21-880: Advanced Stochastic Calculus II – Spring 2009
Homework Assignment 6
(Distributed Monday, April 13, 2009)
(Due by Wednesday, April 29, 2009)

Reading. This homework is based on probabilistic representations of stochastic partial differential equations and reflected diffusions. Read the related lecture notes and Chapter 4 and parts C an D of Chapter 3.6 of K-S

Problem 1. Consider an $N \times N$ matrix D with $D_{ii} = 1$ for every $i = 1, \ldots, N$, and suppose that

$$\sum_{j=1, j \neq i}^{N} |d_{ij}| < 1, \qquad \text{for every } i = 1, \ldots, N. \tag{0.1}$$

The ith column of D represents the direction of "reflection" or constraint that is applicable on the face

$$\{x \in \mathbb{R}^N_+ : x_i = 0\}.$$

Consider the following multi-dimensional analog of the Skorokhod problem. Given an $\mathbb{R}^N$-valued continuous function ψ, show that there exist a unique pair of continuous functions (ϕ, η) that satisfy the following properties:
(a) $\phi(t) \in \mathbb{R}^N_+$ for every $t \in [0, \infty)$;
(b) $\phi(t) = \psi(t) + D\eta(t)$ for every $t \in [0, \infty)$;
(c) for every $i = 1, \ldots, N$, η_i is non-decreasing and

$$\int_0^\infty \phi_i(t) d\eta_i(t) = 0.$$

Hint: Show that there exists a solution (ϕ, η) to the above problem if and only if each coordinate satisfies a certain one-dimensional Skorokhod problem. Use a suitable fixed-point argument to show that a unique solution to the latter exists.

Problem 2. Consider the Dirichlet problem in a domain G with boundary condition f. We showed in class that if f is bounded then any bounded solution to this Dirichlet problem admits the representation

$$u(x) = \mathbb{E}_x[f(X_\tau)]. \tag{0.2}$$

In particular, this establishes uniqueness among bounded solutions. Show by example that there can also exist unbounded solutions that need not be given by (0.2).
Hint: Take $D = \{(x_1, x_2) : x_2 > 0\}$ and $f(x_1, 0) = 0$ for $x_1 \in \mathbb{R}$.

Problem 3. Suppose that $f : (0, \infty) \mapsto \mathbb{R}$ is a Borel-measurable function that satisfies
$$\int_0^\infty e^{-ax^2} |f(x)| \, dx < \infty$$
for some $a > 0$, and define
$$u(t, x) = \mathbb{E}_x[f(|W_t|)], \quad 0 < t < \frac{1}{2a}, x > 0.$$
Show that u is of class $\mathcal{C}^{1,2}$ and satisfies the (forward) heat equation on $(0, 1/2a) \times (0, \infty)$,
$$f(x) = \lim_{t \downarrow 0, y \to x} u(t, y), \quad \forall x \in \mathbb{R},$$
and
$$\lim_{s \to t, x \downarrow 0} \frac{\partial}{\partial x} u(s, x) = 0; \qquad 0 < t < \frac{1}{2a}.$$

Problem 4. (clearly not for submission) Read through Section 4.4.A on the Feynman-Kac formula