2.2 Stopping Times

We are often interested in events that occur at a random time. A random time is simply a $[0,\infty]$-valued random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A very special class of random times is the so-called class of *stopping times.* Many properties of stochastic processes that hold for deterministic times (the martingale property, the Markov property) can be extended to stopping times, though not more general random times. Stopping times are random times that are, in a sense, adapted to the filtration $\{\mathcal{F}_t\}$. Intuitively, a random time is a stopping time for a filtration $\{\mathcal{F}_t\}$ if one can determine whether or not the random time occurs before a certain deterministic time t by observing the past upto time t, which is encoded in the filtration $\{\mathcal{F}_t\}$. We now provide a rigorous definition.

Definition 2.2.1. *Let $(\Omega, \mathcal{F})$ be a measurable space, equipped with a filtration $\mathcal{F}_t, t \in \mathcal{T}$. A random time T is a* stopping time *for the filtration $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ iff $\{T \leq t\} \in \mathcal{F}_t$ for every $t \in \mathcal{T}$. A random time T is said to be an* optional time *for the filtration $\{\mathcal{F}_t\}$ if $\{T < t\} \in \mathcal{F}_t$ for every $t \in \mathcal{T}$.*

Remark 2.2.2. The difference between an optional time and a stopping time disappears if the filtration is right-continuous. Indeed, suppose T is an optional time. Then, for every $t > 0$,

$$\{T \leq t\} = \bigcap_{\varepsilon>0} \{T < t + \varepsilon\} \subset \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t,$$

and so T is an optional time. On the other hand, if T is a stopping time, it is easy to see that it is also an optional time since $\{T < t\} = \bigcup_{n=1}^{\infty} \{T \leq t - 1/n\} \subset \sigma\left(\cup_{n=1}^{\infty} \mathcal{F}_{t-1/n}\right) \subseteq \mathcal{F}_t$. As we have seen, the natural filtration $\{\mathcal{F}_t^B\}$ of Brownian motion is not right-continuous even though Brownian motion is a continuous process. However, it turns out that this filtration can be augmented so as to make it right-continuous, in such a way that Brownian motion still remains a martingale with respect to this filtration. So, in this class, we will mainly deal with processes adapted to filtrations that are right-continuous and, in fact, satisfy the usual conditions, as defined below.

Definition 2.2.3. *Suppose T is a stopping time for the filtration $\{\mathcal{F}_t\}$. The σ-field $\mathcal{F}_T$ is defined to be the set of events $A \in \mathcal{F}$ such that $A \cap \{T \leq t\} \in \mathcal{F}_t$ for every $t \in [0, \infty)$. $\mathcal{F}_T$ can be viewed as the set of events determined prior to the stopping time T. Similarly, if T is an optional time, then the σ-field $\mathcal{F}_{T+}$ is defined to be the set of events $A \in \mathcal{F}$ such that $A \cap \{T \leq t\} \in \mathcal{F}_{t+}$ for every $t \in [0, \infty)$.*

Definition 2.2.4. *A filtration $\{\mathcal{F}_t\}$ is said to* satisfy the usual conditions *if it is right-continuous and $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible events in $\mathcal{F}$.*

Recall the definitions of measurability and progressive measurability of stochastic processes given in Definition 1.2.6. Such restrictions of joint measurability properties of the stochastic process, viewed as a function on $([0, \infty) \times$

$\Omega, \mathcal{B}[0,\infty) \times \mathcal{F})$, are particularly relevant in the context of stopping times, for example to ensure that the "stopped process" is a well-defined stochastic process. We will mainly be considering adapted stochastic processes that a.s. have either right-continuous or left-continuous paths. In this case, they are automatically progressively measurable and so the stopped process always makes sense.

Definition 2.2.5. *Given a stochastic process $\{X_t\}$ and a random time T on $(\Omega, \mathcal{F}, \mathbb{P})$, the function X_T on Ω is defined by*

$$X_T(\omega) \doteq \begin{cases} X_{T(\omega)}(\omega) & \text{if } T(\omega) < \infty \\ \infty & \text{otherwise.} \end{cases}$$

Note that it is a random variable (taking values in the extended reals) on $(\Omega, \mathcal{F}, \mathbb{P})$. Given a progressively measurable stochastic process $\{X_t, \mathcal{F}_t\}$ and a stopping time T, the stopped process $\{X_t^T\}$ is defined by

$$X_t^T = X_{T\wedge t}, \quad t \in [0,\infty).$$

The reader should check that if T is a stopping time, then X_T is $\mathcal{F}_T$-measurable, and if $\{X_t, \mathcal{F}_t\}$ is progressively measurable, then $\{X_t^T, \mathcal{F}_t\}$ is also a progressively measurable stochastic process. Also, the reader should convince themselves that if $S \le T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.

