

## 2. CHANGES OF MEASURE AND GIRSANOV'S THEOREM

We have seen that semimartingales are invariant under various transformations. For example, Itô's formula shows that semimartingales are invariant under composition with  $\mathcal{C}^2$  functions or, more generally, compositions with differences of convex functions. Semimartingales are also invariant under time-changes (although we established this only for continuous semimartingales and suitable time-changes). It turns out that semimartingales are also invariant under absolutely continuous changes of measure. The focus of this section is to establish a version of the latter result for continuous semimartingales. In particular, this often allows one to view a process that is a semimartingale (for example, Brownian motion with drift) under some measure  $\mathbb{P}$  as a Brownian motion under a different measure  $\tilde{\mathbb{P}}$  that is absolutely continuous with respect to  $\mathbb{P}$  on some  $\sigma$ -algebra. One can then often deduce properties of Brownian motion with drift from the corresponding properties for Brownian motion, which are easier to establish due to the special properties and symmetry of Brownian motion.

We first establish some preliminary results concerning exponential martingales and likelihood ratios (or Radon-Nikodym derivatives) in Section 2.1, before stating the main results in Section 2.2. Conditions that allow the verification of the assumptions of the main theorems are presented in Section 2.3 and, finally, some applications of Girsanov's theorem are illustrated in Section 2.4.

### 2.1. Preliminary Results. A. Exponential semimartingales

**Theorem 2.1.** (*Doléans*) *Let  $X$  be a continuous semimartingale with  $X_0 = 0$ , and suppose that  $Z_0$  is some  $\mathcal{F}_0$ -measurable random variable. Then for  $\lambda \in \mathbb{R}$ , there exists a unique (continuous) semimartingale  $Z$  such that*

$$(2.6) \quad Z_t = Z_0 + \lambda \int_{(0,t]} Z_s dX_s.$$

*The unique solution is given explicitly by*

$$(2.7) \quad Z_t = Z_0 \mathcal{E}^\lambda(X)_t,$$

*where*

$$(2.8) \quad \mathcal{E}^\lambda(X)_t \doteq \exp \left( \lambda X_t - \frac{\lambda^2}{2} \langle X \rangle_t \right).$$

*Proof.* Note that  $\mathcal{E}^\lambda(X)_t = f(X_t, \langle X \rangle_t)$ , where  $f(x, y) = e^{\lambda x - \frac{\lambda^2}{2} y}$ . Therefore, applying Itô's formula to  $f(X, \langle X \rangle)$ , and using the fact that

$$\frac{\partial f}{\partial x} = \lambda f, \quad \frac{\partial f}{\partial y} = \frac{\lambda^2}{2} f, \quad \frac{\partial^2 f}{\partial x^2} = \lambda^2 f,$$

we find that

$$\begin{aligned} d(\mathcal{E}^\lambda(X)_t) &= \lambda \mathcal{E}^\lambda(X)_t dX_t - \frac{\lambda^2}{2} \mathcal{E}^\lambda(X)_t d\langle X \rangle_t + \frac{1}{2} (\lambda^2 \mathcal{E}^\lambda(X)_t) d\langle X \rangle_t \\ &= \lambda \mathcal{E}^\lambda(X)_t dX_t. \end{aligned}$$

This shows that  $Z = \mathcal{E}^\lambda(X)$  is a solution of (2.6) with  $Z_0 = 1$ . The fact that (2.7) implies (2.6) for general  $\mathcal{F}_0$ -measurable  $Z_0$  can be deduced in a similar fashion.

