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Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets

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Summary. Let $\mathbf{M}(X)$ be the family of all equivalent local martingale measures Q for some locally bounded d-dimensional process X, and V be a positive process. The main result of the paper (Theorem 2.1) states that the process V is a supermartingale whatever $Q \in \mathbf{M}(X)$, if and only if this process admits the following decomposition:

$$V_t = V_0 + \int_0^t H_s \, dX_s - C_t, \quad t \geqq 0 ,$$

where H is an integrand for X, and C is an adapted increasing process. We call such a representation *optional* because, in contrast to the Doob–Meyer decomposition, it generally exists only with an adapted (*optional*) process C.

We apply this decomposition to the problem of hedging European and American style contingent claims in the setting of *incomplete* security markets.

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1 Introduction

The famous Doob–Meyer decomposition states that each positive supermartingale $V = (V_t)_{t \geqq 0}$ defined on a filtered probability space $(\Omega, \mathscr{F}, \mathbf{F} = (\mathscr{F}_t)_{t \geqq 0}, P)$ has the following representation:

$$V = M - A , \tag{1.1}$$

where $M = (M_t)_{t \geqq 0}$ is a local martingale, $M_0 = V_0$, and $A = (A_t)_{t \geqq 0}$ is an increasing process. Moreover, there exists a decomposition involving a

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predictable process A, with Eq. (1.1) being the unique representation in this case.

Now let $\mathbf{Q} = \{Q\}$ be a family of equivalent probability measures on $(\Omega, \mathscr{F}, \mathbf{F})$, and let V be a positive supermartingale with respect to each measure $Q \in \mathbf{Q}$. We are interested in the decomposition of the form (1.1), with M being a Q-local martingale for all $Q \in \mathbf{Q}$. Even simple examples show that we can generally hope to obtain such a representation only with an adapted (*optional*) increasing process A. With this notation and following El Karoui and Quenez [7], this decomposition will be referred to as an *optional decomposition* of the positive supermartingale V with respect to the family of measures $\mathbf{Q}$.

Note that, in general, such an optional decomposition does not exist. As an example we take $\mathbf{Q}$ to be the family of all equivalent *supermartingale* measures for the process $X_t = \pi_t - t,\ t \geqq 0$. Here π is a Poisson process under the reference probability measure P. We assume that the filtration $\mathbf{F}$ is generated by this Poisson process: $\mathbf{F} = \mathbf{F}^{\pi}$. Then one can show that a local martingale with respect to all $Q \in \mathbf{Q}$ is a constant. Therefore the process X which is a Q-supermartingale for all $Q \in \mathbf{Q}$ cannot be represented as a difference between a local martingale with respect to all $Q \in \mathbf{Q}$ and an increasing process.

The main result of the paper (Theorem 2.1) states that the optional decomposition exists provided that $\mathbf{Q}$ is the family of all equivalent local martingale measures for some locally bounded d-dimensional process X. More exactly, in this case process V which is a positive supermartingale with respect to each measure $Q \in \mathbf{Q}$, admits the representation as follows:

$$V_t = V_0 + (H \bullet X)_t - C_t, \quad t \geqq 0 ,$$

where H is an integrand for X, $(H \bullet X)_t = \int_0^t H_s \, dX_s$ is the stochastic integral of H with respect to X, and C is an adapted increasing process.

In finance such a decomposition leads to a convenient *supermartingale* characterization of wealth and consumption portfolios. We apply this characterization to the problem of hedging European and American style contingent claims in the setting of *incomplete* security markets. This enables us to describe the capital evolution for the corresponding minimal hedging portfolios. The results obtained extend the solutions derived in El Karoui and Quenez [7] for the case when X is the solution of a stochastic differential equation governed by Brownian motion. In addition, they can be considered as a "dynamic" version of Theorem 5.7 in Delbaen and Schachermayer [4]. This theorem implies the existence of a hedging portfolio with initial wealth equal to the upper bound for arbitrage-free option prices at time $t = 0$. On the contrary, Theorems 3.2 and 3.3 below imply the existence of such a hedging portfolio whose capital at *each time instant* t is equal to the upper bound of arbitrage-free option prices at that time.

The proofs in El Karoui and Quenez [7] are mainly based on Girsanov's transformation of probability measures. Apparently this approach cannot be extended to the general setting under consideration. Instead, we apply the arguments based on the Hahn–Banach theorem. The main source for the results presented here was an important paper by Delbaen and Schachermayer [4].

2 Main results

Let $(\Omega, \mathscr{F}, \mathbf{F} = (\mathscr{F}_t)_{t \geqq 0}, P)$ be a filtered probability space which satisfies the conventional conditions of general theory of stochastic processes, i.e., the filtration $\mathbf{F}$ is right-continuous ($\mathscr{F}_t = \mathscr{F}_{t+}$) and the σ-field $\mathscr{F}_0$ contains all null sets from $\mathscr{F}$. For simplicity hereinafter the initial σ-field $\mathscr{F}_0$ is assumed to be the trivial one, i.e., it contains only sets with the measure zero or one.

On this filtered probability space, we consider an RCLL (right-continuous with left limits) d-dimensional random process $X = (X^i)_{i \leqq d}$. We assume that X is a locally bounded process, i.e., there is a sequence of stopping times $(\tau_n)_{n \geqq 1}$ on $(\Omega, \mathscr{F}, \mathbf{F})$ such that the variables τ_n almost surely converge to ∞ as n tends to ∞, and $|X_t^i| \leqq n$ for $t \leqq \tau_n$ and $1 \leqq i \leqq d$.

A probability measure Q is called a *local martingale measure*, if it is equivalent to P and the process X is a Q-local martingale. By $\mathbf{M}(X)$ we denote the set of all local martingale measures for process X. We assume that $\mathbf{M}(X)$ is not empty. Since all further concepts are invariant with respect to equivalent changes of measure, hereafter we assume that $P \in \mathbf{M}(X)$.

The main result of the paper is the following theorem.

Theorem 2.1 (Optional decomposition) *Let $V = (V_t)_{t \geqq 0}$ be a positive process. Then V is a supermartingale for each measure $Q \in \mathbf{M}(X)$ if and only if there exist an X-integrable predictable process $H = (H^i)_{1 \leqq i \leqq d}$ and an adapted increasing process C such that*

$$V_t = V_0 + (H \bullet X)_t - C_t, \quad t \geqq 0 , \tag{2.1}$$

where $(H \bullet X)_t = \int_0^t H_s \, dX_s$ is the stochastic integral of H with respect to X.

Since the process $H \bullet X$ mentioned above is uniformly bounded below, it is a local martingale with respect to all measures $Q \in \mathbf{M}(X)$, see Émery [8] and Ansel and Stricker [1]. Therefore, Eq. (2.1) is indeed an optional decomposition of V with respect to the family $\mathbf{M}(X)$. The proof of Theorem 2.1 is given in Sect. 5.

Theorem 2.1 generalizes the following result obtained by Jacka [10] and Ansel and Stricker [1].

Theorem 2.2 (Jacka, Ansel and Stricker) *Let $M = (M_t)_{t \geqq 0}$ be a positive process. Then M is a local martingale for each measure $Q \in \mathbf{M}(X)$ if and only if there exists an X-integrable predictable process $H = (H^i)_{1 \leqq i \leqq d}$ such that*

$$M_t = M_0 + (H \bullet X)_t, \quad t \geqq 0 .$$

We now consider the question of uniqueness for the decomposition (2.1).

We denote by X^c the continuous martingale part of X with respect to P, $\langle X^c, X^c \rangle$ means the quadratic variation of X^c, $[X,X]$ corresponds to the quadratic variation of X, and $\widetilde{[X,X]}$ designates the compensator of $[X,X]$ with respect to P. For simplicity of notation we formulate the result for the case $d = 1$.

