A. Preliminaries – Superharmonic majorants.

Definition 4.9. *A lower bounded, measurable function* $f : \mathbb{R}^n \to (-\infty, \infty]$ *is called superharmonic (w.r.t.* X*) if, for all* $x \in \mathbb{R}^n$*,*

$$f(x) \geq \mathbb{E}_x\left[f\left(X_\tau\right)\right] \tag{4.42}$$

for all $\mathbb{P}_x$ *a.s. finite stopping times* τ*, and in addition*

$$f(x) = \lim_{k\to\infty} \mathbb{E}_x\left[f\left(X_{\tau_k}\right)\right] \tag{4.43}$$

for any sequence $\{\tau_k\}$ *of stopping times such that* $\tau_k \to 0$ *a.s.* $\mathbb{P}_x$*.*

Remark 4.10. If f satisfies (4.42), it is said to be super mean-valued (w.r.t. X). If f is super mean-valued and, for all $x \in \mathbb{R}^n$ and any sequence $\{\tau_k\}$ of stopping times such that $\tau_k \to 0$ $\mathbb{P}_x$ a.s., the inequality

$$f(x) \leq \liminf_{k\to\infty} \mathbb{E}_x\left[f\left(X_{\tau_k}\right)\right]$$

holds, then f is superharmonic because (4.42) provides the other inequality. In particular, if f is lower semicontinuous and super mean-valued, it is superharmonic.

Let $\mathcal{A}$ be the generator of the diffusion X. Then, from Remark 4.10, Itô's formula and Fatou's lemma, it is easy to deduce that for $f \in \mathcal{C}^2(\mathbb{R}^n)$ with f_x bounded, f is superharmonic (w.r.t. X) if and only if

$$\mathcal{A}f \leq 0.$$

Definition 4.11. *Let* h *be a real measurable function on* $\mathbb{R}^n$*. If* f *is a superharmonic (super mean-valued) function and* $f \geq h$*, we say that* f *is a superharmonic (super mean-valued) majorant of* h *(w.r.t.* X*).*

Lemma 4.12. *Given any set* J *and a family of super mean-valued (w.r.t* X*) functions* $f_j, j \in J$*, the function*

$$f(x) \doteq \inf_{j\in J} f_j(x); \quad x \in \mathbb{R}^n,$$

is super mean-valued w.r.t X*.*

Proof. Suppose that for each $j \in J$, f_j is super mean-valued. Then, for every stopping time τ and $j \in J$,

$$f_j(x) \geq \mathbb{E}_x\left[f_j(X_\tau)\right] \geq \mathbb{E}_x\left[f(X_\tau)\right].$$

Taking the infimum of the left-hand side over $j \in J$, it follows that f is super mean-valued. □

The last result shows that the function

$$\widehat{h}(x) \doteq \inf_f f(x); \quad x \in \mathbb{R}^n, \tag{4.44}$$

where the infimum is over all super mean-valued majorants of h, is again super mean-valued. Thus $\widehat{h}$ is the least super mean-valued majorant of h.

If $\widehat{h}$ is also superharmonic, then $\widehat{h}$ is also the least superharmonic majorant of h.

The proof of Step 1 will rely on an iterative procedure for calculating the least superharmonic majorant $\hat{g}$ of g. This will rely on the following definitions.

Definition 4.13. *A measurable function $f : \mathbb{R}^n \to [0, \infty)$ is called excessive (w.r.t. X) if, for all $x \in \mathbb{R}^n$,*

$$f(x) \geq \mathbb{E}_x [f(X_s)] \quad \forall \ \ s \geq 0,$$

and

$$f(x) = \lim_{s \to 0} \mathbb{E}_x [f(X_s)] .$$

Theorem 4.14. *A non-negative function f is excessive (w.r.t. X) if and only if it is superharmonic (w.r.t. X).*

Proof. Let f be a non-negative function. The implication f superharmonic $\Rightarrow$ f excessive is obvious. We will prove the converse only when $f \in \mathcal{C}^2(\mathbb{R}^k)$ and f_x and f_{xx} are bounded. By Itô's formula, for every $t \geq 0$,

$$\mathbb{E}_x [f(X_t)] = f(x) + \mathbb{E}_x \left[\int_0^t \mathcal{A} f(X_s)\, ds \right] .$$

Thus, if f is excessive, then

$$\mathbb{E}_x \left[\int_0^t \mathcal{A} f(X_s) ds \right] \leq 0.$$

Divide by t, let $t \to 0$ and use the boundedness of $\mathcal{A} f$, the bounded convergence theorem and the fundamental theorem of calculus to conclude that $\mathcal{A} f \leq 0$. By the comment after Remark 4.10, this shows that f is superharmonic. $\square$

We now provide an explicit construction of the least superharmonic majorant of a continuous function.

