21-881: Advanced Stochastic Calculus II – Spring 2009

1. Time Changes

1.1. Deterministic Time Changes.

Consider an increasing,[1] right-continuous function $A : \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$ with $A(0) = 0$ and for $s \geq 0$, define

$$C_s \doteq \inf \{t : A_t > s\}, \tag{1.1}$$

where we set $\inf(\emptyset) = \infty$. The function C is obviously increasing, so that

$$C_{s-} \doteq \lim_{u \uparrow s} C_u$$

is well-defined for every s. Observe that

$$C_{s-} = \inf \{t : A_t \geq s\}.$$

By convention, let $C_{0-} \doteq 0$.

The following lemma shows that A and C play symmetric roles.

Lemma 1.1. *C is right-continuous, $A(C_s) \geq s$ for every s, and*

$$A_t = \inf \{s : C_s > t\}. \tag{1.2}$$

Proof. We have $\{t : A_t > s\} = \bigcup_{\varepsilon > 0} \{t : A_t > s + \varepsilon\}$, and so C is right-continuous. The fact that $A(C_s) \geq s$ follows immediately from the right-continuity of A. To show (1.2), first note that if $C_s > t$, then $t \notin \{u : A_u > s\}$ and $A_t \leq s$. Since A is increasing, it follows that

$$A_t \leq \inf \{s : C_s > t\}. \tag{1.3}$$

On the other hand, $C(A_t) \geq t$ for every t and hence $C(A_{t+\varepsilon}) \geq t + \varepsilon > t$, which implies that for every $\varepsilon > 0$,

$$A_{t+\varepsilon} \geq \inf \{s : C_s > t\}.$$

By the right-continuity of A, this in turn means that

$$A_t \geq \inf \{s : C_s > t\}. \tag{1.4}$$

Relations (1.3) and (1.4) together imply (1.2). □

C is sometimes referred to as the right-continuous inverse of A. The jumps of A correspond to level stretches of C and vice versa. Indeed, there is equality in $A(C_s) \geq s$ only if s is a point of increase of C, that is $C_{s+\varepsilon} > C_s$ for every $\varepsilon > 0$. Moreover, if A is strictly increasing, then C is continuous; if A is continuous and strictly increasing then C is also continuous and strictly increasing and in this case $A_{C_t} = C_{A_t} = t$.

As a consequence of Lemma 1.1, we have the following (deterministic) *change of variables* formula, which enables the explicit computation of some Stieltjes integrals.

[1] Throughout, increasing will refer to non-decreasing, as opposed to strictly increasing.

Proposition 1.2. *If f is a positive Borel function on $[0,\infty)$, then*

$$\int_{[0,\infty)} f(u)\,dA_u = \int_0^\infty f(C_s)\mathbb{I}_{(C_s<\infty)}\,ds. \tag{1.5}$$

Proof. If $f = \mathbb{I}_{[0,v]}$, then the definition of C shows that (1.5) is equivalent to

$$A_v = \int_0^\infty \mathbb{I}_{\{C_s \le v\}}\,ds,$$

which clearly holds by (1.2). By taking differences, the equality holds for indicators of sets of the form $(u,v]$, and by the monotone class theorem, for any Borel-measurable f with compact support. The monotone convergence theorem (MCT) can then be applied to complete the proof. □

Proposition 1.3. *If u is a continuous, increasing function on the interval $[a,b]$, then for a non-negative Borel function f on $[u(a),u(b)]$,*

$$\int_{[a,b]} f(u(s))\,dA_{u(s)} = \int_{[u(a),u(b)]} f(t)\,dA_t.$$

Proof. Define $v_t \doteq \inf\{s : u(s) > t\}$. Then $u(v_t) = t$ since u is continuous, and v is a measurable mapping from $[u(a),u(b)]$ into $[a,b]$. Let dA be the measure on $[u(a),u(b)]$ associated with A and let ν be the image of dA by v. Then dA is the image of ν by u and therefore

$$\int_{[a,b]} f(u(s))\,d\nu(s) = \int_{[u(a),u(b)]} f(t)\,dA_t.$$

In particular, this implies $A(u(b)) - A(u(a)-) = \nu([a,b])$, which proves that ν is associated with the increasing function $s \to A(u(s))$. The proposition is established. □

1.2. **Stochastic Time Changes.** Let $\{\mathcal{F}_t\}$ be a right-continuous filtration, and let A be an increasing, right-continuous, adapted process. For each $s \ge 0$, let C_s be the process defined pathwise by (1.1).

