21-880: Advanced Stochastic Calculus II – Spring 2009
Homework Assignment 4
(Distributed Sunday, March 29, 2009)
(Due Wednesday, April 8, 2009)

Reading. This homework focuses on optimal stopping. References include class notes and Oksendal, Chapter X.

Problem 1. Consider the classical secretary problem (CSP) with n applicants as described in class, but with the modification that the number, K, of applicants is unknown, but is instead assumed to be random with a uniform distribution on $\{1, \ldots, n\}$. The assumptions are the same as in the CSP, but if you pass up an applicant and it turns out that there are no more, you lose. You want to select the best of K applicants.

(a) Find y_j, for $j = 1, \ldots, n$.
(b) Show there is a threshold rule that is optimal.
(c) Find the probability of win using an arbitrary threshold rule.
(d) Find approximately for large n, the optimal threshold rule and the optimal reward.

Problem 2. Prove that the only nonnegative superharmonic functions in $\mathbb{R}$ are the constants and use this to find g^*, when

$$g(x) = \begin{cases} xe^{-x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$

Problem 3. (a) Let Y be the solution to the following SDE:

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t,$$

where b, σ are bounded and continuous. Let $\psi : \mathbb{R}^n \mapsto \mathbb{R}_+$ be continuous and let $\phi : \mathbb{R}^n \mapsto \mathbb{R}_+$ be Lipschitz continuous with at most linear growth. Consider the general optimal stopping problem of finding

$$\gamma(x) = \sup_\tau \mathbb{E}_x \left[\int_0^\tau \phi(Y_s)\, ds + \psi(Y_\tau) \right], \qquad x \in \mathbb{R}^n.$$

Convert this into a time-homogeneous optimal stopping problem: specifically, show that

$$\gamma(x) = \sup_\tau \mathbb{E}_x[g(X_\tau)]$$

for a suitable function g and process X.

(b) Let B be Brownian motion and let $\rho > 0$. Use the formulation in part (a) and a further analysis (similar to what was carried out for the stock-selling problem) to solve the optimal stopping problem

$$\gamma(x) = \sup_{\tau} \mathbb{E}_x \left[\int_0^\tau e^{-\rho t} B_t^2 \, dt + e^{-\rho\tau} B_\tau^2 \right].$$

Problem 4. Let $\{X_t\}$ be an RCLL time-homogeneous $\mathbb{R}^d$-valued Markov process, with infinitesimal generator

$$\mathcal{L}_X f(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}_x[f(X_t)] - f(x)}{t}, \quad x \in \mathbb{R}^d,$$

for $f \in \mathcal{D}_X$, where the domain $\mathcal{D}_X$ is the set of Borel measurable f for which the above limit exists. Let g be a non-negative continuous function such that the following integrability condition is satisfied for a suitable class $\mathcal{N}$ of stopping times:

$$\sup_{\tau \in \mathcal{N}} \mathbb{E}_x \left[\sup_{t \le \tau} |g(X_t)| \right] < \infty.$$

Moreover, define

$$g^*(x) \doteq \sup_{\tau \in \mathcal{N}} \mathbb{E}_x \left[g(X_\tau) \right],$$

$$D \doteq \{x : g(x) < g^*(x)\},$$

and

$$A \doteq \{x : g(x) = g^*(x)\}.$$

(a) Show that if $g^* \in \mathcal{D}_X$, then

$$\begin{aligned} \mathcal{L}_X g^*(x) &\le 0 \quad x \in \mathbb{R}^d \\ g^*(x) &\ge g(x) \quad x \in \mathbb{R}^d \\ \mathcal{L}_X g^*(x) &= 0 \quad x \in D \\ g^*(x) &= g \quad x \in A \end{aligned} \tag{0.1}$$

(b) This problem serves to explain how the above can be used to solve the optimal stopping problem. Suppose there exists a closed set $\overline{A}$ with $\mathbb{P}(\sigma_{\overline{A}} < \infty) = 1$, where

$$\sigma_{\overline{A}} \doteq \inf\{t > 0 : X_t \in \overline{A}\},$$

and a function V such that (0.1) holds with g^* and A replaced by V and $\overline{A}$, respectively. Then show that $V(X_{t \wedge \sigma_{\overline{A}}})$ is a martingale and $V(X_t)$ is a supermartingale. Conclude that $\sigma_{\overline{A}}$ must be an optimal stopping time.

Remark: A common approach to solving optimal stopping problems is then to "guess" what $\overline{A}$ may be, and then define

$$V(x) \doteq \begin{cases} \mathbb{E}_x \left[g(X_{\sigma_{\overline{A}}} \right] & x \in D, \\ g(x) & x \in A. \end{cases}$$

and then verify that V satisfies

$$\begin{aligned} \mathcal{L}_X V &\le 0 \quad x \in \mathbb{R}^d, \\ V &\ge g \quad x \in \mathbb{R}^d. \end{aligned}$$

In particular, when X is continuous, $\overline{A} = \partial D$ and so, for the correct choice of D, V satisfies the Dirichlet problem

$$\begin{aligned} &\mathcal{L}_X V(x) = 0 \quad x \in D, \\ &V(x) = g(x) \quad x \in \partial D. \end{aligned} \tag{0.2}$$

(c) Consider the problem

$$V(x) = \sup_{\tau} \mathbb{E}_x \left[e^{-\lambda_\tau} H(X_\tau) \right],$$

where $\lambda_t = \int_0^t \ell(X_s)\, ds$, for a measurable function $\ell(\cdot)$. Analogous to (0.2), find a PDE that can be used to identify V.