Theorem 4.15. *Let $g = g_0$ be a non-negative continuous function on $\mathbb{R}^n$ and define inductively,*

$$g_n(x) \doteq \max_{s \in S_n} \mathbb{E}_x [g_{n-1}(X_s)] ,$$

where $S_n \doteq \{k \cdot 2^{-n};\ 0 \leq k \leq 4^n\}$, $n = 1, 2, \ldots$. Then $g_n \uparrow \widehat{g}$, where $\widehat{g}$ is the least superharmonic majorant of g.

Proof. Since $0 \in S_n$ for every $n \in \mathbb{N}$, it is clear that $\{g_n\}$ is a sequence of increasing functions. Define $\breve{g}(x) \doteq \lim_{n \to \infty} g_n(x)$, $x \in \mathbb{R}^n$. Then, for every $n \in \mathbb{N}$,

$$\breve{g}(x) \geq g_n(x) \geq \mathbb{E}_x [g_{n-1}(X_s)] , \quad \forall s \in S_n. \tag{4.45}$$

Hence, for every $x \in \mathbb{R}^n$ and $s \in S \doteq \bigcup\limits_{n=1}^{\infty} S_n$, by the monotone convergence theorem,

$$\breve{g}(x) \geq \lim_{n\to\infty} \mathbb{E}_x\left[g_{n-1}\left(X_s\right)\right] = \mathbb{E}_x\left[\breve{g}\left(X_s\right)\right].$$

It can be shown that $x \mapsto \mathbb{E}_x\left[g\left(X_s\right)\right]$ is lower semicontinuous. Thus each g_n is the maximum of a finite number of lower semicontinuous functions, and so is also lower semicontinuous. In turn, this implies that $\breve{g}$, being an increasing limit of lower semicontinuous functions, is also lower semicontinuous. Fix $s \in \mathbb{R}$ and choose a sequence $\{s_k\}_{k\in\mathbb{N}} \subset S$ such that $s_k \to s$ as $k \to \infty$. By (4.45), Fatou's lemma and the lower semicontinuity of $\breve{g}$, we have

$$\breve{g}(x) \geq \lim_{k\to\infty} \mathbb{E}_x\left[\breve{g}\left(X_{s_k}\right)\right] \geq \mathbb{E}_x\left[\lim_{k\to\infty} \breve{g}\left(X_{s_k}\right)\right] \geq \mathbb{E}_x\left[\breve{g}\left(X_s\right)\right]$$

Since g is non-negative, this shows that $\breve{g}$ is excessive. By Theorem 4.14, this implies $\breve{g}$ is superharmonic. Since $\breve{g} \geq g_0 = g$, $\breve{g}$ is a superharmonic majorant of g.

On the other hand, if f is any super mean-valued majorant of g, then by induction

$$f(z) \geq g_n(z) \quad \forall n$$

and so $f(z) \geq \breve{g}(z)$. Thus, $\breve{g}$ is the least super mean-valued majorant of g. Since, in addition, $\widehat{g}$ is superharmonic, this implies $\breve{g} = \widehat{g}$ is the least superharmonic majorant of g. □

We also have the following slight generalization of the above result.