Proposition 1.4. *The family $\{C_s\}_{s\ge 0}$ is an increasing, right-continuous family of stopping times. Moreover, for every t, the random variable A_t is an $\{\mathcal{F}_{C_s}\}$ stopping time.*

Proof. For each $s \ge 0$, C_s is an $\{\mathcal{F}_t\}$ optional time because A is right-continuous and (s,∞) is open (see Problem 2.6 of Chapter 1 of [1]). Since the filtration $\{\mathcal{F}_t\}$ is right-continuous, C_s is also an $\{\mathcal{F}_t\}$ stopping time (see Corollary 2.4 of Chapter 1 of [1]). The right-continuity of the process $\{C_s\}$ follows from Lemma 1.1, and C is trivially pathwise increasing. It is also easy to check that $\{\mathcal{F}_{C_s}\}_{s\ge 0}$ is a right-continuous filtration. (Indeed, using the fact that $C_s = \lim_{\varepsilon\downarrow 0} C_{s+\varepsilon}$, $C_{s+\varepsilon}, \varepsilon > 0$, are $\{\mathcal{F}_t\}$ stopping times and the right-continuity of $\mathcal{F}_t$, show that $\mathcal{F}_{C_s} = \cap_{\varepsilon>0}\mathcal{F}_{C_{s+\varepsilon}}$ – see HW1, Optional Problem 2.) Therefore, using the representation (1.2) for A, the same argument used to show C_s is an $\{\mathcal{F}_t\}$ stopping time also shows that for each t, A_t is an $\{\mathcal{F}_{C_s}\}$ stopping time. □

Definition 1.5. *A time-change C is a family $C_s, s \geq 0$, of stopping times such that the maps $s \mapsto C_s$ are a.s. increasing and right-continuous.*

In what follows, fix a time-change C and define $\widehat{\mathcal{F}}_t \doteq \mathcal{F}_{C_t}$. If X is an $\{\mathcal{F}_t\}$-progressively measurable process, then $\widehat{X}_t \doteq X_{C_t}$ is an $\{\widehat{\mathcal{F}}_t\}$-adapted process. The process $\widehat{X}$ will be called the time-changed process of X.

Given any time-change, one obtains an increasing, adapted, right-continuous process by setting

$$A_t = \inf\{s : C_s > t\},$$

where A could be infinite if $C_\infty \doteq \lim_{s\to\infty} C_s < \infty$. Thus time-changes are essentially the inverses of right-continuous, increasing adapted processes. From the discussion in the previous section, it is clear that if A is continuous and strictly increasing, then so is C and $\widehat{X}_A = X$ on $[0, A_\infty)$.

An important property of semimartingales is that they are invariant under time changes. We will concentrate here only on continuous semimartingales.

We will require the following definition.

Definition 1.6. *If C is a time-change, a process X is said to be C-continuous if X is constant on each interval $[C_{t-}, C_t]$.*

If X is right-continuous and of finite variation, then so is $\widehat{X}$. As a more or less immediate consequence of the deterministic result, Proposition 1.3, we obtain the following relation between the Stieltjes integrals of X and $\widehat{X}$.

Proposition 1.7. *If H is $\mathcal{F}_t$-progressively measurable, then $\widehat{H}$ is $\{\widehat{\mathcal{F}}_t\}$-progressively measurable, and if X is a C-continuous process of finite variation, then $\widehat{H} \cdot \widehat{X} = \widehat{H \cdot X}$, i.e.,*

$$\int_0^{t\wedge A_\infty} H_{C_u}\, dX_{C_u} = \int_0^t \mathbb{I}_{\{C_u<\infty\}} H_{C_u}\, dX_{C_u} = \int_{C_0}^{C_t} H_s\, dX_s.$$

We are now interested in identifying conditions under which the time-change of a continuous local martingale remains a continuous local martingale. A simple example shows that arbitrary time-changes of local martingales need not always remain local martingales. In addition, if C has jumps then $\widehat{X}$ may be discontinuous even if X is continuous. The assumption that X be C-continuous eliminates these possibilities.