To prove uniqueness, define

$$Y_t = \exp\left(-\lambda X_t + \frac{1}{2} \lambda^2 \langle X \rangle_t\right) = \frac{1}{\mathcal{E}^\lambda(X)_t}$$

and note that  $Y_t = g(X_t, \langle X \rangle_t)$ , where  $g(x, y) = e^{-\lambda x + \frac{\lambda^2}{2} y}$ . Another application of Itô's formula to  $g(X, \langle X \rangle)$  shows that

$$\begin{aligned} (2.9) \quad dY_t &= -\lambda Y_t dX_t + \frac{\lambda^2}{2} Y_t d\langle X \rangle_t + \frac{1}{2} (\lambda^2 Y_t) d\langle X \rangle_t \\ &= -\lambda Y_t dX_t + \lambda^2 Y_t d\langle X \rangle_t. \end{aligned}$$

Let  $Z$  be any solution of (2.6). Then (2.6) and (2.9), together, show that

$$(2.10) \quad d\langle Z, Y \rangle_t = -\lambda^2 Y_t Z_t d\langle X \rangle_t.$$

Using the integration-by-parts formula and substituting from (2.6), (2.9) and (2.10), we obtain

$$\begin{aligned} d(Z_t Y_t) &= Z_t dY_t + Y_t dZ_t + d\langle Y, Z \rangle_t \\ &= Z_t (-\lambda Y_t dX_t + \lambda^2 Y_t d\langle X \rangle_t) \\ &\quad + Y_t (\lambda Z_t dX_t) - \lambda^2 Y_t Z_t d\langle X \rangle_t \\ &= 0. \end{aligned}$$

Thus  $Z_t Y_t$  is constant in time, equal to  $Z_0 Y_0 = Z_0$ . In turn, this implies  $Z_t = Z_0 / Y_t = Z_0 \mathcal{E}^\lambda(X)_t$ , which completes the proof.  $\square$

**Remark 2.2.**

1. We call (2.6) an exponential SDE and the solution (2.7) an exponential semimartingale.
2. If  $X = M$  is a continuous local martingale null at zero, then (2.6) shows that  $\mathcal{E}^\lambda(M)$  is a non-negative local martingale and hence (by Fatou's lemma) a supermartingale. In particular, this implies that

$$(2.11) \quad \mathbb{E} \left[ \exp \left( \lambda M_t - \frac{\lambda^2}{2} \langle M \rangle_t \right) \right] \leq 1 \quad \forall t.$$

*B. Likelihood ratios and Radon-Nikodym derivatives.*

In this section, we consider the following setup. Let  $(\Omega, \mathcal{F}^0, \{\mathcal{F}_t^0\})$  be a filtered probability space. Let  $\mathbb{P}$  and  $\tilde{\mathbb{P}}$  be two equivalent probability measures on  $(\Omega, \mathcal{F}^0)$ . Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$  be the augmentation of  $(\Omega, \mathcal{F}^0, \{\mathcal{F}_t^0\})$ . Since  $\mathbb{P}$  and  $\tilde{\mathbb{P}}$  are equivalent, the augmentation with respect to either measure leads to the same  $\{\mathcal{F}_t\}$ ,  $\mathcal{F}$  and, moreover,  $\mathbb{P}$  and  $\tilde{\mathbb{P}}$  are equivalent on  $\mathcal{F}$ .

**Theorem 2.3.** *The following properties hold.*

- (i) *There exists a uniformly integrable  $\mathbb{P}$ -martingale  $D$  such that  $D$  is càdlàg and for all  $t$ ,  $D_t$  is a version of the Radon-Nikodym derivative  $d\tilde{\mathbb{P}}/d\mathbb{P}$  on  $(\Omega, \mathcal{F}_t)$ .*
- (ii)  $\mathbb{P}(\forall t, D_t > 0, D_{t-} > 0) = 1$
- (iii)  *$M$  is a  $\tilde{\mathbb{P}}$  martingale if and only if  $DM$  is a  $\mathbb{P}$  martingale.*
- (iv)  *$M$  is a  $\tilde{\mathbb{P}}$  local martingale if and only if  $DM$  is a  $\mathbb{P}$  local martingale.*

*Proof.* We start with the proof of the first property.

