into $\mathfrak{S}'$ whose range includes $U^{\mathfrak{S}'}$. This principle can be established by means of a Cantor back-and-forth argument, using the U-saturatedness of $\mathfrak{S}$ to get a preimage for each member of $U^{\mathfrak{S}'}$.

Lemma 7.4. *If $\mathfrak{A}$ is a U-special model with U-ranking r and $\mathfrak{A}$ is elementary equivalent to a model $\mathfrak{B}$ of type (ω_1, ω_0), then $\mathfrak{A}$ has a proper elementary extension $\mathfrak{A}'$ having a ranking r' which extends r, and such that $U^{\mathfrak{A}} = U^{\mathfrak{A}'}$.*

Proof. By a short argument from the second part of (7.3.3), we obtain a special extension having the same U and having a ranking extending the given ranking (by a replacement argument, one can assume that f in (7.3.3) is the identity). Call the special extension $\mathfrak{A}^*$ and let r^* be a ranking of it extending r. We now claim that there is a special proper elementary extension $\mathfrak{A}'$ of $\mathfrak{A}^*$ having a ranking r' which extends r^*. This can be argued as follows: Let B' be the universe of a countable elementary substructure of $\mathfrak{B}$ which includes $U^{\mathfrak{B}}$. Take $(B_1, ..., B_1')$ to be a special structure (of cardinality κ) with ranking s which is elementary equivalent to the structure $(\mathfrak{B}, B')$ obtained by adding in B' an additional unary relation. Clearly B_1' is a proper subset of B_1 (the corresponding assertion having been true for the B's) and the structure $(B_1, ..., B_1')$ naturally splits into two structure, namely $\mathfrak{B}_1$, obtained simply by removing the unary relation B_1', and $\mathfrak{B}_1'$, the result of cutting $\mathfrak{B}_1$ down to the universe B_1'. Since reducts and relativized reducts of saturated structures are saturated, both $\mathfrak{B}_1$ and $\mathfrak{B}_1'$ are special, with s and $s|B_1'$ as rankings. Moreover, each is elementarily equivalent to $\mathfrak{B}$. By Lemma 7.3.3, we may identify $\mathfrak{B}^*$ with $\mathfrak{B}_1'$ and r^* with $s|B_1'$. Then $\mathfrak{B}_1$ and s give the desired proper special extension with extension ranking, it being clear from $U^{\mathfrak{B}} \subset B'$ that $\mathfrak{B}_1$ and $\mathfrak{B}_1'$ have the same U.

Lemma 7.5. (Chang [1].) *Assume E is a binary predicate symbol in L. Let T_0 be that theory in L whose models are precisely those structure $\mathfrak{A}$ in which $E^{\mathfrak{A}}$ is an extensional relation, and, for each finite subset H of $U^{\mathfrak{A}}$, there is an element $x \in U^{\mathfrak{A}}$ whose $E^{\mathfrak{A}}$ extension (i.e. $\{y : E^{\mathfrak{A}}yx\}$) is precisely H. Then, for any regular cardinal λ, if $\langle \mathfrak{A}_\alpha : \alpha < \beta \rangle$ is an elementary tower of U-saturated models of T_0 of power λ, $\beta \leq \lambda$, U_α independent of α, the union of this elementary tower is itself U-saturated.*

For the reader's convenience, we outline the proof. We are given a set S of unary formulas, S being finitely satisfiable by elements of U in the union and having power $< \lambda$ and having as parameters elements of the union. For each $\alpha < \beta$, let S_α consist of all formulas of S whose parameters are in $\mathfrak{A}_\alpha$. Since $\mathfrak{A}_\alpha$ is U-saturated, there is an element z_α in U^α (hereafter called U since it is independent of α) which simultaneously satisfies all formulas of S_α. Again since $\mathfrak{A}_\alpha$ is U-saturated, there is an element $w_\alpha \in U$ whose E extension contains all z_τ for $\alpha \leq \tau'$, and all the elements of whose E extension simultaneously satisfy S_α and are in U. Clearly the extensions of the various w_α form a collection having the finite intersection property, any finite number of them having some $z_{\tau'}$ (chosen far enough out) in their intersection. Hence, since $\mathfrak{A}_0$ is U-saturated, there is some element t in the extension of every w_α. This element t simultaneously satisfies S.