Proposition 1.8. *Let C be a.s. finite and X be a continuous $\{\mathcal{F}_t\}$-local martingale.*

(1) *If X is C-continuous, then $\widehat{X}$ is a continuous $\{\widehat{\mathcal{F}}_t\}$-local martingale and*

$$\langle \widehat{X} \rangle = \widehat{\langle X \rangle}.$$

(2) *If, moreover, H is $\{\mathcal{F}_t\}$-progressively measurable and $\int_0^t H_s^2 d\langle X\rangle_s < \infty$ almost surely for every t, then $\int_0^t \widehat{H}_s^2 d\langle \widehat{X}\rangle_s < \infty$ a.s. for every t and*

$$\widehat{H} \cdot \widehat{X} = \widehat{H \cdot X}.$$

Proof. First note that since X is C-continuous, the process $\widehat{X}$ is also continuous. Let T be an $\{\mathcal{F}_t\}$-stopping time such that X^T is bounded. Then X^T is an $\{\mathcal{F}_t\}$ martingale Moreover, $\widehat{T} \doteq \inf\{t : C_t \geq T\}$ is an $\{\widehat{\mathcal{F}}_t\}$-stopping time because, for any $r \geq 0$,

$$\{\widehat{T} \leq r\} = \{C_r \geq T\} \in \mathcal{F}_{C_r} = \widehat{\mathcal{F}}_r.$$

We also have the relations

$$\widehat{X}_t^{\widehat{T}} = \widehat{X}_{\widehat{T}\wedge t} = X_{C_{\widehat{T}\wedge t}} = X_{C_{\widehat{T}} \wedge C_t}.$$

Since X is C-continuous, X is constant on $[T, C_{\widehat{T}}]$ and so we have $X_{C_{\widehat{T}}\wedge C_t} = X_{T\wedge C_t} = X^T_{C_t}$. Together with the previous display, this shows that $\widehat{X}_t^{\widehat{T}} = X^T_{C_t}$, which implies that $\widehat{X}^{\widehat{T}}$ is also bounded and, by the optional stopping theorem applied to the martingale X^T, that $\widehat{X}^{\widehat{T}}$ is an $\{\widehat{\mathcal{F}}_t\}$-martingale. Finally, if $\{T_n\}$ is a sequence of stopping times that increases to infinity a.s. then the corresponding sequence $\{\widehat{T}_n\}$ also increases almost surely to infinity. Thus, we have proved that $\widehat{X}$ is a continuous local martingale with localizing sequence $\{\widehat{T}_n\}$.

Since the intervals of constancy of X and $\langle X\rangle$ are identical (this is a straightforward generalization of the fact that, for any local martingale M, $\langle M\rangle = 0$ if and only if M is constant), the process $\langle X\rangle$ is also C-continuous. The argument in the previous paragraph applied to the continuous local martingale $X^2 - \langle X\rangle$ shows that $\widehat{X}^2 - \widehat{\langle X\rangle}$ is also a continuous local martingale, which completes the proof of property 1.

For the second property, the first part follows from Proposition 1.7. To prove the second part, we need only prove that the increasing process of the local martingale $\widehat{H}\cdot\widehat{X} - \widehat{H \cdot X}$ vanishes identically. However, this is a simple consequence of property 1 and Proposition 1.7. □

The above result shows, in particular, that suitably time-changed Brownian motions are continuous, local martingales. We now establish the important converse to this result.

Theorem 1.9. *Suppose M is an $\{\mathcal{F}_t\}$-continuous local martingale vanishing at 0 and such that $\langle M\rangle_\infty = \infty$, and let*

$$T_t \doteq \inf\{s : \langle M\rangle_s > t\}.$$

Then $B_t \doteq M_{T_t}$ is an $\{\mathcal{F}_{T_t}\}$-Brownian motion, and $M_t = B_{\langle M\rangle_t}$.

Proof. The family $T = \{T_t\}$ is a time-change that is a.s. finite since $\langle M\rangle_\infty = \infty$. Using the fact that the intervals of constancy of M and $\langle M\rangle$ are identical, it follows that M is T-continuous. Thus, by Proposition 1.8, B is a continuous $\{\mathcal{F}_{T_t}\}$-local martingale and

$$\langle B\rangle_t = \langle M\rangle_{T_t} = t.$$

By Lévy's characterization theorem, B is an $\{\mathcal{F}_{T_t}\}$-Brownian motion.

In order to show that $B_{\langle M\rangle} = M$, observe that $B_{\langle M\rangle} = M_{T_{\langle M\rangle}}$ and, although we may have $T > t$, it is always true that $M_{T_{\langle M\rangle_t}} = M_t$ due to the constancy of M on the level stretches of $\langle M\rangle$. □