We will close this section by providing some simple examples of optional and stopping times.

Definition 2.2.6. *Given a stochastic process $\{X_t\}$ taking values in a measurable space $(S, \mathcal{S})$, for $A \in \mathcal{S}$, define the* début *or first-entrance time of A to be*

$$D_A(\omega) \doteq \inf\{t \ge 0 : X_t(\omega) \in A\}$$

and the hitting time *of A to be*

$$H_A(\omega) \doteq \inf\{t > 0 : X_t(\omega) \in A\}.$$

This definition of the hitting time may be a bit bizarre, but it is the one that is required when stating connections with potential theory.

Lemma 2.2.7. *Let $\{X_t\}$ be a continuous, $\mathcal{F}_t$-adapted process taking values in a metric space (S, d). If X is continuous, then the first entrance time into a closed set A is an $\mathcal{F}_t$ stopping time. The first entrance time into an open set by a right-continuous process is an $\mathcal{F}_{t+}$ stopping time.*

Proof. Since the mapping $x \mapsto d(x, A)$ is continuous, the mapping $\omega \mapsto d(X_q(\omega), A)$ is $\mathcal{F}_q$-measurable for every $q \in \mathbb{Q}_+$. But, since X has continuous paths,

$$\{\omega : D_A(\omega) \le t\} = \{\omega : \inf\{d(X_q(\omega), A) : q \in \mathbb{Q}_+ \cap [0,t]\} = 0\},$$

and so $\{D_A \le t\} \in \mathcal{F}_t$.

For the second assertion, note that due to the right-continuity of paths,

$$\{D_A < t\} = \bigcup_{q<t, q\in\mathbb{Q}_+} \{X_q \in A\} \in \mathcal{F}_t,$$

which shows that D_A is an optional time or, equivalently, that D_A is an $\mathcal{F}_{t+}$-stopping time. □

Remark 2.2.8. Note that the second assertion above holds for any topological space $(S, \mathcal{B}(S))$. Also, even if all the paths of X are continuous, D_A need not be an $\mathcal{F}_t$-stopping time when A is open. Indeed, to build intuition, consider the case when X is real-valued and that for some ω, $X_t(\omega) \leq 1$ and that $X_1(\omega) = 1$. Let $A = (1, \infty)$. We cannot tell without looking slightly ahead of time 1 whether or not $D_A = 1$.

2.3 The Optional Sampling Theorem

We now turn to the question of whether the martingale property can be extended from discrete times to random times. As we will see below, under certain additional conditions on the process, the answer is in the affirmative if the random time is a stopping time. Throughout this section, we assume that the filtration satisfies the usual conditions.

Theorem 2.3.1. *Let M be a uniformly integrable martingale and T be a stopping time. Then $\mathbb{E}[M_\infty \mid \mathcal{F}_T] = M_T$ a.s. Moreover, as a result, $\mathbb{E}[|M_T|] < \infty$ and $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.*

Theorem 2.3.2. (Discrete Optional Sampling Theorem) *If M is a uniformly integrable submartingale and S and T are two stopping times such that $S \leq T$ then $\mathbb{E}[M_T \mid \mathcal{F}_S] = M_S$.*

For proofs, refer to [8]. We now state and prove the continuous optional sampling theorem. Recall that $\mathcal{F}_\infty = \sigma\left(\bigcup_{t\in[0,\infty)} \mathcal{F}_t\right)$.

Definition 2.3.3. *A submartingale $\{X_t, \mathcal{F}_t, t \in [0,\infty)\}$ is said to have a last element if there exists an $\mathcal{F}_\infty$-measurable random variable X_∞ such that*

$$\mathbb{E}[X_\infty | \mathcal{F}_t] \geq X_t, \quad \mathbb{P} \text{ a.s.}$$

Theorem 2.3.4. (Continuous Optional Sampling Theorem) *Let $\{X_t, \mathcal{F}_t\}$ be a right-continuous sub-martingale with a last element and let S and T be $\mathcal{F}_t$-stopping times. Then $\mathbb{E}[X_T \mid \mathcal{F}_S] \geq X_S$ a.s. If S and T are optional times then the above holds if we replace $\mathcal{F}_S$ by $\mathcal{F}_{S+}$.*

Proof. The basic idea is to approximate the stopping times by discrete stopping times, use the discrete optional sampling theorem, and then take limits. Define

$$S_n \doteq \begin{cases} \infty & S = \infty \\ k2^{-n} & (k-1)2^{-n} \leq S < k2^{-n} \end{cases}$$

and define T_n similarly. Then S_n and T_n are stopping times, $S_n \searrow S$ and $T_n \searrow S$ and, clearly $S_n \leq T_n$. Moreover, by the discrete optional sampling theorem, $\mathbb{E}[X_{T_n} \mid \mathcal{F}_{S_n}] \geq X_{S_n}$ or, in other words, for all $A \in \mathcal{F}_{S_n}$,

$$\int_A X_{T_n} d\,\mathbb{P} \geq \int_A X_{S_n} d\,\mathbb{P}\,. \tag{2.2}$$

Therefore the above inequality also holds for all $A \in \bigcap_{n=1}^{\infty} \mathcal{F}_{S_n} = \mathcal{F}_{S+}$. Also, since $S \leq S_n$, $\mathcal{F}_S \subseteq \mathcal{F}_{S_n}$.