(i) Let  $D_\infty = d\tilde{\mathbb{P}}/d\mathbb{P}$  on  $\mathcal{F}$ , and let  $D$  be a càdlàg modification of the uniformly integrable  $\mathbb{P}$ -martingale

$$D_t \doteq \mathbb{E}[D_\infty \mid \mathcal{F}_t].$$

Note that the existence of a càdlàg modification follows because the map  $t \mapsto \mathbb{E}[D_t] = \mathbb{E}[D_\infty]$  is continuous and  $\{\mathcal{F}_t\}$  satisfies the usual conditions (see, e.g., Theorem 3.13 of Chapter 1 of [1]). For  $A \in \mathcal{F}_t$ , since  $D$  is a  $\mathbb{P}$ -martingale, we have

$$\mathbb{E}[\mathbb{I}_A D_t] = \mathbb{E}[\mathbb{I}_A D_\infty] = \tilde{\mathbb{P}}(A),$$

so that  $D_t$  is a version of  $d\tilde{\mathbb{P}}/d\mathbb{P}$  on  $(\Omega, \mathcal{F}_t)$ .

(ii) Let

$$\tau \doteq \inf \{t \geq 0 : D_t = 0 \text{ or } D_{t-} = 0\}.$$

Then  $\tau$  is an  $\mathcal{F}_t$ -stopping time because

$$\{\tau \leq t\} = \{D_t = 0\} \cup \left[ \bigcap_{n \in \mathbb{N}} \bigcup_{q \in \mathbb{Q} \cap [0, t]} \left\{ D_q \leq \frac{1}{n} \right\} \right] \in \mathcal{F}_t.$$

Now, for  $n \in \mathbb{N}$ , define  $\theta_n \doteq \inf \{t \geq 0 : D_t < 1/n\}$ . Then  $\theta_n$  is an  $\mathcal{F}_t$ -stopping time and for any rational number  $q > 0$ , clearly  $\theta_n \leq \tau \leq \tau + q$ . Thus, by the optional sampling theorem (which can be applied because  $D$  is a non-negative martingale), for every  $n \in \mathbb{N}$ ,

$$\mathbb{E}[D_{\tau+q}] = \mathbb{E}[D_{\theta_n}] \leq \frac{1}{n}.$$

Sending  $n \rightarrow \infty$ , we conclude that  $\mathbb{P}(D_{\tau+q} = 0) = 1$  and therefore that  $\mathbb{P}(D_{\tau+q} = 0 \forall q \in \mathbb{Q}, q > 0) = 1$ , which in turn implies  $\mathbb{P}(D_t = 0 \text{ for all } t \geq \tau) = 1$  due to the right-continuity of  $D$ . However, for each fixed  $t$ , since  $\tilde{\mathbb{P}}(D_t = 0) = 0$  and  $\mathbb{P}$  is absolutely continuous relative to  $\tilde{\mathbb{P}}$ , we have  $\mathbb{P}(D_t = 0) = 0$ . Hence  $\mathbb{P}(\tau = \infty) = 1$ .

(iii) Let  $M$  be a  $\tilde{\mathbb{P}}$ -martingale. Then for  $s \leq t$  and  $A \in \mathcal{F}_s$ , we have

$$\tilde{\mathbb{E}}[M_t \mathbb{I}_A] = \tilde{\mathbb{E}}[M_s \mathbb{I}_A],$$

which implies  $\mathbb{E}[D_t M_t \mathbb{I}_A] = \mathbb{E}[D_s M_s \mathbb{I}_A]$ . Thus,  $DM$  is a  $\mathbb{P}$ -martingale, and the “only if” part is proved. The “if” part is proved in a similar fashion with  $(\mathbb{P}, \tilde{\mathbb{P}}, D^{-1})$  replacing  $(\tilde{\mathbb{P}}, \mathbb{P}, D)$ .