Definition 7.6. Suppose r and r' are U-rankings of U-special $\mathfrak{A}$, $\mathfrak{A}'$, respectively.

(7.6.1) We write $(\mathfrak{A}, r) \subset\subset (\mathfrak{A}', r')$ if $\mathfrak{A}$ is an elementary substructure of $\mathfrak{A}'$ and r' extends r, and $U^{\mathfrak{A}} = U^{\mathfrak{A}'}$.

(7.6.2) We write $(\mathfrak{A}, r) \subset_\gamma (\mathfrak{A}', r')$ if, whenever $x \in |\mathfrak{A}|$, $r(x) < \gamma \to r'(x) < \gamma$, and $\gamma \leq r(x) \to r'(x) = r(x)$ and $U^{\mathfrak{A}} = U^{\mathfrak{A}'}$ and $r | U^{\mathfrak{A}} = r' | U^{\mathfrak{A}}$. Finally, we write $(\mathfrak{A}, r) \subset : (\mathfrak{A}', r')$ if there exists γ such that $(\mathfrak{A}, r) \subset_\gamma (\mathfrak{A}', r')$.

We propose now to complete the proof of the principal theorem. Recall that we are assuming that the **GCH** holds beneath κ, which is a singular cardinal of cofinality τ, $G(\alpha)$ being a strictly increasing sequence of regular cardinals beginning with 0 and a cardinal greater than ω which tends from below to κ as α tends to τ. Further we are assuming the combinatorial principle $\square_\kappa$ of Jensen which, in the formulation we intend to use, asserts that there is a sequence $\langle S_\alpha : \alpha \text{ is a limit ordinal} < \kappa^+ \rangle$ such that each S_α is a closed subset of α, indeed closed cofinal if $\omega < \mathrm{cf}(\alpha)$, that each S_α has cardinality less than κ (since κ is singular, the case κ itself doesn't arise), and finally, such that a coherence condition holds: if $\beta \in S_\alpha$, then $S_\beta = \beta \cap S_\alpha$. Under these assumptions, we wish to establish the two-cardinal proposition

$$(\omega_1, \omega_0) \to (\kappa^+, \kappa) .$$

L is a countable first-order language with equality, having a unary predicate symbol U. We are given an L-structure $\mathfrak{B}$ of type (ω_1, ω_0) and we seek an elementarily equivalent structure of type (κ^+, κ). There is no loss of generality in assuming that L has a binary predicate symbol E and that $\mathfrak{B}$ is a model of the theory T_0 defined in the statement of Lemma 7.5 (since any structure can be augmented to be a model of T_0).

It is our plan to form a sequence $\langle(\mathfrak{A}_\alpha, r_\alpha) : \alpha < \kappa^+\rangle$ such that the $\mathfrak{A}$'s form a strictly increasing elementary tower of models elementarily equivalent to $\mathfrak{B}$ and have the same U (i.e. $U^{\mathfrak{A}_\alpha}$ is independent of α), such that each $\mathfrak{A}_\alpha$ has cardinality κ, that each $\mathfrak{A}_\alpha$ is U-special and r_α is a U-ranking of $\mathfrak{A}_\alpha$, and finally, such that the following three conditions are fulfilled:

(i) if $\alpha < \beta$, then $(\mathfrak{A}_\alpha, r_\alpha) \subset: (\mathfrak{A}_\beta, r_\beta)$;

(ii) if $\alpha \in S_\beta$, then $(\mathfrak{A}_\alpha, r_\alpha) \subset\subset (\mathfrak{A}_\beta, r_\beta)$;

(iii) if α is the $G(\alpha')$ th element of S_β (i.e. $\alpha \in S_\beta$ and $\alpha \cap S_\beta$ has order type $G(\alpha')$), then for all $x \in |\mathfrak{A}_\beta| - |\mathfrak{A}_\alpha|$, $r_\beta(x)$ is at least α'.