Lemma 2.1 *Assume that there is a positive predictable process* $h = (h_t(\omega))_{t \geqq 0}$ *such that*

$$[\widetilde{X,X}]_t(\omega) = \int_0^t h_s(\omega)\, d\langle X^c, X^c\rangle_s(\omega)$$

almost surely. Then the processes $H \bullet X$ *and* C *in the decomposition* (2.1) *are uniquely defined.*

Proof. Let H and G be predictable X-integrable processes, and A and B be adapted RCLL processes of bounded variation such that

$$H \bullet X + A = G \bullet X + B := V .$$

We must show that $H \bullet X = G \bullet X$ or, equivalently,

$$HI\{|H| \leqq N\} \bullet X = GI\{|G| \leqq N\} \bullet X , \tag{2.2}$$

for any $N \geqq 1$.

Because the continuous martingale part of V equals

$$V^c = H \bullet X^c = G \bullet X^c ,$$

we deduce that

$$\int_0^\infty (H_s - G_s)^2\, d\langle X^c, X^c\rangle = 0 .$$

Then the condition of the lemma implies:

$$\int_0^\infty (H_s - G_s)^2\, d[\widetilde{X,X}] = 0 ,$$

and Eq. (2.2) follows from the Doob inequality for locally square integrable martingales. □

Note 2.1. The condition of the lemma is invariant with respect to an equivalent change of measure P.

Finally, we state the result that links the "optional" decomposition with the "predictable" decomposition of Doob and Meyer.

Let V be a bounded below Q-supermartingale for all $Q \in \mathbf{M}(X)$. We denote by $\mathscr{B}$ the set of predictable increasing RCLL processes B, such that $B_0 = 0$ and $V + B$ is a Q-supermartingale for all $Q \in \mathbf{M}(X)$. We introduce an ordering $\prec$ on $\mathscr{B}$ indicating that A is *less* than B ($A \prec B$) if $B - A$ is an increasing process.

Lemma 2.2 *The maximal element* $\hat{B}$ *in the ordered set* $\mathscr{B}$ *exists and is unique.*

Proof. We start with an intermediate claim.

Claim. *Let* A *and* B *be the elements of* $\mathscr{B}$. *Then there exists* $C \in \mathscr{B}$ *such that* $C - A$ *and* $C - B$ *are increasing processes.*

Hahn's decomposition for increasing predictable processes (see [12, Chap. I, Proposition 3.13]) implies the existence of a predictable process h with values in $\{-1,1\}$ such that

$$\int_0^t |dA_s - dB_s| = \int_0^t h_s(dA_s - dB_s), \quad t \geqq 0 .$$

Denote

$$C_t = \frac{1}{2}\left(\int_0^t (1+h_s)\,dA_s + \int_0^t (1-h_s)\,dB_s\right), \quad t \geqq 0 .$$

Then the processes $C-A$ and $C-B$ are increasing processes. Moreover, because

$$V + C = \frac{1}{2}\left(\int_0^t (1+h_s)\,d(V_s+A_s) + \int_0^t (1-h_s)\,d(V_s+B_s)\right), \quad t \geqq 0$$

and the processes $V+A$ and $V+B$ are supermartingales with respect to all $Q \in \mathbf{M}(X)$, we deduce that $V+C$ is a supermartingale with respect to all $Q \in \mathbf{M}(X)$ and, therefore, $C \in \mathscr{B}$. The claim is proved.

Now let $b = \sup_{B\in\mathscr{B}} EB_\infty$ and $(B^n)_{n\geqq 1}$ be a sequence in $\mathscr{B}$ such that the expectations EB^n_∞ tend to b as n tends to ∞. Using the claim, we can construct this sequence so as to make the processes $B^{n+1} - B^n$ increasing for any $n \geqq 1$. Then the sequence $(B^n)_{n\geqq 1}$ converges to a process $\widehat{B}$ in $[0,+\infty]$ uniformly. It can be easily seen that $\widehat{B} \in \mathscr{B}$. Finally, the claim above and the fact

$$E\widehat{B}_\infty = b = \sup_{B\in\mathscr{B}} EB_\infty$$

imply that $\widehat{B}$ is the unique maximal element in $\mathscr{B}$. □

Note 2.2. When $\mathbf{M}(X)$ is a singleton, the process $\widehat{B}$ is exactly the process that appears in the Doob–Meyer decomposition.

3 Applications in finance

1. In finance the process $X = (X^i)_{i\leqq d}$ is interpreted as a discounted price process of d assets in a security market. As above, we assume that X is a locally bounded RCLL process and that the set $\mathbf{M}(X)$ of local martingale measures for X is not empty. This corresponds to the absence of arbitrage opportunities on the security market, see the recent paper by Delbaen and Schachermayer [4] for precise statement.

We are reminded that a *wealth* and *consumption* portfolio can be described as a triple $\Pi = (v, H, C)$, where v is the *initial wealth* of the portfolio, $H = (H^i)_{i\leqq d}$ is a predictable X-integrable process of *numbers* of assets, and $C = (C_t)_{t\geqq 0}$ is an adapted increasing right-continuous process of *consumption*. A *capital* process $V = (V_t)_{t\geqq 0}$ of portfolio Π equals

$$V_t = v + \int_0^t H_s\,dX_s - C_t, \quad t \geqq 0 . \tag{3.1}$$

This equation has a clear economic interpretation: changes in portfolio wealth are caused by changes in asset prices and by consumption. In particular, when $C \equiv 0$, Eq. (3.1) means that the portfolio Π is developing in a *self-financing* way. Portfolio Π is called an *admissible* strategy if $V_t \geqq 0$, $t \geqq 0$. Theorems 2.1 and 2.2 immediately lead to the following important characterization of admissible portfolios.

Theorem 3.1 *Let $(V_t)_{t \geqq 0}$ be a positive process. Then*

(i) *V is the capital of a self-financing portfolio if and only if V is a local martingale with respect to all $Q \in \mathbf{M}(X)$.*
(ii) *V is the capital of a wealth and consumption portfolio if and only if V is a supermartingale with respect to all $Q \in \mathbf{M}(X)$.*

2. Let now f be a positive random variable on $(\Omega, \mathscr{F})$. We interpret f as the value of a contingent claim or as the payment of a European option with maturity at time $T = \infty$. An admissible strategy Π with wealth V is called the *hedging* portfolio for f if $V_\infty \geqq f$. Moreover, a hedging portfolio $\widehat{\Pi}$ with wealth $\widehat{V}$ is called the *minimal* hedge for f if $\widehat{V}_t \leqq V_t$ almost surely, for any $t \geqq 0$ and hedging strategy Π with wealth V.

The supermartingale property of wealth V of hedging strategy Π implies that

$$V_t \geqq \operatorname{ess} \sup_{Q \in \mathbf{M}(X)} E_Q[V_\infty \,|\, \mathscr{F}_t] \geqq \operatorname{ess} \sup_{Q \in \mathbf{M}(X)} E_Q[f \,|\, \mathscr{F}_t], \quad t \geqq 0\,. \tag{3.2}$$

The following theorem states that the lower bound in (3.2) is achieved and is equal to the wealth of the minimal hedge. For continuous price processes this result was proved in the paper by El Karoui and Quenez [7].

Theorem 3.2 *Let f be a positive random variable such that $\sup_{Q \in \mathbf{M}(X)} E_Q f < \infty$. Then the minimal hedging strategy $\widehat{\Pi} = (\widehat{v}, \widehat{H}, \widehat{C})$ exists and its wealth $\widehat{V}$ equals*

$$\widehat{V}_t = \widehat{v} + (\widehat{H} \bullet X)_t - \widehat{C}_t = \operatorname{ess} \sup_{Q \in \mathbf{M}(X)} E_Q[f \,|\, \mathscr{F}_t]\,.$$

Proof. The proof follows from Theorem 3.1 above and from the fact that the process $(\operatorname{ess}\sup_{Q \in \mathbf{M}(X)} E_Q[f \,|\, \mathscr{F}_t])_{t \geqq 0}$ is a supermartingale for all $Q \in \mathbf{M}(X)$, see Proposition 4.2 in Sect. 4. □

If τ is a finite stopping time, i.e. $P(\tau < +\infty) = 1$, and f is a $\mathscr{F}_\tau$-measurable function, then from Theorem 3.2 we deduce that $\widehat{V}_\infty = f$. We notice that for a general $\mathscr{F} = \sigma(\bigcup_{t \geqq 0} \mathscr{F}_t)$-measurable claims f this equality does not hold. To demonstrate this, we use the following simple example.

