21-880: Advanced Stochastic Calculus II – Spring 2009
Homework Assignment 5
(Distributed Wednesday, April 8, 2009)
(Due Monday, April 20, 2009)

Reading. This homework focuses on large deviations, optimal control and stochastic filtering. References include class notes and Oksendal, Chapters VI and XI. Only Problems 1–5 are for submission.

Problem 1. Suppose $\{X_i\}$ is a sequence of i.i.d. random variables, with $X_1 \sim \mathcal{N}(\mu, \sigma^2)$ for some $\mu, \sigma^2 > 0$, and let $S_n = \sum_{i=1}^n X_i$. Find

$$J \doteq -\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(|S_n| > 2\mu n).$$

Problem 2. (a) Suppose that $\mathcal{S}$ and $\mathcal{R}$ are two separable metric spaces and $f : \mathcal{S} \mapsto \mathcal{R}$ is a continuous function. Let I be a good rate function $I : \mathcal{S} \mapsto [0, \infty)$. Suppose that $\{\mu_n\}$ is a sequence of probability measures on $\mathcal{S}$ that satisfy an LDP on $\mathcal{S}$ with good rate function I. Show that the sequence of measures $\{\mu_n \circ f^{-1}\}$ satisfy an LDP on $\mathcal{Y}$. Explicitly identify the rate function of $\{\mu_n \circ f^{-1}\}$ in terms of I and f.

(b) Use the last problem to solve Problem 4b of the mid-term exam.

Problem 3. Consider the standard diffusion control problem, and let X^u denote the controlled process, when a Markov control u is being used. Suppose the objective is to minimize the long-run average cost per unit time:

$$V(x) = \inf_{u\in\mathcal{U}} \lim_{t\to\infty} \frac{\mathbb{E}_x\left[\int_0^t k(X_s^u)\, ds\right]}{t},$$

where $k : \mathbb{R}^n \mapsto \mathbb{R}_+$ is a continuous bounded function. Assuming some minimal regularity conditions on V (explicitly state the conditions you assume), derive the HJB equation that V should satisfy.

Problem 4. Write down the HJB equation for the problem

$$V(t, x) \doteq \inf_u \mathbb{E}_{t,x}\left[\int_t^\infty e^{-\alpha s}\left(\phi(X_s) + u_s^2\right)\, ds\right],$$

where, with B a standard 1-dimensional Brownian motion,

$$dX_t = u_t\, dt + dB_t.$$

Show that if $V \in \mathcal{C}^2$ and the optimal control u^* exists, then

$$u^*(t, x) = -\frac{1}{2} e^{\alpha t} \frac{\partial V}{\partial x}.$$

Problem 5. Consider a filtering problem in which the signal is a (constant in time) random-variable: $dX_t = 0$, $\mathbb{E}[X_0] = 0$ and $\mathbb{E}[X^2] = a^2 > 0$, while the observation process is

$$dZ_t = G(t)X_t\,dt + dV_t, \quad Z_0 = 0,$$

where V is a one-dimensional Brownian motion. A filter is said to provide an *exact asymptotic estimation* of the signal if $\mathbb{E}[|X_t - \hat{X}_t|^2] \to 0$ as $t \to \infty$. Find a criterion on G for the above filter to provide an exact asymptotic estimation.

Problem 6. (*not for submission*) Show that the stopping rules N' and N'' in the proofs of Lemmas 6.1 and 6.2 of the notes are regular.