21-880: Advanced Stochastic Calculus II – Spring 2009
Homework Assignment 3
(Distributed Friday, February 20, 2009)
(Due Monday, March 2, 2009)

Reading. This homework is based on applications to finance, large deviations and existence and uniqueness of stochastic differential equations (with reflection).

Problem 1. Suppose $b : [0,\infty) \mapsto \mathbb{R}$ and $\sigma : [0,\infty) \mapsto \mathbb{R}$ are such that b and σ are Lipschitz continuous on every finite interval. Given a standard Brownian motion W on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{F}_t$ the augmented filtration generated by Brownian motion, state whether there exists an $\mathcal{F}_t$-adapted process X such that $\mathbb{P}$ a.s.,

$$X(t) = X(0) + \int_0^t b(X(s))\,ds + \int_0^t \sigma(X(s))\,dW(s) + Y(t),$$

$X(t) \geq 0$ for all $t \in [0,\infty)$, $Y(0) = 0$, Y is non-decreasing and

$$\int_{(0,\infty)} X(s)\,dY(s) = 0.$$

State whether X is pathwise unique or unique in distribution. Provide rigorous proofs of your answer (you may appeal to specific theorems or estimates in the book or theorems covered in class to make your proof concise).

Problem 2. (Black and Scholes (1973) option pricing formula) Suppose P_0 and P_1 represent the price of a bond and a stock, respectively. Given $T < \infty$, $\sigma, r > 0$, suppose that P_0 and P_1 evolve on $[0,T]$ according to

$$P_0(t) = p_0 e^{rt}$$

and

$$dP_t(t) = b_1(t)P_1(t)\,dt + \sigma P_1(t) dW_t = rP_1(t)\,dt_\sigma P_1(t) d\tilde{W}_t.$$

Write down the valuation process $\{X_t\}$ associated with the option price with strike price q and maturity T. (The valuation process is the wealth process associated with the hedging strategy against the contingent claim.) Suppose v is a function on $[0,T] \times \mathbb{R}_+$ that satisfies the Cauchy problem

$$-\frac{\partial v}{\partial t} + rv = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x}$$

on $[0,T) \times (0,\infty)$ with

$$v(T,x) = (x-q)^+; \qquad x \geq 0,$$

and the polynomial growth condition

$$\max_{t\in[0,T]} |v(t,x)| \leq M(1+|x|^{2\mu}, x \in \mathbb{R},$$

for some $M > 0$ ad $\mu \geq 1$. Then use Itô's formula and the Marov property to show that

$$X_t = v(t, P_1(t)), \qquad t \in [0,T].$$

Black and Scholes explicitly solved for v.

Problem 3. A stationary O-U process is a Gaussian process with covariance function $C(t,t') = c\exp^{-\beta|t-t'|}$, with $c, \beta > 0$: β is called the parameter and c the size of the OU process.

(a) Prove that the process

$$X_t = e^{-\lambda t} W_{\exp(2\lambda t)}$$

is a stationary Ornstein-Uhlenbeck (OU) process. Compute its parameter and its size.

(b) Let X be a continuous O-U process with parameter 1/2 and size 1, and set

$$\beta_t = \begin{cases} X_t + \dfrac{1}{2}\displaystyle\int_0^t X_u\,du & t \geq 0 \\ X_t - \dfrac{1}{2}\displaystyle\int_0^t X_u\,du & t < 0.. \end{cases}$$

Prove that β is a Gaussian process with continuous paths. Is β a Brownian motion ?

Remark. The OU process was originally introduced as a model for the velocity of "physical" Brownian motion. The location of a particle with this velocity is then given by the integral of the OU process. The last exercise helps explore the relation between the "physical model" of Brownian motion as the integral of the OU process and the usual "mathematical model" of Brownian motion.

Problem 4.