Example 3.1. Let $(\Omega, \mathscr{F}, \mathbf{F}, P)$ be a filtered probability space with a Wiener process W. We set $X \equiv 1$ and define claim f as $f = I(\tau < \infty)$, where stopping time τ equals

$$\tau = \inf\{t \geqq 0 : W_t \geqq e^t\}\,.$$

In this case $\mathbf{M}(X)$ is the set of all probability measures equivalent to P. It can be easily seen that $P(\tau < \infty) < 1$, and

$$\widehat{V}_t = \operatorname{ess} \sup_{Q \in \mathbf{M}(X)} Q(\tau < \infty \mid \mathcal{F}_t) = 1, \quad t \geqq 0 .$$

Therefore, $\widehat{V}_\infty = 1 > I(\tau < +\infty)$ with positive probability.

The question of particular interest is to know whether the minimal hedging strategy $\widehat{\Pi}$ is a self-financing portfolio. This is connected with the problem of *attainability* of contingent claims, see papers by Jacka [10], Ansel and Stricker [1] and Delbaen and Schachermayer [5]. From these papers we deduce that the minimal hedge $\widehat{\Pi}$ with wealth $\widehat{V}$ is a self-financing strategy if and only if there is a measure $Q \in \mathbf{M}(X)$ such that $\widehat{V}$ is a Q-uniformly integrable martingale on $[0, +\infty[$.

3. Let now $f = (f_t)_{t \geqq 0}$ be an adapted positive process. We interpret f as the reward process of an American type option. The wealth and consumption portfolio $\Pi = (v, H, C)$ with capital process $V = (V_t)_{t \geqq 0}$ is called a *hedging* strategy for f if

$$V_t \geqq f_t, \quad t \geqq 0 .$$

The portfolio $\widetilde{\Pi} = (\widetilde{v}, \widetilde{H}, \widetilde{C})$ with capital process $\widetilde{V} = (\widetilde{V}_t)_{t \geqq 0}$ is termed the *minimal* hedging portfolio if

$$V_t \geqq \widetilde{V}_t \geqq f_t ,$$

for any $t \geqq 0$ and hedging portfolio Π with capital V.

The following theorem can be considered as generalization of the results obtained by Bensoussan [2] and Karatzas [13] in the setting of incomplete markets.

Denote by $\mathcal{M}_t$ the set of stopping times τ with values in $[t, +\infty)$.

Theorem 3.3 *Let $f = (f_t)_{t \geqq 0}$ be an adapted positive process such that*

$$\sup_{\tau \in \mathcal{M}_0} \sup_{Q \in \mathbf{M}(X)} E_Q f_\tau < +\infty .$$

Then the minimal hedging portfolio $\widetilde{\Pi} = (\widetilde{v}, \widetilde{H}, \widetilde{C})$ exists, and its capital at time $t \geqq 0$ equals

$$\widetilde{V}_t = \widetilde{v} + (\widetilde{H} \bullet X)_t - \widetilde{C}_t = \operatorname{ess} \sup_{Q \in \mathbf{M}(X),\, \tau \in \mathcal{M}_t} E_Q[f_\tau \mid \mathcal{F}_t] .$$

Proof. The proof follows from Theorem 3.1 and from the fact that the process $(\operatorname{ess}\sup_{Q \in \mathbf{M}(X),\, \tau \in \mathcal{M}_t} E_Q[f_\tau \mid \mathcal{F}_t])_{t \geqq 0}$ is a supermartingale for any $Q \in \mathbf{M}(X)$, see Proposition 4.3 in Sect. 4. □

4 Auxiliary facts and results

1. First we recall some facts and definitions from the theory of stochastic integration, for which we refer to [6, 12, 17].

Suppose X is a real-valued process; then the *maximal* function $(X)_t^*$ is defined as $\sup_{0 \leqq s \leqq t} |X_s|$.

Suppose X and Y are semi-martingales; then the *Émery distance* between X and Y equals

$$D(X,Y) = \sup_{|H| \leqq 1} \left(\sum_{n \geqq 1} 2^{-n} E[\min(|(H \bullet X)_n|, 1)] \right) ,$$

where sup is taken over the set of all predictable processes H bounded by 1. For this metric the space of semi-martingales is complete, see Emery [9]. The corresponding topology is called a *semi-martingale* or *Émery* topology.

In particular, if A and B are predictable processes of bounded variation, the Émery distance between A and B equals

$$D(A,B) = \sum_{n \geqq 1} 2^{-n} E \left[\min \left(\int_0^n |dA_s - dB_s|, 1 \right) \right] ,$$

where $\int_0^n |dA_s - dB_s|$ is the total variation of $A - B$ on $[0,t]$. This is a consequence of the following Hahn decomposition: there exists a predictable process h with values in $\{-1,+1\}$ such that

$$\int_0^t |dA_s - dB_s| = \int_0^t h_s(dA_s - dB_s), \quad t \geqq 0 ,$$

see Jacod and Shiryaev [12, Chap. I, Proposition 3.13].

Now let H be a predictable process, and X be a semi-martingale. The process H is called *X-integrable* if there exists a local martingale M and a process A of bounded variation such that $X = M + A$ and

1. the process $\int_0^t |H_s| |dA_s|$ has a bounded variation,
2. the increasing process $(\int H_s^2 \, d[M,M]_s)^{1/2}$ is locally integrable, where $[M,M]$ is the quadratic variation of the local martingale M.

In this case, $H \bullet A$ is a Lebesgue–Stieltjes integral; the stochastic integral $H \bullet M$ exists as a stochastic integral with respect to a local martingale, and is a local martingale. The stochastic integral $H \bullet X$ equals $H \bullet A + H \bullet M$ and does not depend on any particular choice of M and A.

If a predictable process H is locally bounded, this process is integrable with respect to all semi-martingales. If H is unbounded, then by Theorem 1 in [3] process H is X-integrable if and only if the sequence $HI\{|H| \leqq n\} \bullet X$, $n \geqq 1$, converges in semi-martingale topology. Moreover, in this case the limit of the sequence equals $H \bullet X$.

An X-integrable process H is called an *admissible* integrand if there exists a constant a such that $a + (H \bullet X)_t \geqq 0$, $t \geqq 0$. A counter-example in Émery [8] shows that a stochastic integral with respect to a local martingale can be not a local martingale. However, if M is a local martingale and H is an *admissible* integrand for M, then $H \bullet M$ is a local martingale, see [1].

A semi-martingale X is called a *special* semi-martingale if it can be decomposed as $X = M + A$, where M is a local martingale and A is a *predictable* process of bounded variation. Then such a decomposition is unique. For the sequel we need the following proposition on special semi-martingales. For the proof we refer to [3], where this result was called the Jeulin theorem.

Proposition 4.1 *Let X be a special semi-martingale with the canonical decomposition $X = M + A$, and H be a predictable X-integrable process. Then $H \bullet X$ is a special semi-martingale if and only if*

1. *H is M-integrable in the sense of stochastic integrals of local martingales,*
2. *H is A-integrable in the sense of Lebesgue–Stieltjes integrals.*

In this case, the canonical decomposition of $H \bullet X$ is given as $H \bullet X = H \bullet M + H \bullet A$.

We will also need a technical lemma whose formulation is taken from the paper by Mémin [16].