One must remember that, for any $\alpha' < \tau$, there are exactly $G(\alpha')$ elements of rank $< \alpha'$ in the structure. Condition (iii) is designed to insure that, at certain limit stages, we do not have too many elements of rank $< \alpha'$.

We move to the induction step in the definition of the promised sequence. (The case $\alpha = 0$ obviously presents no difficulties, since (7.3.2) gives us the existence of a special model.) Suppose that the sequence $\langle \mathfrak{A}_\alpha, r_\alpha) : \alpha < \alpha_0\rangle$ satisfies all of the above conditions, including (i)-(iii), for $\alpha, \beta < \alpha_0$ and for $\alpha' < \tau$. We wish to define $\mathfrak{A}_{\alpha_0}$ and r_{α_0}. Three cases arise.

Case 1. α_0 is a successor ordinal, say $\alpha_0 = \alpha + 1$. We use Lemma 7.4 to find $\mathfrak{A}_{\alpha_0}$ and r_{α_0} such that r_{α_0} is a U-ranking of $\mathfrak{A}_{\alpha_0}$ which is a proper elementary extension of $\mathfrak{A}_\alpha$, and $(\mathfrak{A}_\alpha, r_\alpha) \subset\subset (\mathfrak{A}_{\alpha_0}, r_{\alpha_0})$. It is quite easy to check that all the above conditions hold for $\alpha < \alpha_0 + 1$ (no new cases of (ii) and (iii) arising, for example).

Case 2. α_0 is a limit ordinal and S_{α_0} is cofinal in α_0. We take $\mathfrak{A}_{\alpha_0}$ to be the union of all the preceding $\mathfrak{A}_\alpha$'s, which is the same as $\bigcup\{\mathfrak{A}_\alpha : \alpha \in S_{\alpha_0}\}$, and $r_{\alpha_0} = \bigcup\{r_\alpha : \alpha \in S_{\alpha_0}\}$. Note that the latter equation does define a function by condition (ii) for α and β less than α_0 and by the coherence of the S_α's, which latter implies that if $\alpha < \beta$ are in S_{α_0}, then $\alpha \in S_\beta$. What does require scrutiny is the claim that r_{α_0} is a U-ranking of

$\mathfrak{A}_{\alpha_0}$. In the first place, we want to see that $\{x : r_{\alpha_0}(x) < \alpha'\}$ has cardinality exactly $G(\alpha')$ for every $\alpha' < \tau$. This is a straight-forward consequence of condition (iii) above, which guarantees that only the first $G(\alpha')$ terms in the sequence $\langle \mathfrak{A}_\alpha : \alpha \in S_{\alpha_0} \rangle$ give us elements having rank less than α'. Hence, it remains only to show that the set of elements having rank less than α' forms a saturated elementary submodel of $\mathfrak{A}_{\alpha_0}$. But this can be represented in the following way as an increasing union of length $\delta = \min(G(\alpha'), \text{order type } S_{\alpha_0})$ of saturated structures each of cardinality $G(\alpha')$, forming an elementary tower of elementary substructures of $\mathfrak{A}_{\alpha_0}$ and having the same U throughout (all of which means that Lemma 7.5 is applicable):

$$\bigcup_{\beta < \delta} \{x \in |\mathfrak{A}_\beta| : r_\beta(x) < \alpha'\} .$$

Case 3. α_0 is a limit ordinal but S_{α_0} is not cofinal in α_0. Hence, as was specified in the formulation of $\square_\kappa$ being used, α_0 has cofinality ω. Let β_0 be the least upper bound of S_{α_0} (which owing to the closure of the latter set, is a member of S_{α_0}) and let β_i be a strictly increasing ω-sequence of ordinals beginning with β_0 which converges from below to α_0. Further let α' be the least ordinal such that $G(\alpha')$ exceeds the order type of S_{α_0}. We now adopt the

Convention. We write $\mathfrak{A}^*_\alpha$ for $(\mathfrak{A}_\alpha, r_\alpha)$.

