Thus $\mathbb{E}[\sup_n Y_n] = \infty$. However, since $\mathbb{E}\left[Y_{n+1} \mid \mathcal{F}_n\right] \geq Y_n \quad \forall n$, it is always optimal for the player to continue playing. But, if you continue playing for ever, then you receive $Y_\infty = 0$. Thus, there is no stopping rule that achieves $\mathbb{E}[\sup_n Y_n]$.

2. $Y_0 = 0, Y_n = 1 - \frac{1}{n}$ for $n = 1, 2, \cdots, Y_\infty = 0$. Here A1 is satisfied but A2 is violated. Again, as in the last example, it is always better to continue, but if you wait forever you win nothing. Thus, there is no optimal stopping rule.

• Sufficiency of Conditions

Definition: A stopping rule N is said to be **regular** if

$$\mathbb{E}\left[Y_N \mid \mathcal{F}_N\right] \geq Y_n \text{ a.s. on } \{N > n\} \quad \forall n.$$

Lemma 4.1. *Assume A1. Given any stopping rule $N, \exists$ a regular stopping rule N' such that*

$$\mathbb{E}\left[Y_{N'}\right] \geq \mathbb{E}\left[Y_N\right].$$

Proof. Define $N' = \min\left\{n \geq 0 : \mathbb{E}\left[Y_N \mid \mathcal{F}_N\right] \leq Y_N\right\}$. Clearly, N' is a stopping time and $N' \leq N$. Moreover, on $\{N' = n\}$, we have $\mathbb{E}\left[Y_N \mid \mathcal{F}_n\right] \leq Y_n$, while on $\{N' = \infty\}$, we have $Y_N = Y_{N'} = Y_\infty$ a.s. Hence,

$$\begin{aligned}
\mathbb{E}[Y_{N'}] &\overset{A1}{=} \sum_{n=0}^{=\infty} \mathbb{E}[\mathbb{I}_{\{N'=n\}} Y_n] \\
&\geq \sum_{n=0}^{\infty} \mathbb{E}\left[\mathbb{I}_{\{N'=n\}} \mathbb{E}[Y_N \mid \mathcal{F}_n]\right] + \mathbb{E}[\mathbb{I}_{\{N'=\infty\}} Y_\infty] \\
&= \sum_{n=0}^{=\infty} \mathbb{E}[\mathbb{I}_{\{N'=n\}} Y_N] \overset{A1}{=} \mathbb{E}[Y_N]
\end{aligned}$$

The proof of regularity is left to the reader. □

Lemma 4.2. *Assume A1. If N and N' are regular stopping rules, then so is $N'' = \max(N, N')$ and in this case*

$$\mathbb{E}[Y_{N''}] \geq \max\{\mathbb{E}[Y_N], \mathbb{E}[Y_{N'}]\} \tag{4.39}$$

Proof. $N'' = N$ except on sets of the form $\{N = n\} \cap \{N' > n\}$, in which case $\mathbb{E}[Y_{N''} \mid \mathcal{F}_n] = \mathbb{E}[Y_{N'} \mid \mathcal{F}_n] \geq Y_n$. Hence,

$$\begin{aligned} \mathbb{E}[Y_{N''}] &= \sum_{n=0}^{=\infty} \mathbb{E}[\mathbb{I}_{\{N=n\}} Y_{N''}] \\ &= \sum \mathbb{E}[\mathbb{I}_{\{N=n\}} \mathbb{E}[Y_{N''} \mid \mathcal{F}_n]] \\ &\geq \sum \mathbb{E}[\mathbb{I}_{\{N=n\}} Y_n] \\ &= \mathbb{E}[Y_N] \end{aligned}$$

By symmetry, $\mathbb{E}[Y_{N''}] \geq \mathbb{E}[Y_{N'}]$ and (4.39) follows. The proof of regularity is left to the reader. □

Theorem 4.3. *Under A1 and A2, there exists a stopping rule N^* s.t.*

$$\mathbb{E}[Y_{N^*}] = V^* = \sup_N \mathbb{E}[Y_N].$$

Proof. If $V^* = -\infty$, the result is trivial. So assume $-\infty < V^* < \infty$. Let $N_1, N_2, \cdots$ be a sequence of stopping rules s.t. $\mathbb{E}[Y_N] \to V^*$ as $N \to \infty$. Let $N'_1, N'_2, \cdots$ be the regularized versions, as in Lemma 4.1. Let $N''_j = \max(N'_1, \cdots, N_j)$. By Lemma 4.2 and 4.1,