Lemma 4.1 *Let X be a semi-martingale, such that the quadratic variation $[X,X]_\infty^{1/2}$ belongs to $L^p(\Omega, \mathscr{F}, P)$ for $p \geqq 1$. Then X is a special semi-martingale, and there exists a universal constant a_p such that*

$$\| [A,A]_\infty^{1/2} \|_{L^p} \leqq a_p \| [X,X]_\infty^{1/2} \|_{L^p} ,$$

$$\| [M,M]_\infty^{1/2} \|_{L^p} \leqq (a_p + 1) \| [X,X]_\infty^{1/2} \|_{L^p} ,$$

where $X = M + A$ is a canonical decomposition of X; in particular, for $p = 2$ we can take $a_2 = 1$.

2. Let now X be a local martingale, and f be a positive function on $(\Omega, \mathscr{F}, P)$. We denote by $\mathbf{M}(X)$ the set of local martingale measures for X. The following proposition is adapted from the paper by El Karoui and Quenez [7].

Proposition 4.2 *Let f be a positive variable such that $\sup_{Q \in \mathbf{M}(X)} E_Q f < +\infty$. There is an RCLL process $V = (V_t)_{t \geqq 0}$ such that*

$$V_t = \operatorname{ess} \sup_{Q \in \mathbf{M}(X)} E_Q[f \mid \mathscr{F}_t], \quad t \geqq 0 .$$

The process V is a Q-supermartingale whatever $Q \in \mathbf{M}(X)$.

The Proposition 4.2 is a particular case of the following proposition used in the proof of Theorem 3.3.

Proposition 4.3 *Let $f = (f_t)_{t \geqq 0}$ be a positive adapted RCLL process such that*

$$\sup_{Q \in \mathbf{M}(X),\, \tau \in \mathscr{M}_0} E_Q f_\tau < +\infty$$

There is an RCLL process $V = (V_t)_{t \geqq 0}$ such that for all $t \geqq 0$

$$V_t = \operatorname{ess} \sup_{Q \in \mathbf{M}(X),\, \tau \in \mathscr{M}_t} E_Q[f_\tau \mid \mathscr{F}_t] .$$

The process $V = (V_t)_{t \geqq 0}$ is a Q-supermartingale whatever $Q \in \mathbf{M}(X)$.

Proof. For each time instant t we define the variable $\widetilde{V}_t$ as

$$\widetilde{V}_t = \operatorname{ess} \sup_{Q \in \mathbf{M}(X),\, \tau \in \mathcal{M}_t} E_Q[f_\tau \mid \mathcal{F}_t]\,. \tag{4.1}$$

We have to show that the process $\widetilde{V} = (\widetilde{V}_t)_{t \geqq 0}$ is a Q-supermartingale for all $Q \in \mathbf{M}(X)$ and that $\widetilde{V}$ admits an RCLL modification.

Let the probability measure P be an element of $\mathbf{M}(X)$. By $\mathcal{Z}_t$ we denote the set of processes $z = (z_t)_{t \geqq 0}$ such that

1. z is the density process of some measure $Q \in \mathbf{M}(X)$ with respect to P,
2. $z_s = 1,\ s \leqq t$.

Eq. (4.1) can be rewritten as

$$\widetilde{V}_t = \operatorname{ess} \sup_{z \in \mathcal{Z}_t,\, \tau \in \mathcal{M}_t} E[f_\tau z_\tau \mid \mathcal{F}_t]\,,$$

where E is the expectation with respect to the measure P.

Let us fix positive numbers s and t, $s < t$, and show that

$$E[\widetilde{V}_t \mid \mathcal{F}_s] = \operatorname{ess} \sup_{z \in \mathcal{Z}_t,\, \tau \in \mathcal{M}_t} E[f_\tau z_\tau \mid \mathcal{F}_s]\,. \tag{4.2}$$

First we have

(4.3)

$$E[\widetilde{V}_t \mid \mathcal{F}_s] = E\left[\operatorname{ess} \sup_{z \in \mathcal{Z}_t,\, \tau \in \mathcal{M}_t} E[f_\tau z_\tau \mid \mathcal{F}_t] \mid \mathcal{F}_s\right] \geqq \operatorname{ess} \sup_{z \in \mathcal{Z}_t,\, \tau \in \mathcal{M}_t} E[f_\tau z_\tau \mid \mathcal{F}_s]\,.$$

To prove the reverse inequality we take the sequence $(y^n, \sigma_n)_{n \geqq 1}$ in $(\mathcal{Z}_t, \mathcal{M}_t)$ such that

$$\widetilde{V}_t = \sup_{n \geqq 1} E[f_{\sigma_n} y^n_{\sigma_n} \mid \mathcal{F}_t]\,.$$

Using this sequence we construct a new sequence $(z^n, \tau_n)_{n \geqq 1}$ as follows

$$(z^1, \tau_1) = (y^1, \sigma_1)$$

and for $n \geqq 1$

$$(z^{n+1}, \tau_{n+1}) = \begin{cases} (z^n, \tau_n), & \text{if } E[f_{\tau_n} z^n_{\tau_n} \mid \mathcal{F}_t] \geqq E[f_{\sigma_{n+1}} y^{n+1}_{\sigma_{n+1}} \mid \mathcal{F}_t]\,, \\ (y^{n+1}, \sigma_{n+1}) & \text{if } E[f_{\tau_n} z^n_{\tau_n} \mid \mathcal{F}_t] < E[f_{\sigma_{n+1}} y^{n+1}_{\sigma_{n+1}} \mid \mathcal{F}_t]\,. \end{cases}$$

We have $(z^n, \tau_n)_{n \geqq 1} \subseteq (\mathcal{Z}_t, \mathcal{M}_t)$ and

$$E[f_{\tau_n} z^n_{\tau_n} \mid \mathcal{F}_t] = \max_{k \leqq n} E[f_{\sigma_k} y^k_{\sigma_k} \mid \mathcal{F}_t] \uparrow \widetilde{V}_t\,.$$

Now from the theorem on monotone convergence we deduce

$$E[\widetilde{V}_t \mid \mathcal{F}_s] = E\left[\lim_{n \to \infty} E[f_{\tau_n} z^n_{\tau_n} \mid \mathcal{F}_t] \mid \mathcal{F}_s\right] = \lim_{n \to \infty} E[f_{\tau_n} z^n_{\tau_n} \mid \mathcal{F}_s]$$

$$\leqq \operatorname{ess} \sup_{z \in \mathcal{Z}_t,\, \tau \in \mathcal{M}_t} E[f_\tau z_\tau \mid \mathcal{F}_s]\,.$$

Together with inequality (4.3) this proves Eq. (4.2).

Since $\mathscr{Z}_t \subseteq \mathscr{Z}_s$, $\mathscr{M}_t \subseteq \mathscr{M}_s$ for $s \leqq t$, the equality (4.2) implies the supermartingale property for the process $\widetilde{V}$:

$$E[\widetilde{V}_t \mid \mathscr{F}_s] \leqq \widetilde{V}_s, \quad s \leqq t.$$

To finish the proof of Proposition 4.3 we must show that the process $\widetilde{V}$ admits an RCLL modification. This is the case if and only if the function $(E\widetilde{V}_t)_{t \geqq 0}$ is right-continuous (see, for example, Theorem 3.1 in [15]).

When $s = 0$, the equality (4.2) takes the form

$$E\widetilde{V}_t = \sup_{z \in \mathscr{Z}_t,\, \tau \in \mathscr{M}_t} E[f_\tau z_\tau]. \tag{4.4}$$

Let $t, (t_n)_{n \geqq 1}$ be positive numbers such that $t_n \downarrow t$, $n \to +\infty$, and $t_n < t+1$, $n \geqq 1$. Because $\widetilde{V}$ is a supermartingale, we have

$$E\widetilde{V}_t \geqq \lim_{n \to \infty} E\widetilde{V}_{t_n}. \tag{4.5}$$

To prove the reverse inequality we fix $\varepsilon > 0$ and choose the stopping time $\sigma = \sigma(\varepsilon)$ from $\mathscr{M}_t$ and the process $z = z(\varepsilon)$ from $\mathscr{Z}_t$ such that

$$E\widetilde{V}_t < Ef_\sigma z_\sigma + \varepsilon \quad \text{and} \quad P(\sigma > t) = 1. \tag{4.6}$$

This is possible by Eq. (4.4) and the right-continuity of the process f.