(iv) Let  $M$  be a  $\tilde{\mathbb{P}}$ -local martingale. By replacing  $M$  by  $\{M_t - M_0 : t \geq 0\}$ , assume that  $M_0 = 0$ . Let  $\{T_n\}$  be a localizing sequence for  $M$  under  $\tilde{\mathbb{P}}, D$  so that  $M^{T_n}$  is a  $\tilde{\mathbb{P}}$ -martingale. Since  $\mathbb{P} \ll \tilde{\mathbb{P}}$ , under  $\mathbb{P}$ ,  $\{T_n\}$  is again a sequence of stopping times with  $T_n \uparrow \infty$ . By part (iii),  $DM^{T_n}$  is a  $\mathbb{P}$ -martingale, so that

$$(DM)^{T_n} = (DM^{T_n})^{T_n}$$

is a  $\mathbb{P}$ -martingale, and the result follows.  $\square$

## 2.2. Girsanov-Cameron-Martin Theorem.

### A. Continuous Semimartingales under Changes of Measure.

Suppose we are in the setting of Section 2.1B. By assertions (i) and (ii) of Theorem 2.3, we know that

$$(2.12) \quad Z_t = \left. \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \right|_{\mathcal{F}_t}, \quad t \in [0, \infty),$$

is a strictly positive uniformly integrable martingale. Suppose, in addition, that  $Z$  is continuous. Then the process  $Z^{-1}$  is well-defined, locally bounded and continuous (and hence progressively measurable). Hence, the stochastic integral

$$(2.13) \quad X_t \doteq \int_0^t Z_s^{-1} dZ_s$$

is well-defined. Then

$$Z_t = Z_0 + \int_0^t Z_s dX_s.$$

In other words,  $Z$  is the solution  $Z = Z_0 \mathcal{E}(X)$  of the exponential SDE driven by the continuous local martingale  $X$ .

**Theorem 2.4.** *Suppose the process  $Z$  defined in (2.12) is continuous and let  $X$  be defined as in (2.13). Given any continuous  $\mathbb{P}$  local martingale  $M$ , the process*

$$(2.14) \quad N_t \doteq M_t - \langle M, X \rangle_t = M_t - \int_0^t Z_s^{-1} d\langle M, Z \rangle_s$$

*defines a continuous  $\tilde{\mathbb{P}}$ -local martingale and, moreover,*

$$\langle N \rangle = \langle M \rangle.$$

*Proof.* Due to Theorem 2.3(iv), in order to show that  $N$  is a  $\tilde{\mathbb{P}}$  local martingale we need only verify that  $NZ$  is a  $\mathbb{P}$ -local martingale. By the integration-by-parts formula,

$$d(M_t Z_t) = M_t dZ_t + Z_t dM_t + d\langle M, Z \rangle_t.$$

Also,

$$\begin{aligned} d(\langle M, X \rangle_t Z_t) &= \langle M, X \rangle_t dZ_t + Z_t d\langle M, X \rangle_t \\ &= \langle M, X \rangle_t dZ_t + Z_t (Z_t^{-1} d\langle M, Z \rangle_t,) \end{aligned}$$

where the second equality follows by substituting  $dX = Z^{-1}dZ$  from (2.13). Note that there is no Itô correction term here because  $\langle M, X \rangle$  is a finite variation process. Subtracting the second equation from the first and using the fact that  $N = M - \langle M, X \rangle$ , we obtain

$$\begin{aligned} d(N_t Z_t) &= M_t dZ_t - \langle M, X \rangle_t dZ_t + Z_t dM_t \\ &= N_t dZ_t + Z_t dM_t, \end{aligned}$$

so that

$$N_t Z_t = N_0 Z_0 + \int_0^t N_s dZ_s + \int_0^t Z_s dM_s.$$

Each of the terms on the right-hand-side is a stochastic integral with respect to a  $\mathbb{P}$ -local martingale, which implies  $NZ$  is itself a  $\mathbb{P}$ -local martingale. The fact that  $\langle N \rangle = \langle M \rangle$  follows from (2.14) and the fact that  $\langle M, X \rangle$  is a finite variation process.  $\square$