$$\mathbb{E}[Y_{N''_j}] \geq \mathbb{E}[Y_{N'_j}] \geq \mathbb{E}[Y_{N_j}]$$

and consequently

$$\lim_{n\to\infty} \mathbb{E}[Y_{N''_j}] = V^*.$$

Now $\{N''_j\}$ is a monotone sequence of stopping rules, which converges to $N^* = \sup\{N'_1, N'_2, \cdots\}$. Moreover, either $N''_j \to \infty$ or N''_j is fixed integer from some j on. Thus $\lim_{j\to\infty} Y_{N''_j} \leq Y_{N^*}$ a.s. from A2. From the Fatou-Lebesgue theorem, $Y_{N''_j}$ are bounded above by $\sup_n Y_n$ which is integrable by A1.

So, $V^* = \limsup \mathbb{E}[Y_{N''_j}] \leq \mathbb{E}[\limsup Y_{N''_j}] \leq \mathbb{E}[Y_{N^*}]$ since $\mathbb{E}[Y_{N^*}] \leq V^*$ by definition of V^*, we are done. □

• **Principle of Optimality**

Intuition: It is optimal to stop at the initial stage if and only if $y_0 = V_*$. If we observed $X_1 = x_1, \cdots, X_n = x_n$, define

$$V_n^*(x_1, \cdots, x_n) = \sup_{N\geq n} \mathbb{E}[Y_N \mid X_1 = x_1, \cdots X_n = x_n],$$

where here the supremum is over all stopping rules N such that $\mathbb{P}(N \geq n) = 1$. Note $V_0^* = V^*$. We expect that it will be optimal to stop at n if and only if $y_n(x_1, \cdots, x_n) = V_n^*(x_1, \cdots x_n)$. This is the *principle of optimality.* It holds here, under conditions A1 and A2 but, for the technical reason that the supermum over an uncountable number of random variables (here the supremum of $\mathbb{E}[Y_n|\mathcal{F}_n]$ over the set of stopping rules $N \geq n$) may not be a random variable, it requires a slight modification. Instead, we use the essential supremum.

Definition. Let $X_t, t \in T$ be a collection of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The random variable Z is an essential supremum of $(X_t)_{t\in T}$ and denoted $Z = \text{ess sup} X_t$ if and only if

(1) $\mathbb{P}(Z \geq X_t) = 1$ for all $t \in T$, and
(2) if Z' is any other random variable such that $\mathbb{P}(Z' \geq X_t) = 1$ for all $t \in T$, then $\mathbb{P}(Z' \geq Z) = 1$.

For an example where

$$\operatorname*{ess\,sup}_{t\in T} X_t \neq \sup_{t\in T} X_t,$$

consider $T = [0,1]$, and let $X_t = \mathbb{I}_{\{u=t\}}$ where U unif. r.v. $\sim \mathcal{U}[0,1]$. Then $\sup_{t\in T} X_t = 1$ but $\operatorname*{ess\,sup}_{t} X_t = 0$.

We will use the following fact without proof.

Lemma 4.4. *There exists a countable subset $C \subset T$ s.t. $Z = \sup_{t\in C} X_t$ is an essential supremum.*

In order to continue with our optimality principle, we will need to formulate conditional versions of regularity.

Definition: A stopping rule $N \geq n$ is **regular from n on** if $\forall k \geq n$,

$$\mathbb{E}[Y_N \mid \mathcal{F}_k] \geq Y_k \quad \text{on} \quad \{N > k\}.$$

We now state results analogous to Lemmas 4.1 and 4.2 for these versions (the proofs are straightforward adaptations of the proofs of the other two lemmas, and so are omitted).

Lemma 4.5. *Under A1, for any stopping rule $N \geq n$, there exists a stopping rule $N' \geq n$ regular from n on, such that*

$$\mathbb{E}[Y_{N'}|\mathcal{F}_n] \geq \mathbb{E}[Y_N|\mathcal{F}_n].$$

Lemma 4.6. *Under A1, if $N \geq n$ and $N' \geq n$ are both regular from n on, then so is $N'' = max(N, N')$ and $\mathbb{E}[Y_{N'}|\mathcal{F}_n] \geq \mathbb{E}[Y_N|\mathcal{F}_n]$).*

We now state the *optimality equation* of dynamic programming.

$$V_n^* = \operatorname*{ess\,sup}_{N\geq n} \mathbb{E}[Y_N \mid \mathcal{F}_n].$$