Corollary 4.16. *Define* $h_0 \doteq g$ *and inductively*

$$h_n(x) = \sup_{s\geq 0} \mathbb{E}_x\left[h_{n-1}\left(X_s\right)\right], \quad n = 1, 2, \dots .$$

Proof. Let $\overline{h} \doteq \lim h_n$, which is well-defined since the sequence $\{h_n\}$ is non-decreasing. Then, clearly, $\overline{h} \geq \breve{g} = \widehat{g}$, where the last equality follows from Theorem 4.15. On the other hand, since $\widehat{g}$ is excessive by Theorem 4.14 and $\widehat{g} \geq g$, we have

$$\widehat{g}(x) \geq \sup_{s\geq 0} \mathbb{E}_x\left[\widehat{g}\left(X_s\right)\right] \geq \sup_{s\geq 0} \mathbb{E}_x\left[g\left(X_s\right)\right].$$

By induction, this implies that $\widehat{g} \geq h_n$ for all $n \in \mathbb{N}$, and so $\widehat{g} \geq \overline{h}$. Together with the opposite inequality, this shows $\widehat{g} = \overline{h}$ and the proof is complete. □

The following additional property will also be useful in the subsequent analysis.

Lemma 4.17. *If* f *is super mean-valued and* H *is a Borel set, then* $\tilde{f}(x) = \mathbb{E}_x\left[f\left(X_{\tau_H}\right)\right]$ *is super mean-valued. If, in addition,* f *is lower semicontinuous, then* $\tilde{f}$ *is superharmonic.*

Proof. Suppose $\tilde{f}$ is super mean-valued. By the strong Markov property we have, for any stopping time α,

$$\begin{aligned} \mathbb{E}_x\left[\tilde{f}\left(X_\alpha\right)\right] = \mathbb{E}_x\left[\mathbb{E}_{X_\alpha}\left[f\left(X_{\tau_H}\right)\right]\right] &= \mathbb{E}_x\left[\mathbb{E}_x\left[f\left(X_{\tau_H} \circ \theta^\alpha\right) \mid \mathcal{F}_\alpha\right]\right] \\ &= \mathbb{E}_x\left[f\left(X_{\tau_H} \circ \theta^\alpha\right)\right] \\ &= \mathbb{E}_x\left[f\left(X_{\tau_H^\alpha}\right)\right] \end{aligned} \tag{4.46}$$

where θ^α is the shift operator associated with the stopping time α and $\tau_H^\alpha \doteq \inf\{t > \alpha : Z_t \notin H\}$. Since $\tau_H^\alpha \geq \tau_H$ and f is super mean-valued, we have by Problem 1(b) of HW4,

$$\mathbb{E}_x\left[\tilde{f}\left(X_\alpha\right)\right] = \mathbb{E}_x\left[f\left(X_{\tau_H^\alpha}\right)\right] \leq \mathbb{E}_x\left[f\left(X_{\tau_H}\right)\right] = \tilde{f}(x).$$

So $\tilde{f}$ is super mean-valued.

From (4.46) and Fatou's lemma, we see that if f is lower semicontinuous and α_k are stopping times such that $\alpha_k \to 0$ a.s. $\mathbb{P}^x$, then, with $\sigma_k \doteq \tau_H^{\alpha_k}$,

$$\begin{aligned} \lim_{k\to\infty} \mathbb{E}_x\left[\tilde{f}\left(X_{\alpha_k}\right)\right] = \lim_{k\to\infty} \mathbb{E}_x\left[f\left(X_{\sigma_k}\right)\right] &\geq \mathbb{E}_x\left[\liminf_{k\to\infty} f\left(X_{\sigma_k}\right)\right] \\ &\geq \mathbb{E}_x\left[f\left(X_{\tau_H}\right)\right] \\ &= \tilde{f}(x). \end{aligned}$$

By Remark 4.10, this shows $\tilde{f}$ is superharmonic. □

B. Rigorous Proof of the Optimal Stopping Theorem

We now rigorously establish the claims made in the rough outline of the proof earlier. We start by showing that $g^* = \widehat{g}$. One inequality is easily deduced.

Lemma 4.18. *$\widehat{g}(x) \geq g^*(x)$ for every $x \in \mathbb{R}^n$.*

Proof. Let $g \geq 0$ and let f be a super mean-valued majorant of g. If τ is a stopping time, then

$$f(x) \geq \mathbb{E}_x\left[f\left(X_\tau\right)\right] \geq \mathbb{E}_x\left[g\left(X_\tau\right)\right].$$

Therefore,

$$f(x) \geq \sup_\tau \mathbb{E}_x\left[g\left(X_\tau\right)\right] = g^*(x).$$

Taking the infimum of the left-hand side over all super mean-valued majorants f, the lemma follows. □

The reverse inequality is proved as part of the following more general existence theorem.