Now for $n \geqq 1$ we define the stopping time $\sigma_n \in \mathscr{M}_{t_n}$ and the process $z^n \in \mathscr{Z}_{t_n}$ as

$$\sigma_n = \begin{cases} \sigma, & \sigma \geqq t_n \\ t+1, & \sigma < t_n \end{cases}, \qquad z_t^n = \begin{cases} z_t/z_{t_n}, & \sigma \geqq t_n \text{ and } t \geqq t_n, \\ 1, & \sigma < t_n \text{ or } t < t_n. \end{cases}$$

We have $\sigma_n \to \sigma$ and $z_{\sigma_n}^n \to z_\sigma$ almost surely as n tends to ∞. Now we deduce from Fatou's lemma and (4.4) and (4.6) that

$$E\widetilde{V}_t \leqq \lim_{n \to \infty} Ef_{\sigma_n} z_{\sigma_n}^n + \varepsilon \leqq \lim_{n \to \infty} E\widetilde{V}_{t_n} + \varepsilon.$$

Since ε is an arbitrary positive number and by Eq. (4.5) we deduce that the function $(E\widetilde{V}_t)_{t \geqq 0}$ is right-continuous. This completes the proof of Proposition 4.3. □

3. The next proposition is a slight modification of Theorem 5.7 in [4] and we only sketch the proof. We use the notation of Proposition 4.2.

Proposition 4.4 *Let τ and σ be stopping times on $(\Omega, \mathscr{F}, \mathbf{F}, P)$ such that $\tau \leqq \sigma$, and f be a bounded $\mathscr{F}_\sigma$-measurable random variable. Denote $V_\tau =$ ess $\sup_{Q \in \mathbf{M}(X)} E_Q[f \mid \mathscr{F}_\tau]$. There is an admissible integrand H such that*

$$(H \bullet X)_t = 0, \quad t \leqq \tau, \quad \textit{and} \quad V_\tau + (H \bullet X)_\sigma \geqq f.$$

Proof. For simplicity we consider the case $\tau = 0$ and $\sigma = \infty$.

Following Delbaen and Schachermayer [4] we define the sets:

$$K_0 = \{V_0 + (H \bullet X)_\infty : \ H \text{ is an admissible integrand}\} ,$$
$$C_0 = K_0 - L^0_+ ,$$
$$C = C_0 \bigcap L^\infty ,$$

where L^0_+ and L^∞ are the sets of positive and bounded random functions on $(\Omega, \mathscr{F}, P)$ respectively.

We need to prove that $f \in C$. We proceed on a well known path. By Theorem 4.2 in [4] (see also the remark after Corollary 1.2 in [4]) the set C is $\sigma(L^\infty, L^1)$-closed. Therefore, if $f \notin C$, by the Hahn–Banach theorem there is a signed measure $R \in L^1$ such that

$$\sup_{g \in C} E_R g < E_R f .$$

This inequality and the construction of the set C imply that R is a positive measure such that $E_R(H \bullet X)_\infty = 0$, if H and $-H$ are admissible processes. If we normalize the measure R as $R(\Omega) = 1$, we obtain that R is an absolutely continuous probability such that X is a local R-martingale. Therefore for each $\varepsilon > 0$ measure $R^\varepsilon = \varepsilon P + (1 - \varepsilon)R$ belongs to $\mathbf{M}(X)$. We easily deduce that

$$\sup_{g \in C} E_{R^\varepsilon} g = V_0 = \sup_{Q \in \mathbf{M}(X)} E_Q f \geqq E_{R^\varepsilon} f .$$

Therefore

$$E_R f = \lim_{\varepsilon \to 0} E_{R^\varepsilon} f \leqq V_0 \leqq \sup_{g \in C} E_R g$$

and we come to a contradiction. □

4. Finally we prove a technical lemma to be used later on several occasions. This lemma can be considered as an extension of Lemma A 1.1 from [4] in the setting of increasing processes.

Suppose $\mathscr{A}$ is a family of random processes; then the notation $B \in \operatorname{conv} \mathscr{A}$ means that the process B is a finite convex combination of elements in $\mathscr{A}$.

Lemma 4.2 *Let $(A^n)_{n \geqq 1}$ be a sequence of positive increasing adapted processes on a filtered probability space $(\Omega, \mathscr{F}, \mathbf{F}, P)$. There exists a sequence $B^n \in \operatorname{conv}(A^n, A^{n+1}, \ldots)$, $n \geqq 1$, and a $[0, +\infty]$-valued increasing process B such that*

$$B_t = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} B^n_{t+\varepsilon} = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} B^n_{t+\varepsilon} .$$

If there are numbers $T > 0$ and $\varepsilon > 0$ such that for all $n : P(A^n_T > \varepsilon) > \varepsilon$, then $P(B_T > 0) > 0$.

Proof. Let $(t_i)_{i \geqq 1}$ be a dense subset of $[0, +\infty)$. Application of Lemma A 1.1 from [4] and diagonalization procedure results in a sequence $B^n \in \operatorname{conv}(A^n, A^{n+1}, \ldots)$, $n \geqq 1$, such that for all $i \geqq 1$ the sequence $(B^n_{t_i})_{n \geqq 1}$ converges almost surely to a $[0, +\infty]$-valued variable B'_{t_i}.

We now define the process $B = (B_t)_{t \geqq 0}$ as

$$B_t = \inf_{t_i > t} B'_{t_i} .$$

It can be easily seen that the sequence $(B^n)_{n \geqq 1}$ and the process B are the processes required by the lemma.

Finally, if $P(A^n_T > \varepsilon) > \varepsilon$ then

$$E(B_T \wedge 1) \geqq \liminf_{n \to \infty} (EB^n_T \wedge 1) \geqq \liminf_{n \to \infty} (EA^n_T \wedge 1) > \varepsilon^2$$

and the result follows. □

5 Proof of the main theorem

We start with two auxiliary lemmas.

We are in the setting of Theorem 2.1. Let V be a positive supermartingale for all $Q \in \mathbf{M}(X)$. By $\mathscr{C}$ we denote the set of increasing processes C such that $C_0 = 0$ and the process $V + C$ is a supermartingale for all $Q \in \mathbf{M}(X)$. We introduce an order relation $\prec$ on $\mathscr{C}$ saying that C_1 is *less* than $C_2 (C_1 \prec C_2)$ if $C_2 - C_1$ is an increasing process.

Lemma 5.1 *There exists a maximal element $\widehat{C}$ on the ordered set $\mathscr{C}$.*

Proof. Kuratowski's lemma (see [14, Theorem 25, p. 33]) implies the existence of a maximal chain $\widetilde{\mathscr{C}} \in \mathscr{C}$. Denote

$$a = \sup_{C \in \widetilde{\mathscr{C}}} EC_\infty .$$

If $C \in \mathscr{C}$, the process $V + C$ is a P-supermartingale, and hence

$$EC_\infty \leqq E(V_\infty + C_\infty) \leqq V_0 .$$

It follows that $a < +\infty$.

Now we find an *ordered* sequence $(C^n)_{n \geqq 1}$ in $\widetilde{\mathscr{C}}$ such that expectations EC^n_∞ tend to a as n tends to ∞, and define the process $\widehat{C}$ as the limit: $C^n \uparrow \widehat{C}$, $n \to \infty$. Notice that the convergence here is uniform on $[0, +\infty]$.

It is easy to see that $\widehat{C} \in \mathscr{C}$. Moreover, since $\widetilde{\mathscr{C}}$ is the maximal chain in $\mathscr{C}$, the process $\widehat{C}$ is the maximal element of $\mathscr{C}$ if and only if it is the maximal element of $\widetilde{\mathscr{C}}$. Let $C \in \widetilde{\mathscr{C}}$. Because all elements of $\widetilde{\mathscr{C}}$ are comparable between each other, there are two possibilities:

1. $C \prec C^{n_0}$ for some n_0 and then $C \prec \widehat{C}$,
2. $C^n \prec C$ for all $n \geqq 1$ and then $\widehat{C} \prec C$. However the theorem on monotone convergence implies that

$$E\widehat{C}_\infty = \lim_{n \to \infty} EC^n_\infty = a \geqq EC_\infty .$$

Therefore $\widehat{C}_\infty = C_\infty$ and hence $\widehat{C} = C$.