Theorem 4.7. *Under A1, $V_n^* = \max(Y_n, \mathbb{E}[V_{n+1}^* \mid \mathcal{F}_n])$.*

Proof. Let $N \geq n$ be an arbitrary stopping rule. On $\{N > n\}$, $\mathbb{E}[Y_N \mid \mathcal{F}_{n+1}] \leq V_{n+1}^*$ and so on $\{N > n\}$.

$$\begin{aligned} \mathbb{E}[Y_N \mid \mathcal{F}_n] &= \mathbb{E}[\mathbb{E}[Y_N \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n] \\ &\leq \mathbb{E}[V_{n+1}^* \mid \mathcal{F}_n]. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}[Y_N \mid \mathcal{F}_n] &= \mathbb{I}_{\{N=n\}} Y_n + \mathbb{I}_{\{N>n\}} \mathbb{E}[Y_N \mid \mathcal{F}_n] \\ &\leq \max(Y_n, \mathbb{E}[V^*_{n+1} \mid \mathcal{F}_n]) \end{aligned}$$

To show the reverse inequality,

$$Y_n \leq V^*_n$$

trivially. By Lemma 4.4, $\exists$ a sequence $N_1, N_2, \cdots$ of stopping rules with each $N_k \geq n+1$ such that

$$V^*_{n+1} = \sup_k \mathbb{E}[Y_{N_k} \mid \mathcal{F}_{n+1}]. \tag{4.40}$$

By Lemma 4.5, $\exists$ for each k, a stopping rule $N'_k \geq n+1$, regular from $n+1$ on such that

$$\mathbb{E}[Y_{N'_k} \mid \mathcal{F}_{n+1}] \geq \mathbb{E}[Y_{N_k} \mid \mathcal{F}_{n+1}].$$

Let $N''_k = \max(N'_1, \cdots, N'_k)$. Then

$$\begin{aligned} V^*_n &\geq \mathbb{E}[Y_{N''_k} \mid \mathcal{F}_n] = \mathbb{E}[\mathbb{E}[Y_{N''_k} \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n] \\ &\geq \mathbb{E}[\max_{1\leq j\leq k} \mathbb{E}[Y_{N_j} \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n] \quad \text{by Lemma4.6} \\ &\geq \mathbb{E}[\max_{1\leq j\leq k} \mathbb{E}[Y_{N_j} \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n] \\ &\xrightarrow[k\to\infty]{\substack{\text{cond. MCT} \\ \&(4.40)}} \mathbb{E}[V^*_{n+1} \mid \mathcal{F}_n]. \end{aligned}$$

□

• **Markov Models** We now specialize to the case of Markov models, in which the rewards are additionally assumed to have a certain Markov structure. Specifically, we assume
• Observations $X_1, \cdots, X_n, \cdots, \mathcal{F}_n = \sigma(X_1, \cdots, X_n)$.
• Rewards $Y_n = u_n(z_n)$, where z_n is $\mathcal{F}_n$ -measurable and $\{z_n\}$ forms a Markov chain.

In this case, one can show that in the optimality equation, V^*_n is also a function of z_n, and so we can write $V^*_n = v_n(z_n)$, and the optimal stopping time has the form

$$N^* = \min\{n \geq 0 \;\; v_n(z_n) \geq u_n(z_n)\}.$$

We illustrate this by means of an example.
Example: House-selling problem.
• $X_1, X_2, \cdots$ iid $\sim F$.
• (without recall) $Y_n = X_n - nc$. $Y_0 = Y_\infty = -\infty$ ①
(with recall) $Y_n = Mn - nc$. $Mn = \sup_{i\leq n} X_i, Y_0 = Y_\infty = -\infty$ ②

Theorem 4.8. *$X, \{X_i\}$ identically distributed. Then Y_n is defined as in ① or ②*
If $\mathbb{E}[X^+] < \infty$, then $\sup_n Y_n < \infty$ a.s. and $Y_n \to -\infty$ a.s.

If $\mathbb{E}[(X^+)^2] < \infty$*, then* $\mathbb{E}[\sup_n Y_n] < \infty$*.*

Proof. Since $Mn - nc = \max(X_1 - nc, \cdots, X_n - nc) \leq \max(X_1 - c, \cdots, X_n - nc)$ we have $\sup_n (X_n - nc) = \sup_n (Mn - nc)$. So, without loss of generality, can assume Y_n is defined by ①.