Theorem 4.19. *The following properties hold:*

(a) $g^* = \widehat{g}$.

(b) *For $\varepsilon > 0$, let*

$$D_\varepsilon \doteq \{x \in \mathbb{R}^n : g(x) < \widehat{g}(x) - \varepsilon\}$$

and define $\tau_\varepsilon \doteq \tau_{D_\varepsilon}$. If g is bounded, then for every $x \in \mathbb{R}^n$,

$$|g^*(x) - \mathbb{E}_x [g(X_{\tau_\varepsilon})]| < 2\varepsilon$$

(c) *For any continuous g, let*

$$D \doteq \{x \in \mathbb{R}^n : g(x) < g^*(x)\},$$

and for $N \in \mathbb{N}$, define $g_N \doteq g \wedge N$,

$$D_N \doteq \{x : g_N(x) < \widehat{g_N}(x)\}$$

and $\sigma_N \doteq \tau_{D_N}$. If $\mathbb{P}_x$ a.s. $\sigma_N < \infty$ for all N, then

$$g^*(x) = \lim_{N\to\infty} \mathbb{E}_x [g(X_{\sigma_N})].$$

In particular, if $\tau_D < \infty$ $\mathbb{P}_x$ a.s. and the family $\{g(X_{\sigma_N})\}_{N\in\mathbb{N}}$ is $\mathbb{P}_x$ uniformly integrable, then

$$g^*(x) = \mathbb{E}_x [g(X_{\tau_D})]$$

and $\tau^* = \tau_D$ is an optimal stopping time.

Proof. First, assume that g is bounded and define

$$h_\varepsilon(x) = \mathbb{E}_x [\widehat{g}(X_{\tau_\varepsilon})]$$

It is not hard to see from the explicit construction that $\widehat{g}$ is lower semicontinuous. By Lemma 4.17, this implies h_ε is superharmonic. We now show that

$$g(x) \le h_\varepsilon(x) + \varepsilon \quad \forall x. \tag{4.47}$$

Indeed, suppose

$$\beta \doteq \sup_x [g(x) - h_\varepsilon(x)] > \varepsilon.$$

Then for any $\delta > 0$, we can find x_0 such that

$$g(x_0) - h_\varepsilon(x_0) \ge \beta - \delta.$$

On the other hand, since $h_\varepsilon + \beta$ is a super mean-valued majorant of g, we have

$$\widehat{g}(x_0) \le h_\varepsilon(x_0) + \beta.$$

Together, the last two relations show that

$$\widehat{g}(x_0) \le g(x_0) + \delta. \tag{4.48}$$

We now consider two possible cases.

Case 1: $\tau_\varepsilon > 0$ $\mathbb{P}_{x_0}$, a.s. By (4.48) and the definition of D_ε,

$$\begin{aligned} g(x_0) + \delta \ge \widehat{g}(x_0) \ge \mathbb{E}_{x_0} [\widehat{g}(X_{t\wedge\tau_\varepsilon})] &\ge \mathbb{E}_{x_0} \left[(g(X_t) + \varepsilon)\, \mathbb{I}_{\{t<\tau_\varepsilon\}}\right] \\ &\to g(x_0) + \varepsilon, \text{ as } t \downarrow 0 \end{aligned}$$

This leads to a contradiction for $\delta < \varepsilon$.

Case 2: $\tau_\varepsilon = 0$ $\mathbb{P}_{x_0}$ a.s. Then $h_\varepsilon(x_0) = \widehat{g}(x_0)$, and so $g(x_0) \leq h_\varepsilon(x_0)$, which contradicts the definition of β. Thus (4.47) is proved, and we conclude that $h_\varepsilon + \varepsilon$ is a superharmonic majorant of g. Therefore,

$$\begin{aligned} \widehat{g} \leq h_\varepsilon + \varepsilon &= \mathbb{E}\left[\widehat{g}\left(X_{\tau_\varepsilon}\right)\right] + \varepsilon \\ &\leq \mathbb{E}\left[(g+\varepsilon)(X_{\tau_\varepsilon})\right] + \varepsilon \\ &\leq g^* + 2\varepsilon. \end{aligned} \tag{4.49}$$

Since ε is arbitrary, we have $\widehat{g} \leq g^*$. When combined with Lemma 4.18, this shows that $\widehat{g} = g^*$ for bounded g.