The proof of Lemma 5.1 is finished. □

Let $\widehat{C}$ be the maximal element of $\mathscr{C}$ given by Lemma 5.1. Define the process $U = (U_t)_{t \geqq 0}$ by

$$U_t = V_t + \widehat{C}_t, \quad t \geqq 0 .$$

Let also T be a positive number, and $\mathscr{T}$ be a dense subset of $[0, T]$.

Lemma 5.2 *Let $(G^n)_{n \geqq 1}$ be a sequence of admissible integrands, and $(A^n)_{n \geqq 1}$ be a sequence of adapted increasing processes such that $A_0^n = 0$, $n \geqq 1$, and*

$$U_0 + (G^n \bullet X)_t \geqq -a, \quad t \in [0, T],\ n \geqq 1 \text{ for some } a \geqq 0$$

and

$$\limsup_{n \to \infty} |U_0 + (G^n \bullet X)_t - A_t^n - U_t| = 0, \quad t \in \mathscr{T} ,$$

where the process U and the set $\mathscr{T}$ are defined before the formulation of the lemma.

Then

(1) *the variables A_T^n tend to 0 in probability as n tends to ∞ and*

(2) *the maximal functions $(U_0 + G^n \bullet X - U)_T^*$ tend to 0 in probability as n tends to ∞.*

Proof. (1) Assume that there exists an increasing sequence $(n_k)_{k \geqq 1}$ and a positive number ε such that $P(A_T^{n_k} > \varepsilon) > \varepsilon$, $k \geqq 1$. Then Lemma 4.2 implies the existence of a sequence $B^k \in \operatorname{conv}(A^{n_k}, A^{n_{k+1}}, \ldots)$, $k \geqq 1$ and of an increasing adapted process B such that $P(B_T > 0) > 0$ and

$$B_t = \lim_{\delta \to 0} \limsup_{k \to \infty} B_{t+\delta}^k = \lim_{\delta \to 0} \liminf_{k \to \infty} B_{t+\delta}^k, \quad t \geqq 0 .$$

We come to a contradiction with the maximality of $\widehat{C}$, if we show that $B_0 = 0$ and the process $U + B$ is a supermartingale on interval $[0, T]$ for all $Q \in \mathbf{M}(X)$.

By H^k we denote the convex combination of $(G^{n_k}, G^{n_{k+1}}, \ldots)$ obtained with the same weights as $B^k \in \operatorname{conv}(A^{n_k}, A^{n_{k+1}}, \ldots)$. We have

$$\limsup_{k \to \infty} |U_0 + (H^k \bullet X)_t - B_t^k - U_t| = 0, \quad t \in \mathscr{T}$$

and

$$U_0 + (H^k \bullet X)_t \geqq -a, \quad t \in [0, T],\ k \geqq 1 .$$

Now we fix $t \geqq 0$ and define a sequence $(t_l)_{l \geqq 1}$ in $\mathscr{T}$ such that $t_l \downarrow t$, $l \to \infty$. We deduce

$$U_t + B_t = \lim_{l \to \infty} \liminf_{k \to \infty} (U_{t_l} + B_{t_l}^k) = U_0 + \lim_{l \to \infty} \liminf_{k \to \infty} (H^k \bullet X)_{t_l} .$$

If $t = 0$, Fatou's lemma and the supermartingale property of $H^k \bullet X$ imply that

$$B_0 = EB_0 = E \lim_{l \to \infty} \liminf_{k \to \infty} (H^k \bullet X)_{t_l}$$

$$\leqq \liminf_{l \to \infty} \liminf_{k \to \infty} E(H^k \bullet X)_{t_l} \leqq 0 .$$

Since $B \geqq 0$, it follows that $B_0 = 0$.

Further, for $t > 0$, $s \in \mathscr{T}$, $s < t$ and $Q \in \mathbf{M}(X)$ Fatou's lemma and the supermartingale property of $H^n \bullet X$ imply

$$E_Q[U_t + B_t \mid \mathscr{F}_s] \leqq U_0 + \liminf_{l \to \infty} \liminf_{k \to \infty} E_Q[(H^k \bullet X)_{t_l} | \mathscr{F}_s]$$

$$\leqq U_0 + \liminf_{k \to \infty} (H^k \bullet X)_s$$

$$= \liminf_{k \to \infty} (U_s + B_s^k) \leqq U_s + B_s .$$

Since the set $\mathscr{T}$ is dense in $[0, T]$, we deduce that the process $U + B$ is a supermartingale on $[0, T]$ for all $Q \in \mathbf{M}(X)$ and come to a contradiction. The first assertion of the lemma is proved.
(2) We must prove that the maximal functions $((G^n - G^m) \bullet X)_T^*$ tend to 0 in probability as n and m tend to ∞. If this were not the case, we could find two increasing sequences $(i_k, j_k)_{k \geqq 1}$ and a positive number ε such that

$$P\left(\sup_{0 \leqq t \leqq T} ((G^{i_k} - G^{j_k}) \bullet X)_t \geqq \varepsilon\right) > \varepsilon, \quad k \geqq 1 .$$

We define the stopping time τ_k as

$$\tau_k = \inf\{t \geqq 0 : \ ((G^{i_k} - G^{j_k}) \bullet X)_t \geqq \varepsilon\} .$$

We have

$$\left\{\omega : \ \sup_{t \leqq T} ((G^{i_k} - G^{j_k}) \bullet X)_t \geqq \varepsilon\right\} = \{\omega : \ \tau_k(\omega) \leqq T\} .$$

Therefore

$$P(\tau_k \leqq T) \geqq \varepsilon, \quad k \geqq 1 . \tag{5.1}$$

Now for $k \geqq 1$ we define the integrand L^k and the increasing process C^k as

$$L^k = G^{i_k} I\{t \leqq \tau_k\} + G^{j_k} I\{t > \tau_k\} ,$$

$$C^k = ((G^{i_k} - G^{j_k}) \bullet X)_{\tau_k} I_{[\tau_k, +\infty)} .$$

We have

$$(L^k \bullet X)_t - C_t^k = (G^{i_k} \bullet X)_t I_{[0, \tau_k)} + (G^{j_k} \bullet X)_t I_{[\tau_k, +\infty)} .$$

It follows that

$$U_0 + (L^k \bullet X)_t \geqq -a, \quad t \in [0, T], \ k \geqq 1 .$$

Part (1) of the lemma implies that the variables $A_T^{i_k}$ and $A_T^{j_k}$ tend to 0 in probability as k tends to ∞. Passing to a subsequence we can assume that this convergence holds almost surely. Then

$$\limsup_{k \to \infty} (|U_0 + (G^{i_k} \bullet X)_t - U_t| + |U_0 + (G^{j_k} \bullet X)_t - U_t|) = 0, \quad t \in \mathscr{T}$$

and therefore

$$\limsup_{k\to\infty} |U_0 + (L^k \bullet X)_t - C_t^k - U_t| = 0, \quad t \in \mathscr{T} .$$

Now we deduce from part (1) of the lemma that the variables C_T^k tend to 0 in probability as k tends to ∞. But

$$C_T^k \geqq \varepsilon I_{\{\tau_k \leqq T\}}$$

and we come to a contradiction with (5.1). □

Proof of Theorem 2.1. Let $\widehat{C}$ be a maximal element of $\mathscr{C}$ given by Lemma 5.1. By $U = (U_t)_{t\geqq 0}$ we define the process

$$U_t = V_t + \widehat{C}_t, \quad t \geqq 0 .$$

To prove Theorem 2.1 we have to construct an admissible integrand L such that

$$U_t = U_0 + (L \bullet X)_t, \quad t \geqq 0 .$$

Notice that it is sufficient to prove this representation only for any finite time interval $[0, T]$. In the sequel we consider the case where t belongs to $[0, 1]$. To make the proof more readable we divide it into a number of steps.