It is possible to find a strictly increasing sequence of ordinals φ_i less than τ and (if $i > 0$) greater than α' such that $\varphi_0 = 0$ and, for each i, $\mathfrak{A}^*_{\beta_i} \subset_{\varphi_{i+1}} \mathfrak{A}^*_{\beta_{i+1}}$. The possibility of forming such a sequence follows from the observation that if the relation $\subset_\varphi$ holds between two structures and φ' is an ordinal between φ and τ, then the relation $\subset_{\varphi'}$ also holds. Of course we are also using the inductive assumption that $\subset$: holds between any two structures – or rather structures with rankings thus far defined and recalling the definition of $\subset$: given in Definition 7.6.

As before, we take $\mathfrak{A}_{\alpha_0}$ to be the union of all the preceding $\mathfrak{A}_\alpha$'s, which is the same as the countable union $\bigcup \mathfrak{A}_{\beta_i}$. It remains to define the rank function r_{α_0} (which we also refer to as the α_0-rank function). If

$x \in |\mathfrak{A}_{\alpha_0}|$, define $i(x)$ to be the least j such that $x \in |\mathfrak{A}_{\beta_j}|$. We set

$$r_{\alpha_0}(x) = \max(\varphi_{i(x)}, r_{\beta_i(x)}(x)) .$$

We claim that if $\varphi_i < \varphi \leq \varphi_{i+1}$ then $\{x \in |\mathfrak{A}_{\alpha_0}| : r_{\alpha_0}(x) < \varphi\}$ is the same as $\{x \in |\mathfrak{A}_{\beta_i}| : r_{\beta_i}(x) < \varphi\}$ and hence is (or, more precisely, determines) a saturated model of cardinality $G(\varphi)$. Two considerations establish this euqality: (1) By definition, no new elements of α_0-rank $< \varphi$ appear after structure $\mathfrak{A}_{\beta_i}$; (2) Since the relation $\subset_{\varphi_i}$ holds between any pair of structures from the list $\mathfrak{A}^*_{\beta_0}, ..., \mathfrak{A}^*_{\beta_i}$, an element, once having had rank $< \varphi$ in one of the earlier structures, retains rank $< \varphi$ in $\mathfrak{A}_{\beta_i}$, though the exact rank may change. On the other hand, if φ exceeds all of the φ_i, then we can apply Lemma 7.5 since the set of elements having α_0-rank less than φ is just the union over i of the elements having β_i-rank less than φ, an increasing union (since the property of having rank less than φ is preserved by all of the relevant extensions) in which U remains the same throughout.