If g is not bounded, let $g_N \doteq \min(N, g)$ for $N \in \mathbb{N}$. Then, $g^* \geq (g_N)^* = \widehat{g_N} \uparrow r$ as $N \to \infty$, where $r \geq \widehat{g}$ since r is a superharmonic majorant of g (note that r is superharmonic by Problem 1(c) of HW 4). Thus $r = \widehat{g} = g^*$, which proves (a) for general g.

Property (b) follows from (a) and (4.49).

Finally, to obtain (c), let us again first assume g is bounded. Then, since

$$\tau_\varepsilon \uparrow \tau_D \text{ as } \varepsilon \downarrow 0,$$

and $\tau_D < \infty$ $\mathbb{P}_x$ a.s., we have

$$\mathbb{E}_x\left[g\left(X_{\tau_\varepsilon}\right)\right] \to \mathbb{E}_x\left[g\left(X_{\tau_D}\right)\right] \quad \text{as } \varepsilon \downarrow 0$$

Hence, by (a) and (4.49),

$$g^*(x) = \mathbb{E}_x\left[g\left(X_{\tau_D}\right)\right] \tag{4.50}$$

if g is bounded.

On the other hand, if g is not bounded, as before, define $g_N = N \wedge g$ and

$$r \doteq \lim_{N\to\infty} \widehat{g_N}.$$

Then r is superharmonic by Problem 1(c) of HW 4, and since $\widehat{g_N} \leq \widehat{g}$ for all N, we have $r \leq \widehat{g}$. On the other hand, $g_N \leq \widehat{g_N} \leq r$ for all N and therefore $g \leq r$. Since $\widehat{g}$ is the least superharmonic majorant of g, it follows that $r = \widehat{g}$. Hence, using (a) and (4.50),

$$\begin{aligned} g^*(x) = \widehat{g}(x) = \lim_{N\to\infty} \widehat{g_N}(x) &= \lim_{N\to\infty} \mathbb{E}_x\left[g_N\left(X_{\sigma_N}\right)\right] \\ &\leq \lim_{N\to\infty} \mathbb{E}_x\left[g\left(X_{\sigma_N}\right)\right] \\ &\leq g^*(x), \end{aligned}$$

and the first display in (c) follows.

Now, suppose $\tau_D < \infty$ $\mathbb{P}_x$ a.s. It is easy to see that D is the increasing union of the sets D_N. Therefore, $\tau_D = \lim_{N\to\infty} \sigma_N$. So, by the first display in

(c) and the uniform integrability condition, we have

$$\begin{aligned}\widehat{g}(x) = \lim_{N\to\infty} \widehat{g_N}(x) &= \lim_{N\to\infty} \mathbb{E}_x\left[g_N\left(X_{\sigma_N}\right)\right] \\ &= \mathbb{E}_x\left[\lim_{N\to\infty} g_N\left(X_{\sigma_N}\right)\right] = \mathbb{E}_x\left[g\left(X_{\tau_D}\right)\right].\end{aligned}$$

This completes the proof of the theorem. □

Remark 4.20. The entire argument above can be extended to the more general case of an inhomogenous diffusion Y that satisfies

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t$$

and a continuous reward function $g : \mathbb{R} \times \mathbb{R}^n \mapsto [0, \infty)$ that depends both on time and space, with the optimal expected reward now defined as

$$g^*(t, x) \doteq \sup_\tau \mathbb{E}_{t,x}\left[g(\tau, X_\tau)\right],$$

where $\mathbb{E}_{(t,x)}$ represents expectation conditioned on the process $Y_s \doteq (t + s, X_s)$, starting at $y = (t, x)$. Note that Y_s is simply the graph of $X_s = X_s^x$ shifted to start at (t, x). If X takes values in $\mathbb{R}^n$, then Y takes values in $\mathbb{R}^{n+1}$ and the entire procedure above can be applied to the diffusion Y.

