21-880: Advanced Stochastic Calculus II – Spring 2009
Homework Assignment 2

(Distributed Wednesday, February 18, 2009)
(Due Wednesday, February 25, 2009)

Reading. This homework is based on the Girsanov change of measure, time changes and existence and uniqueness of stochastic differential equations (with reflection). Submit only the first four problems.

Problem 1. Let X be a d-dimensional Brownian motion with drift $\mu \neq 0$ and define

$$T \doteq \inf\{t > 0 : |X|(t) = 1\}.$$

Show that X_T is independent of T.
Hint: First establish the analogous result for the case of Brownian motion without drift. Use Girsanov's theorem to show that if the property is true for Brownian motion without drift, then it is true for Brownian motion with drift.

Problem 2. Let W be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}_t$ the associated augmented filtration and let $\tau_0 \doteq \inf\{t > 0 : W(t) = 0\}$. Then it can be shown that for any $z \in \mathbb{R}_+$, $R \in (0, \infty)$ and $\varepsilon \in [0, 1)$,

$$\mathbb{E}_z \left[\int_0^{t \wedge \tau_0} \frac{\mathbb{I}_{[0,R]}(W(s))}{W^{1+\varepsilon}(s)} \, ds \right] < \infty, \tag{0.1}$$

where $\mathbb{E}_z$ denotes expectation under $\mathbb{P}$, conditioned on $W(0) = z$. Let b be an $\mathcal{F}_t$-adapted, real-valued process and let M be a continuous $\mathcal{F}_t$-martingale with a continuously differentiable quadratic variation process $V = \langle M \rangle$, such that for some constant $\theta < \infty$,

$$\sup_{s \in [0,\infty)} b(s) \leq \theta \qquad \text{and} \ \frac{1}{\theta} \leq \inf_{s \in [0,\infty)} V'(s) < \sup_{s \in [0,\infty)} V'(s) \leq \theta,$$

where V' is the pathwise derivative of V. Define

$$B \doteq B(0) + \int_0^{\cdot} b(s) \, ds + M,$$

and $T_0 \doteq \inf\{t \geq 0 : B(t) = 0\}$. Use (0.1), along with time-change arguments and Girsanov's theorem, to show that for any $z \in \mathbb{R}_+$ and $R \in (0, \infty)$,

$$\mathbb{E}_z \left[\int_0^{t \wedge T_0} \frac{\mathbb{I}_{[0,R]}(B(s))}{B(s)} \, ds \right] < \infty \qquad \text{for } t \in [0, \infty),$$

where $\mathbb{E}_z$ denotes expectation conditioned on $B(0) = z$. This problem illustrates how (under suitable non-degeneracy conditions) estimates for general diffusions can often be reduced to an estimate involving just Brownian motion.

Problem 3. In this problem, you will show the remarkable fact that the Bessel-3 process can be viewed as "Brownian motion always conditioned to be positive."

Suppose $\{B_s\}$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with $B_0 = x > 0$. Let $M_t = B_t$, let $T_0 \doteq \inf\{s > 0 : B_s = 0\}$ and let $\{\mathcal{F}_s\}$ be the augmentation of the filtration generated by $\{B_s\}$. For $t \in (0, \infty)$, let $\mathbb{Q}_t$ be the measure on path space such that

$$\frac{d\mathbb{Q}_t}{d\mathbb{P}}_{|\mathcal{F}_t} = \frac{1}{x} M_{t \wedge T_0},$$

and let $\mathbb{Q}_\infty$ be the measure on $\mathcal{F}_\infty$ that coincides with $\mathbb{Q}_t$ on $\mathcal{F}_t$.

(a) Argue that $\mathbb{Q}_t$ (respectively, $\mathbb{Q}_\infty$) is a well-defined probability measure on $\mathcal{F}_t$ (respectively, $\mathcal{F}_\infty$).

(b) Show that, under $\mathbb{Q}_t$, $\{B_s\}$ is a 3-dimensional Bessel process on $[0, t]$.

(c) Show that $\mathbb{Q}_\infty$ coincides with the conditional measure that usual Brownian motion is conditioned to stay positive. For this, first show that the meaasure $\mathbb{Q}_t$ is concentrated on paths with $T_0 > t$ and so, for any $r \in (x, \infty)$ and $T_r = \inf\{t > 0 : B_t = r\}$. the measure $\mathbb{Q}_t$ is concentrated on paths with $t \wedge T_r < T_0$. Next, note that

$$\lim_{t \to \infty} \mathbb{Q}_t(T_r < t) = 1.$$

Then, show that this is the same as the conditional measure on usual Brownian motion given that $T_r < T_0$. Lastly, complete the argument.

Problem 4. Let B be Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{\mathcal{F}_t\}$ be the associated augmented filtration. Suppose $\{M_t, \mathcal{F}_t\} \in \mathcal{M}^{c,loc}$ is such that $\exp\left(M_t - \frac{1}{2}\langle M \rangle_t\right)$ is a martingale under $\mathbb{P}$, and define $\tilde{\mathbb{P}}_T(A) = \mathbb{E}[\mathbb{I}_A Z_T]$ for $A \in \mathcal{F}_T$. Show that, for any $\mathcal{F}_t$-measurable random variable Y such that $\tilde{E}[|Y|] < \infty$, and $0 \leq s \leq t \leq T$,

$$\tilde{\mathbb{E}}[Y | \mathcal{F}_s] - \frac{1}{Z_s} \mathbb{E}[Y Z_t | \mathcal{F}_s].$$