That $\mathfrak{A}^*_{\beta_0} \subset\subset \mathfrak{A}^*_{\alpha_0}$ (and more generally, as a consequence of the transitivity of $\subset\subset$ and induction assumptions, that $\mathfrak{A}^*_\alpha \subset\subset \mathfrak{A}^*_{\alpha_0}$ for all $\alpha \in S_{\alpha_0}$) is immediate from the definition of r_{α_0}. Thus condition (ii) remains valid for all ordinals $< \alpha_0 + 1$. Condition (iii) remains true because it was true for the case $\beta = \beta_0$, because $S_{\alpha_0} = S_{\beta_0} \cup \{\beta_0\}$, and by the stipulation that φ_i for i non-zero be greater than what we called α', which is not quite the same as any α' figuring explicitly in the statement of (iii). To check that condition (i) remains valid, it will suffice to see that $\mathfrak{A}^*_{\beta_i} \subset : \mathfrak{A}^*_{\alpha_0}$ for all i ($\subset :$ being a transitive relation). In fact, $\mathfrak{A}^*_{\beta_i} \subset_{\varphi_i} \mathfrak{A}^*_{\alpha_0}$ as can be seen from considering these two cases: (a) if $i(x) = i$, then by definition the α_0-rank of x is the same as the β_i-rank; (b) if $i(x) = j < i$ but the β_i-rank of x is at least φ_i, then, since $\mathfrak{A}^*_{\beta_j} \subset_{\varphi_i} \mathfrak{A}^*_{\beta_i}$, the β_j-rank of x is also at least φ_i and is in fact equal to the β_i-rank of x. Hence the α_0-rank of x equals its β_j-rank equals its β_i-rank, as desired.

Thus we are able to form a sequence $\langle(\mathfrak{A}_\alpha, r_\alpha) : \alpha < \kappa^+\rangle$ satisfying the conditions (i)-(iii). Let $\mathfrak{A} = \bigcup \{\mathfrak{A}_\alpha : \alpha < \kappa^+\}$. $|\mathfrak{A}|$ has cardinality κ^+ because the structures $\mathfrak{A}_\alpha$ form a strictly increasing tower (and because each structure has power κ). Since the relation $\subset :$ obtains between any two $(\mathfrak{A}_\alpha, r_\alpha)$, U remains fixed throughout, whence $U^{\mathfrak{A}}$ has cardinality κ^+. $\mathfrak{A}$, being the union of an elementary tower of structures each elementarily equivalent to $\mathfrak{B}$, is itself elementarily equivalent to $\mathfrak{B}$. $\mathfrak{A}$, then, is a model of type (κ^+, κ) elementarily equivalent to $\mathfrak{B}$, completing the proof.

Before concluding, let me digress for a moment to answer this question of Vaught: Does the transfer result $(\lambda^+, \lambda) \to (\kappa^+, \kappa)$ still hold, under the above hypotheses on κ, if the language is allowed to have κ many symbols? It does in virtue of the following fact: If every structure of type (λ^+, λ) having countable similarity type (i.e. appropriate for a countable language) is elementarily equivalent to some structure of type (κ^+, κ) having a universal elementary substructure of cardinality κ, [12] then every structure of type (λ^+, λ) appropriate for a language of cardinality κ is elementarily equivalent to some structure of type (κ^+, κ). The hypothesis certainly holds under our above assumptions on κ (that the **GCH** hold beneath κ and $\Box_\kappa$ hold) because the model $\mathfrak{A}_0$, though special and hence universal, is indeed an elementary submodel of the final model.

We now sketch the proof of the above 'fact'. Suppose $\mathfrak{B}$ is a structure of type (λ^+, λ) whose language has power κ. An easy argument shows that, for present purposes, we may replace $\mathfrak{B}$ by a structure having countable many relations and κ distinguished elements: Let u_α, $\alpha < \kappa$, be a one-to-one list of elements from $|\mathfrak{B}|$, and let R_α, $\alpha < \kappa$, list all the relations in the structure $\mathfrak{B}$. If R_α is n-ary, set

