Appendix A. Convex Analysis

A.1. **Properties of the Fenchel-Legendre transform.** Suppose $X_1 \sim \mu$, where $\mu \in M_1(\mathbb{R}^d)$ and let $\Lambda(\cdot)$ be the cumulant generating function (equivalently, log moment generating function).

Definition A.1. *The Fenchel-Legendre transform of* $\Lambda(\cdot)$ *is*

$$\Lambda^*(x) \doteq \sup_{\lambda \in \mathbb{R}^d} \left[\langle \lambda, x \rangle - \Lambda(\lambda) \right], \qquad x \in \mathbb{R}^d.$$

We collect some useful properties of Λ and Λ^* in the following proposition.

Proposition A.2. *The following properties hold when* $d = 1$*.*

(a) Λ *is a convex function.*

(b) Λ^* *is a convex, lower semi-continuous function. Moreover, if* $0 \in \mathcal{D}_\Lambda^\circ$*, then* Λ^* *has compact level sets.*

(c) *If* $\Lambda(\lambda) < \infty$ *for some* $\lambda > 0$*, then* $\overline{x} < \infty$ *(possibly* $\overline{x} = -\infty$*) and for all* $x \geq \overline{x}$*,*

$$\Lambda^*(x) = \sup_{\lambda \geq 0} [\lambda x - \wedge(\lambda)] \tag{A.62}$$

and Λ^* *is non-decreasing on* (x, ∞)*. Similarly, if* $\Lambda(\lambda) < \infty$ *for some* $x < 0$*, then* $\overline{x} > -\infty$ *(possibly* $\overline{x} = \infty$*) and for all* $x \leq \overline{x}$*.*

$$\Lambda(x) = \sup_{\lambda = 0} [\lambda x - \Lambda(\lambda)], \tag{A.63}$$

and Λ^* *is non-increasing on* $(-\infty, \overline{x})$*.*

(d) *If* $\overline{x} \in (-\infty, \infty)$*, then*

$$\Lambda^*(\overline{x}) = 0 \quad \text{and} \quad \inf_{x \in \mathbb{R}} \Lambda^*(x) = 0. \tag{A.64}$$

(e) $\Lambda(\cdot)$ *is differentiable in* D_Λ°*, with*

$$\frac{d\Lambda}{d\lambda}(\lambda) = \frac{1}{M(\lambda)} \int_{\mathbb{R}} x e^{\lambda x} \, d\mu \tag{A.65}$$

and, if $\eta = \arg\inf_\lambda \Lambda(\lambda)$ *then*

$$\frac{d\Lambda}{d\lambda}(\eta) = 0. \tag{A.66}$$

(f) *If* $0 \in \mathcal{D}_\Lambda^\circ$*, then* $\overline{x} = \int_{\mathbb{R}} x \, d\mu \in (-\infty, \infty)$*.*

References

[1] Karatzas, I. and Shreve, S. *Brownian motion and Stochastic Calculus*, Springer-Verlag, New York, 1988.