1°. Suppose $\mathbb{E}[|X^+|] < \infty$. Then

$$\begin{aligned}\mathbb{P}\left(\sup_{n\geq 1} Y_n > z\right) &\leq \sum_{n=1}^{\infty} \mathbb{P}(Y_n > z) = \sum_{n=1}^{\infty} \mathbb{P}\left(\tfrac{X_z}{c} > n\right) \\ &\leq \mathbb{E}\left[\tfrac{(X-z)^+}{c}\right] \to 0 \text{ as } z \to \infty.\end{aligned}$$

Thus $\sup_n Y_n < \infty$ a.s. Moreover,

$$\begin{aligned}Y_n &\leq Mn - nc = \left(Mn - \tfrac{nc}{2}\right) - \tfrac{nc}{2} \\ \sup_n Y_n &\leq u - \tfrac{nc}{2} \to -\infty \text{ as } n \to \infty \text{ since} \\ u &:= \sup_n \left(M_n - \tfrac{nc}{2}\right) < \infty\end{aligned}$$

by the last calculation with $\frac{c}{2}$ instead of c.

2°. Suppose $\mathbb{E}[|X^+|^2] < \infty$. Then

$$\begin{aligned}\mathbb{E}\left[\sup_{n\geq 1} Y_n\right] &\leq \textstyle\int_0^\infty \mathbb{P}\left(\sup_{n\geq 1} Y_n > z\right) dz \\ &\leq \textstyle\int_0^\infty \mathbb{E}\left[\tfrac{(X-z)^+}{c}\right] dz \\ &= \textstyle\int_0^\infty \int_z^\infty \tfrac{x-z}{c} dF(x) dz \\ &= \textstyle\int_0^\infty \left(\int_0^x \tfrac{x-z}{c} dz\right) dF(x) \\ &= \textstyle\int_0^\infty \tfrac{x^2}{2c} dF(x) = \tfrac{\mathbb{E}[|X^+|^2]}{2c} < \infty.\end{aligned}$$

□

The Principle of Optimality now says that

$$N^* = \min\{n > 0 : X_n \geq V^*\}.$$

To compute V^*, note that

$$\begin{aligned}V^* &= \mathbb{E}_x[\max\{X_1, V^*\}] - c \\ &= \int_{-\infty}^{V^*} V^* dF(x) + \int_{V^*}^{\infty} x dF(x) - c\end{aligned}$$

Rearranging terms, we obtain

$$\mathbb{E}[(x - v^*)^*] = c.$$

Since the l.h.s. is continuous in V^*, decreasing from $-\infty$ to 0. Hence, there exists a unique solution.

4.3. **Optimal Stopping: Continuous Time – the Markov Case.** Suppose $b : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \mapsto \mathbb{R}^{n\times m}$ are bounded and measurable functions. Let W be an m-dimensional standard Brownian motion on the filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and let X be an $\mathcal{F}_t$-adapted process that, together, form a weak solution to the SDE

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t. \tag{4.41}$$

As usual, let $\mathbb{E}_x$ denote the expectation conditional on $X_0 = x$. The optimal stopping problem is defined as follows.

The Optimal Stopping Problem. Given X satisfying (4.41) and a non-negative, continuous, real-valued function g on $\mathbb{R}^n$, find a stopping time τ^* for $\{X_t\}$ such that

$$\mathbb{E}_x\left[g\left(X_{\tau^*}\right)\right] = \sup_{\tau} \mathbb{E}_x\left[g\left(X_{\tau}\right)\right]$$

where the supremum is over all stopping times and we interpret $g(X_{\tau(\omega)}(\omega)) = 0$ if $\tau(\omega) = \infty$. Also, identify the corresponding optimal expected reward

$$g^*(x) = \mathbb{E}_x\left[g\left(X_{\tau^*}\right)\right].$$

In Section 4.4 we describe the solution to this problem under general conditions and in Section 4.5 we provide concrete examples of the solution.

4.4. **Solution of the Optimal Stopping Problem.** We start by roughly outlining the main steps to solving the problem.

Rough Idea of Approach:

Step 1: Identify g^* with the least superharmonic majorant $\hat{g}$ of g w.r.t. X.

Step 2: Define the region $D \subset \mathbb{R}^n$ by

$$D \doteq \{x \in \mathbb{R}^n : g(x) < g^*(x)\}.$$

The region D is referred to as the "continuation region". Then the first exit time $\tau^* = \tau_D$ from D for X solves the optimal stopping problem. Hence,

$$g^*(x) = \mathbb{E}_x\left[g\left(X_{\tau_D}\right)\right].$$

In order to present rigorous proofs of the above statements, we need to first introduce the definition and describe some properties of the least superharmonic majorant of a function.