$$R'_n u_\alpha z_1 \dots z_n \quad \text{if} \quad R_\alpha z_1 \dots z_n$$

for every sequence of z's in $|\mathfrak{B}|$. For brevity, then, we simply assume that $\mathfrak{B}$ itself has countably many relations and κ distinguished elements. Let $\mathfrak{B}'$ be the structure obtained by deleting the distinguished elements from $\mathfrak{B}$, i.e. $\mathfrak{B}'$ is the reduct of $\mathfrak{B}$ to the similarity type corresponding to all of the relations and none of the distinguished elements in $\mathfrak{B}$. By hypothesis, there is a structure $\mathfrak{A}'$ of type (κ^+, κ) elementarily equivalent to $\mathfrak{B}'$ which has a universal elementary submodel of cardinality κ, say $\mathfrak{A}^*$. $\mathfrak{A}^*$ being universal, if $\mathfrak{B}''$ is an elementary submodel of $\mathfrak{B}'$ of power κ which contains all the distinguished elements of $\mathfrak{B}$, then there is an elementary embedding of $\mathfrak{B}''$ into $\mathfrak{A}^*$, call the elementary embedding f. We propose to expand $\mathfrak{A}'$ to a structure $\mathfrak{A}$ which shall be elementarily equivalent to $\mathfrak{B}$. We must specify a denotation for each of the individual constants. If the individual constant c denotes u in $\mathfrak{A}$, let c denote $f(u)$ in $\mathfrak{A}$. Using the fact that f is an elementary embedding, it is easy to check that $\mathfrak{A}$ is indeed elementarily equivalent to $\mathfrak{B}$.

Notes

[1] This has been independently worked out by Gandy in [4].

[2] For another elegant treatment (of the case ω_ω) see the final section of [2].

[3] There are analogous definitions for the case $|M| = V$. When the context permits, we frequently write Σ_n instead of $\Sigma_n(M)$, etc.

[4] By Property 0.1 we can replace Π_{n-1} by Σ_n if M is closed under finite subsets.

[5] So in particular, every simple function is Σ_0.

[6] So there are functions f such that f is Σ_0, i.e. R, defined by $R(y, x)$ iff $y = f(x)$, is Σ_0 relation; hence R is rud, but f is not rud, and χ_R is rud.

[7] Also $\pi(y) = \{\pi(z) | z \in y \cap X\}$ and π is the identity on transitive subsets of X.

[8] Rebholz noted that the above proof can be shortened, since after showing that $\beta = \alpha$, note that $f = \pi \cdot \tilde{h} \cdot g$ is $\Sigma_n(J_\alpha)$ in the parameters $\pi(p)$, $\pi(q)$ and maps a subset of $\omega\gamma$ onto $J_\beta = J_\alpha$. (Also π is $\Sigma_n(J_\alpha)$ in the parameters $\pi(p)$, $\pi(q)$, p, q.)

[9] $(\pi_\tau^{-1} \cdot \pi_\nu)(x) = \pi_\tau^{-1}(\pi_\nu(x))$.

[10] Then $\nu_0 =$ the least ν such that $\beta \cap \alpha_\nu \in J_{\beta_\nu}$.

[11] # was first proved by Jensen, using a more difficult argument.

[12] Equivalently, the (κ^+, κ) model is κ-universal, i.e. every elementarily equivalent structure of power κ can be elementarily embedded into it.

References

[1] G. Boolas and H. Putnam, Degrees unsolvability of constructible sets of integers.

[2] C.C. Chang, A note on the two cardinals problem, Proc. Amer. Math. Soc. 16 (1965) 1148–1155.

[3] C.C. Chang, What's so special about saturated models?, to appear.

[4] R.O. Gandy, Set theoretic functions for elementary syntax, in: D. Scott (ed.), Axiomatic set theory, Proc. Symp. in Pure Math., Vol. XIII, Part II, to appear.

[5] R. Jensen and C. Karp, Primitive recursive set functions, in: D. Scott (ed.), Axiomatic set theory, Proc. Symp. in Pure Math., Vol. XIII, Part I, 143–176.

[6] C. Karp, A proof of the relative consistency of the continuum hypothesis, in: Sets, models, and recursion theory (North-Holland, Amsterdam, 1967) 1–32.

[7] R.L. Vaught, The Lövenheim–Skolem theorem, in: Y. Bar-Hillel (ed.), Logic, methodology and philosophy of science (North-Holland, Amsterdam, 1965) 81–89.