Step 1 *There are admissible integrands* $(K^n)_{n\geqq 1}$ *such that*

$$\limsup_{n\to\infty} (U_0 + (K^n \bullet X)_s - U_s)_1^* = 0 , \tag{5.2}$$

$$U_0 + (K^n \bullet X)_t \geqq 0, \quad t \in [0, 1], \ n \geqq 1 \tag{5.3}$$

$$\sup_{n\geqq 1} [K^n \bullet X, K^n \bullet X]_1 < +\infty , \tag{5.4}$$

where all relations hold almost surely.

Proof. First we construct a sequence of admissible integrands $(G^n)_{n\geqq 1}$ such that

$$\limsup_{n\to\infty} (U_0 + (G^n \bullet X)_s - U_s)_2^* = 0 , \tag{5.5}$$

$$U_0 + (G^n \bullet X)_t \geqq 0, \quad t \in [0, 2], \ n \geqq 1 . \tag{5.6}$$

The desired sequence $(K^n)_{n\geqq 1}$ will be obtained later as a sequence of appropriate convex combinations of $(G^n)_{n\geqq 1}$.

By $\mathscr{T}(n)$ we denote the set of numbers of the form $i \cdot 2^{-n}, 0 \leqq i \leqq 2^{n+1}$. We have that $\mathscr{T}(n) \subseteq \mathscr{T}(n+1)$ and that the limiting set $\mathscr{T}(\infty) = \bigcup_{n\geqq 1} \mathscr{T}(n)$ is dense in $[0, 2]$. For $n \geqq 1$ we define the process $U^n = (U_t^n)_{t\geqq 0}$ as $U_t^n = \min(U_t, n)$. It is clear that U^n is a Q-supermartingale for all $Q \in \mathbf{M}(X)$.

For $n \geqq 1$ and $0 \leqq i \leqq 2^{n+1} - 1$ Proposition 4.4 implies the existence of an admissible integrand G^{ni} such that

$$(G^{ni} \bullet X)_t = 0, \quad t \leqq i \cdot 2^{-n}$$

and

$$U^n_{i \cdot 2^{-n}} + (G^{ni} \bullet X)_{(i+1) \cdot 2^{-n}} \geqq U^n_{(i+1) \cdot 2^{-n}} \, .$$

Now we define the integrand G^n and the increasing process A^n as

$$G^n = \sum_{i=0}^{2^{n+1}-1} G^{ni} I\{i \cdot 2^{-n} < t \leqq (i+1) \cdot 2^{-n}\} \, ,$$

$$A^n = \sum_{i=0}^{2^{n+1}-1} (U^n_{i \cdot 2^{-n}} + (G^{ni} \bullet X)_{(i+1) \cdot 2^{-n}} - U^n_{(i+1) \cdot 2^{-n}}) I_{[(i+1) \cdot 2^{-n}, +\infty[} \, .$$

We have

(5.7) $$U^n_0 + (G^n \bullet X)_{i \cdot 2^{-n}} - A^n_{i \cdot 2^{-n}} = U^n_{i \cdot 2^{-n}}, \quad 0 \leqq i \leqq 2^{n+1} \, .$$

The process G^n as a finite sum of admissible integrands is an admissible integrand. Therefore $G^n \bullet X$ is a supermartingale. Since

$$U^n_0 + (G^n \bullet X)_2 = U^n_2 + A^n_2 \geqq 0 \, ,$$

we deduce that $U^n_0 + (G^n \bullet X)_t \geqq 0$, $t \in [0,2]$. This proves inequality (5.6).

Further we deduce from (5.7) that

$$\limsup_{n \to \infty} \sup_{t \in \mathscr{T}(n)} |U_0 + (G^n \bullet X)_t - A^n_t - U_t| = 0 \, ,$$

because maximal functions $(U^n - U)^*_2$ tend to 0 almost surely as n tends to ∞. Now part (2) of Lemma 5.2 implies the convergence of the maximal functions $(U_0 + (G^n \bullet X)_s - U_s)^*_2$ to 0 in probability. Passing, if necessary, to a subsequence, we can suppose that this convergence holds almost surely. It follows that Eq. (5.5) for the sequence $(G^n)_{n \geqq 1}$ takes place.

Since $(U)^*_2 < +\infty$, we deduce from Eq. (5.5) that

$$\sup_{n \geqq 1} (G^n \bullet X)^*_2 < +\infty \, .$$

Therefore, the probability of the stopping time

$$\sigma_m = \inf_{n \geqq 1} \inf\{t \geqq 0\colon \, |(G^n \bullet X)_t| \geqq m\}$$

being below 2 tends to 0 as m tends to ∞. Accounting for the supermartingale property of $G^n \bullet X$ we obtain that

$$E[U_0 + (G^n \bullet X)^*_{\sigma_m}] \leqq m + E[U_0 + (G^n \bullet X)_{\sigma_m}] \leqq m + U_0 \, .$$

Now the Davis inequality implies the existence of a constant $D_m < +\infty$ such that

$$E[G^n \bullet X, G^n \bullet X]^{1/2}_{\sigma_m} \leqq D_m \, .$$

According to Lemma 4.2 there is a sequence of increasing processes $C^n \in \operatorname{conv}([G^n \bullet X, G^n \bullet X]^{1/2}, [G^{n+1} \bullet X, G^{n+1} \bullet X]^{1/2}, \ldots)$, $n \geqq 1$, and an increasing process C such that

$$C_t = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} C^n_{t+\varepsilon} = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} C^n_{t+\varepsilon}, \quad t \geqq 0 \, .$$

From Fatou's lemma we deduce that

$$EC_1 I(\sigma_m \geqq 2) \leqq \liminf_{n\to\infty} EC^n_{\sigma_m} I(\sigma_m \geqq 2) \leqq D_m \, .$$

Since $\lim_{m\to\infty} P(\sigma_m \geqq 2) = 1$, we have that $C_1 < +\infty$ almost surely and therefore $\sup_{n\geqq 1} C^n_1 < +\infty$.

Now we define the desired sequence $(K^n)_{n\geqq 1}$ as the sequence of convex combinations of $(G^n)_{n\geqq 1}$ obtained in much the same way as the sequence $(C^n)_{n\geqq 1}$ was obtained from $([G^n \bullet X, G^n \bullet X]^{1/2})_{n\geqq 1}$. Now the proof of Step 1 follows from (5.5), (5.6), and the "Minkowski" inequality: $[K^n \bullet X, K^n \bullet X]^{1/2}_1 \leqq C^n_1$ (see [6, Chap. VII, Eq. (54.1)]). The proof of Step 1 is finished. □

If the sequence $([K^n \bullet X, K^n \bullet X])_{n\geqq 1}$ is bounded not only in probability but in L^1_Q-norm for some measure $Q \in \mathbf{M}(X)$:

$$\sup_{n\geqq 1} E_Q[K^n \bullet X, K^n \bullet X]_1 < +\infty \, ,$$

then by standard arguments we can find a sequence $L^n \in \operatorname{conv}(K^n, K^{n+1}, \ldots)$, $n \geqq 1$, such that martingales $(L^n \bullet X)_{n\geqq 1}$ converge in the space $\mathscr{M}^2(Q, [0,1])$ of square integrable martingales. The limiting martingale has the form $L \bullet X$ for some integrand L (see, for example, [11, Chap. 4]), and we finish.

Under the weaker condition (5.4), the construction of the desirable integrand L is much more complex. We proceed in a similar way to that in [4].

Denote $\rho = \sup_{n\geqq 1}[K^n \bullet X, K^n \bullet X]_1$, and define the probability R on $(\Omega, \mathscr{F})$ such that

$$\frac{dR}{dP} = \frac{e^{-\rho}}{Ee^{-\rho}} \, .$$

Then the inequality (5.4) in Step 1 implies that $R \sim P$.

