21-880: Advanced Stochastic Calculus II – Spring 2008
Homework Assignment 1

(Distributed Monday, February 2, 2009)
(Due Wednesday, February 11, 2008)

Reading. This homework is based on time-changes and representations for continuous martingales (read Section 3.4 of Karatzas-Shreve). Also, read the handout on time-changes – in particular, Proposition 1.8 states a more general result than the time-change theorem (Theorem 1.9) that was discussed in class.

Please submit the first four problems only.

Prob. 1. This problem serves to demonstrate that continuous local martingales need not remain continuous local martingales under arbitrary time-changes – i.e., the conditions in Proposition 1.5 are necessary. Let $\{X_t, \mathcal{F}_t\}$ be a Brownian motion. Identify an adapted process $\{A_t, \mathcal{F}_t\}$ such that, with C being the time-change

$$C_t \doteq \inf\{s > 0 : A_s > t\}, \quad t \in [0, \infty),$$

and $\widehat{X}_t \doteq X_{C_t}$, $\widehat{F}_t \doteq \widehat{F}_{C_t}$ for $t \in [0, \infty)$, $\{\widehat{X}_t, \widehat{F}_t\}$ is not a local martingale. State whether it is a semimartingale.

Prob. 2. Let B be standard Brownian motion starting at 0, and let $X_t = B_t + t$, $t \in [0, \infty)$.

(a) Show that the process $M_t = \exp(-2X_t)$ is a time-changed Brownian motion starting at 1, which is killed (stopped) when it first hits zero.
(b) Define $\gamma = \inf_t X_t$. Use the result in (a) to prove that $-\gamma$ is exponentially distributd with parameter 2.
(c) Can you provide another proof of (b), which does not use the time-change ?

Prob. 3. In what follows, for any process H, H^* represents the associated supremum process. Suppose you are told that there exist (universal) constants $c_p, C_p \in (0, \infty)$ such that for any Brownian motion $\{B_t, \mathcal{G}_t\}$ and a $\mathcal{G}_t$-stopping time T,

$$c_p \mathbb{E}[T^{p/2}] \leq \mathbb{E}[(B_T^*)^p] \leq C_p \mathbb{E}[T^{p/2}]. \tag{0.1}$$

Conclude from this the following important BDG inequalities:
For every $p \in (0, \infty)$, there exist two constants $c_p, C_p \in (0, \infty)$ such that, for all continuous local martingales M with $M(0) = 0$,

$$c_p \mathbb{E}\left[\langle M \rangle_\infty^{p/2}\right] \leq \mathbb{E}\left[(M_\infty^*)^p\right] \leq C_p \mathbb{E}\left[\langle M \rangle_\infty^{p/2}\right]. \tag{0.2}$$

Remark: This is a simple exercise, but it is important for you to remember the BDG inequalities (0.2). The student that wants to be challenged may try to prove (0.1) – if you want to try this, come to me for hints.

Prob. 4. (*Review Problem.*) Let X be Brownian motion with drift μ. Let $f = f(x,t)$ be a $\mathcal{C}^\infty$ function (in both x and t). Under what conditions on f is the process $Z_t = f(|X_t|, t)$ (a) a local martingale; (b) a supermartingale? Justify your answer.

Prob. 5. (*Review Problem.*) Show that a non-negative local martingale is a supermartingale.

Prob. 6. (*Review Problem.*) Justify why the optional sampling theorem can be applied to a non-negative martingale.

Prob. 7. Let $(\Omega, \mathcal{F}^0, \{\mathcal{F}^0_t\})$ be a filtered probability space, let $\tilde{\mathbb{P}}$ and $\mathbb{P}$ be equivalent measures on $(\Omega, \mathcal{F}^0)$ and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ be the $\mathbb{P}$-augmentation of $(\Omega, \mathcal{F}^0, \{\mathcal{F}^0_t\})$. Convince yourself that $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent on $(\Omega, \mathcal{F})$. Define

$$Z_t \doteq \left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t}, \quad t \in [0, \infty).$$

Show that $\{Z_t\}$ is a uniformly integrable $(\mathbb{P}, \{\mathcal{F}_t\})$ martingale.