**Remark 2.5.** Note that Theorem 2.4 shows that  $M$  remains a semimartingale under  $\tilde{\mathbb{P}}$ , and explicitly identifies its decomposition. Thus Theorem 2.4 shows that continuous semimartingales remain continuous semimartingales under an absolutely continuous change of measure in which the likelihood process  $Z$  is continuous. In fact, the more general result that a semimartingale (not necessarily continuous) remains a semimartingale under an absolutely continuous change of measure is true, but the proof of this fact lies beyond our current scope.

#### B. Specializing to Brownian Filtrations.

We now consider a specific setting of the above result, which will be important in applications. Suppose

$$(2.15) \quad \Omega \doteq \mathcal{C}([0, \infty) : \mathbb{R}^n) \quad W_t(\omega) = \omega(t) \quad \mathcal{F}_t^0 = \sigma(B_{s, s \leq t}),$$

let  $\mathbb{P}$  be Wiener measure on  $\Omega$ , so that  $W$  is a Brownian motion under  $\mathbb{P}$  and let  $\mathcal{F}_t$  be the  $\mathbb{P}$ -augmentation of  $\mathcal{F}_t^0$ .

**Theorem 2.6. (Cameron-Martin, Girsanov)** *The following properties hold:*

- (i) Let  $\tilde{\mathbb{P}}$  be a law on  $(\Omega, \mathcal{F})$  which is equivalent to  $\mathbb{P}$ . Then there exists a predictable  $\mathbb{R}^n$ -valued process  $\{H_t\}$  such that

$$(2.16) \quad Z_t \doteq \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \exp \left( \int_0^t H_s dW_s - \frac{1}{2} \int_0^t |H_s|^2 ds \right)$$

and, under  $\tilde{\mathbb{P}}$ ,

$$\tilde{W}_t \doteq W_t - \int_0^t H_s ds$$

is a Brownian motion.

- (ii) Let  $\gamma$  be a predictable  $\mathbb{R}^n$ -valued process such that

$$\zeta_t \doteq \exp \left( \int_0^t \gamma_s dW_s - \frac{1}{2} \int_0^t |\gamma_s|^2 ds \right)$$

defines a uniformly integrable  $\mathbb{P}$ -martingale. Define a measure  $\tilde{\mathbb{P}}$  on  $(\Omega, \mathcal{F})$  by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \zeta_\infty.$$

Then, under  $\tilde{\mathbb{P}}$ ,

$$\tilde{W}_t \doteq W_t - \int_0^t \gamma_s ds,$$

$t \in [0, \infty)$ , is a Brownian motion.

*Proof.* (i) The process  $Z$  is a  $\mathbb{P}$ -martingale adapted to the Brownian filtration  $\mathcal{F}_t$  and is thus continuous by the martingale representation theorem (see, e.g., Theorem 4.15 of Chapter 3 of [1]). Moreover, by the zero-one law,  $Z_0 = 1$ . Another application of the martingale representation theorem shows that the continuous  $\mathbb{P}$  local martingale

$$X_t = \int_0^t Z_s^{-1} dZ_s$$

admits an integral representation

$$X_t = \int_0^t H_s dW_s$$

and  $Z$  solves the exponential SDE

$$Z_t = 1 + \int_0^t Z_s dX_s$$

whose unique solution  $\mathcal{E}(X)$  is the right-hand-side of (2.16). However, by Theorem 2.4

$$\tilde{W}_t^i \doteq W_t^i - \langle W^i, X^i \rangle_t = W_t^i - \int_0^t H_s^i ds$$

defines a continuous  $\tilde{\mathbb{P}}$ -local martingale  $\tilde{W}^i$ . Moreover, since

$$\langle \tilde{W}^i, \tilde{W}^j \rangle_t = \langle W^i, W^j \rangle = \delta_{ij}t,$$