The process X is a locally bounded P-martingale. Therefore, it is a special semi-martingale with respect to measure R. Let $X = M + A$ be the canonical decomposition of X, where M is an R-local martingale and A is a process of R-integrable variation. Since the definition of the stochastic integral is invariant with respect to equivalent changes of measure, the stochastic integral $K^n \bullet X$ exists on $(\Omega, \mathscr{F}, \mathbf{F}, R)$ and is a semi-martingale. Since

$$E_R \sup_{n\geqq 1} [K^n \bullet X, K^n \bullet X]_1 < +\infty \, , \tag{5.8}$$

Lemma 4.1 and Proposition 4.1 imply that $K^n \bullet X$ is a special semi-martingale on $[0,1]$ with the canonical decomposition

$$(K^n \bullet X)_t = (K^n \bullet M)_t + (K^n \bullet A)_t, \quad 0 \leqq t \leqq 1 \, .$$

Step 2 *There is a sequence* $L^n \in \operatorname{conv}(K^n, K^{n+1}, \ldots)$, $n \geqq 1$, *such that the sequence* $(L^n \bullet M)_{n\geqq 1}$ *converges in semi-martingale topology on* $[0,1]$.

Proof. Lemma 4.1 and Eq. (5.8) imply that $\sup_{n\geqq 1} E_R\, [K^n \bullet M, K^n \bullet M]_1 < +\infty$. Therefore the sequence $(K^n \bullet M)_{n\geqq 1}$ is bounded in the space $\mathscr{M}^2(R, [0,1])$

of square integrable martingales. Hence there is a sequence $L^n \in \mathrm{conv}(K^n, K^{n+1}, \ldots)$, $n \geqq 1$, such that the R-martingales $(L^n \bullet M)_{n\geqq 1}$ converge in $\mathscr{M}^2(R,[0,1])$ and therefore converge also in semi-martingale topology on $[0,1]$. □

Step 3 *The sequence* $(L^n \bullet A)_{n\geqq 1}$, *where the integrands* $(L^n)_{n\geqq 1}$ *are given in Step* 2, *converges in semi-martingale topology on* $[0,1]$.

Proof. The proof of Step 3 follows the same lines as the proof of Lemma 4.11 in [4]. We have to show that the variances $\int_0^1 |L_t^n - L_t^m||dA_t|$ tend to 0 as n and m tend to ∞. If this were not the case, we could find two increasing sequences $(i_n, j_n)_{n\geqq 1}$ and a number $\varepsilon > 0$ such that $P(A_1^n > \varepsilon) > \varepsilon$, $n \geqq 1$, where $A_t^n = \frac{1}{2}\int_0^t |L_s^{i_n} - L_s^{j_n}||dA_s|$.

Hahn's decomposition for increasing predictable processes implies the existence of predictable processes h^n with values in $\{-1,1\}$ such that

$$A_t^n = \frac{1}{2}\int_0^t h^n(L^{i_n} - L^{j_n})\,dA\,.$$

We define the integrand G^n as

$$G^n = \tfrac{1}{2}(L^{i_n} + L^{j_n} + h^n(L^{i_n} - L^{j_n}))\,.$$

We have

$$G^n \bullet X = \tfrac{1}{2}(L^{i_n} + L^{j_n}) \bullet X + A^n\,.$$

Since L^n is a convex combination of integrands $(K^n, K^{n+1}, \ldots)$ given in Step 1, we deduce that

$$\limsup_{n\to\infty}(U_0 + G^n \bullet X - A^n - U)_1^* = 0\,. \tag{5.9}$$

By construction of h^n and since

$$(G^n - L^{i_n}) \bullet A = \tfrac{1}{2}(h^n - 1)(L^{i_n} - L^{j_n}) \bullet A\,,$$

$$(G^n - L^{j_n}) \bullet A = \tfrac{1}{2}(h^n + 1)(L^{i_n} - L^{j_n}) \bullet A\,,$$

we deduce that the processes $(G^n - L^{i_n}) \bullet A$ and $(G^n - L^{j_n}) \bullet A$ are increasing processes. Moreover, since

$$(G^n - L^{i_n}) \bullet M = \tfrac{1}{2}(h^n - 1)(L^{i_n} - L^{j_n}) \bullet M$$

we deduce that the maximal functions $((G^n - L^{i_n}) \bullet M)_1^*$ tend to 0 in probability, because the processes $(L^{i_n} - L^{j_n}) \bullet M$ tend to 0 in semi-martingale topology on $[0,1]$. The same holds for $((G^n - L^{j_n}) \bullet M)_1^*$. Taking, if necessary, a subsequence, we can assume that these maximal functions converge almost surely. Then the stopping times τ_n defined as

$$\tau_n = \inf_{m\geqq n}\inf\{t \geqq 0\colon\ (G^m \bullet M)_t < \max((L^{i_m} \bullet M)_t, (L^{j_m} \bullet M)_t) - 1\}$$

form an increasing sequence such that

$$\lim_{n\to\infty} I(\tau_n < 1) = 0\,. \tag{5.10}$$

For $t < \tau_n$ and $m \geqq n$ we have

$$\begin{aligned}(G^m \bullet X)_t &= (G^m \bullet A)_t + (G^m \bullet M)_t \\ &\geqq \max((L^{im} \bullet A)_t, (L^{jm} \bullet A)_t) + \max((L^{im} \bullet M)_t, (L^{jm} \bullet M)_t) - 1 \\ &\geqq \max((L^{im} \bullet X)_t, (L^{jm} \bullet X)_t) - 1 .\end{aligned}$$

At time τ_n the jump $\Delta(G^m \bullet X)$ is either $\Delta(L^{im} \bullet X)$ or $\Delta(L^{jm} \bullet X)$, and hence the inequality

$$(G^m \bullet X)_t \geqq \max((L^{im} \bullet X)_t, (L^{jm} \bullet X)_t) - 1$$

holds for $t \leqq \tau_n$ and $m \geqq n$. Because L^n is a convex combination of the variables $(K^n, K^{n+1}, \ldots)$ given in Step 1, we deduce that

$$U_0 + (G^m \bullet X)_t \geqq -1, \quad 0 \leqq t \leqq \tau_n, \ m \geqq n .$$

Now we define the integrand H^n as $H^n = G^n I_{[0,\tau_n]}$. We obtain that H^n is an admissible integrand on $[0,1]$:

$$U_0 + (H^n \bullet X)_t \geqq -1, \quad 0 \leqq t \leqq 1 .$$

Moreover, Eqs. (5.9) and (5.10) imply that

$$\limsup_{n \to \infty} (U_0 + (H^n \bullet X)_s - A_s^n - U_s)_1^* = 0 . \tag{5.11}$$

Now Lemma 5.2 implies that the variables A_1^n tend to 0 in probability as n tends to ∞, and we come to a contradiction. The proof of Step 3 is finished. □

Now we are able to finish the proof of Theorem 2.1. Steps 2 and 3 imply the existence of a sequence of admissible integrands $(L^n)_{n \geqq 1}$ such that the sequences $(L^n \bullet M)_{n \geqq 1}$ and $(L^n \bullet A)_{n \geqq 1}$ converge in semi-martingale topology on $[0,1]$. Therefore, the sequence of stochastic integrals $(L^n \bullet X)_{n \geqq 1}$ also converges in semi-martingale topology on $[0,1]$. Now Mémin's theorem (see [16]) implies the existence of a predictable process L such that integrals $L^n \bullet X$ converge to $L \bullet X$ in semi-martingale topology on $[0,1]$. In particular, maximal functions $(L^n \bullet X - L \bullet X)_1^*$ tend to 0 in probability as n tends to ∞. Since L^n is a convex combination of the variables $(K^m)_{m \geqq n}$ given in Step 1, we deduce that

$$U_t = U_0 + (L \bullet X)_t, \quad 0 \leqq t \leqq 1$$

and finish the proof of Theorem 2.1. □

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