C. A Representation Result

The following example shows that we need not have existence in general. Suppose

$$X_t = t, \quad t \geq 0,$$

and

$$g(t, x) = \frac{x^2}{1 + x^2}, \;\; x \in \mathbb{R}.$$

Then by direct inspection it is clear that $g^*(x) = 1$, but there is no stopping time τ such that

$$\mathbb{E}_{t,x}\left[g\left(X_\tau\right)\right] = 1.$$

However, we now show that if an optimal stopping time τ^* exists, then it is of the form presented in Theorem 4.19.

Theorem 4.21. *Let $D \doteq \{x : g(x) < g^*(x)\} \subset \mathbb{R}^n$. Suppose there exists an optimal time τ^* for the optimal stopping problem. Then*

$$\tau^* \geq \tau_D, \tag{4.51}$$

where τ_D is the first exit time of X from D, and

$$g^*(x) = \mathbb{E}_x\left[g\left(X_{\tau_D}\right)\right] \text{ for all } x. \tag{4.52}$$

Hence, τ_D is an optimal stopping time.

Proof. Let τ be a stopping time for X and assume that $\mathbb{P}_x(\tau < \tau_D) > 0$. Since $g\left(X_\tau\right) < g^*\left(X_\tau\right)$ if $\tau < \tau_D$ and $g \leq g^*$ always, we have

$$\begin{aligned}
\mathbb{E}_x\left[g\left(X_\tau\right)\right] &= \int_{\tau<\tau_D} g(X_\tau) d\mathbb{P}_x + \int_{\tau\geq\tau_D} g\left(X_\tau\right) d\mathbb{P}_x \\
&< \int_{\tau<\tau_D} g^*(X_\tau) d\mathbb{P}_x + \int_{\tau\geq\tau_D} g^*(X_\tau) d\mathbb{P}_x \\
&= \mathbb{E}_x\left[g^*\left(X_\tau\right)\right] \\
&\leq g^*(x).
\end{aligned}$$

since g^* is superharmonic by Theorem 4.19. This proves (4.51).

To obtain (4.52), note that by the definition of g^*, the fact that $\tau^* \geq \tau_D$ by (4.51), $\widehat{g}$ is superharmonic and Problem 1(b) of HW 4, and the definition of D and the fact that $g^* = \widehat{g}$ by Theorem 4.19(a), we have

$$g^*(x) = \mathbb{E}_x\left[g\left(X_{\tau^*}\right)\right] \leq \mathbb{E}_x\left[\widehat{g}\left(X_{\tau^*}\right)\right] \leq \mathbb{E}_x\left[\widehat{g}\left(X_{\tau_D}\right)\right] \leq \mathbb{E}_x\left[g\left(X_{\tau_D}\right)\right] \leq g^*(x),$$

which completes the proof of the theorem. □

4.5. **Examples.**

Example 1.

Let $X_t = B_t$ be 2-dimensional standard Brownian motion. Using the fact that B is recurrent (since we are in 2 dimensions), it is possible to show that the only non-negative superharmonic functions in $\mathbb{R}^2$ are the constants. Therefore,

$$g^*(x) = \|g\|_\infty = \sup\left\{g(y) : y \in \mathbb{R}^2\right\} \quad \forall x.$$

So, if g is unbounded, $g^* = \infty$ and no optimal stopping time exists. If g is bounded, then let

$$D \doteq \left\{x : g(x) < \|g\|_\infty\right\}.$$

If $\tau_D = \infty$ a.s. (from potential theory, this can be shown to hold if and only if ∂D is a polar set, or equivalently, $\text{cap}(\partial D) = 0$, where $\text{cap}(\partial D)$ represents the so-called logarithmic capacity of ∂D), no optimal stopping time exists, while if $\tau_D < \infty$ (which happens if and only if cap $(\partial D) > 0$), then

$$\mathbb{E}_x\left[g\left(B_{\tau_D}\right)\right] = \|g\|_\infty = g^*.$$

and $\tau^* = \tau_D$ is optimal.

Example 2. Application to selling stocks.

This was covered in detail in class, and is thus omitted. Other examples were also covered in HW 4.