Observe that since S_n is a decreasing sequence, $\{X_{S_n}, \mathcal{F}_{S_n}\}$ is a backward sub-martingale and, moreover, $\mathbb{E}[X_{S_n}]$ is decreasing and bounded below by $\mathbb{E}[X_0]$. Therefore, by Theorem 2.1.7, the sequence $\{X_{S_n}\}$ is uniformly integrable, and likewise it can also be argued that $\{X_{T_n}\}$ is uniformly integrable. Also, since the process X is right continuous, $X_S = \lim_{n\to\infty} X_{S_n}$, $X_T = \lim_{n\to\infty} X_{T_n}$, and so we can take limits in the equation above as $n \to \infty$ and interchange the limits with expectation/integration due to the proved uniform integrability to conclude that $\mathbb{E}[X_T \mid \mathcal{F}_{S+}] \geq X_S$. If S and T are stopping times, X_S is $\mathcal{F}_S$-measurable, and so taking expectations with respect to $\mathcal{F}_S$ on both sides of the last equality, we obtain $\mathbb{E}[X_T \mid \mathcal{F}_S] \geq X_S$. ☐

We now provide some applications of the optional sampling theorem.

Lemma 2.3.5. *Let* $\tau \doteq \inf\{t \geq 0 \mid B_t \notin (a,b)\}$, *where* $a < 0 < b$. *Then*

1. $\mathbb{P}(B_\tau = b) = \dfrac{-a}{b-a}$;
2. $\mathbb{E}[\tau] = -ab = |ab|$.

Proof. Since B is a continuous process and the complement of (a,b) is closed, the first entrance time τ into the complement of (a,b) is a stopping time. In addition, since the paths of Brownian motion are a.s. unbounded, $\mathbb{P}(\tau < \infty) = 1$. However, we cannot naively apply the Optional Sampling Theorem (OST) since Brownian motion does not have a last element (indeed, this would imply that Brownian motion is uniformly integrable, which is easy to see that it is not by a direct calculation using the normal density function). Instead, we look at the *stopped process* $\{B_{t\wedge\tau}\}$, which is a right continuous martingale that has a last element (since it is in fact uniformly bounded). So, applying the OST to the stopped process, we get

$$0 = \mathbb{E}[B_0] = \mathbb{E}[B_{\tau\wedge n}] = b\,\mathbb{P}[B_\tau = b, \tau \leq n] + a\,\mathbb{P}[B_\tau = a, \tau \leq n] + \mathbb{E}[B_n; \tau > n].$$

Taking limits as $n \to \infty$ and using the fact that $\mathbb{P}(\tau < \infty) = 1$, we obtain

$$0 = b\,\mathbb{P}[B_\tau = b] + a\,\mathbb{P}[B_\tau = a]$$

and the first property is established.

For the second property, we first observe that $M_t := (B_t - a)(b - B_t) + t$ is a martingale since it equals $t - B_t^2 + (a+b)B_t - ab$ since it is a linear combination of the martingales $B_t^2 - t$ and B_t. Moreover, for every $n \in \mathbb{N}$, the stopped process $\{M_{\tau\wedge t}, t \leq n\}$ is clearly uniformly bounded and thus uniformly integrable. Thus, applying the OST to this stopped process, we conclude that

$$-ab = \mathbb{E}[M_0] = \mathbb{E}[M_{\tau\wedge n}] = \mathbb{E}[\tau \wedge n] + \mathbb{E}[(B_{\tau\wedge n} - a)(b - B_{\tau\wedge n})].$$

Now $\tau \wedge n$ is monotonic in n and $(B_{\tau\wedge n} - a)(b - B_{\tau\wedge n})$ is uniformly bounded. Therefore, taking limits as $n \to \infty$, using the fact that $\mathbb{P}(\tau < \infty) = 1$ and applying the Monotonic Convergence and Bounded Convergence Theorems, we obtain

$$-ab = \mathbb{E}[\tau] + \mathbb{E}[(B_\tau - a)(b - B_\tau)].$$

However, the last expectation is zero since $B_\tau \in \{a, b\}$, and so the resutl follows. □

We now consider another application. Let $X_t = B_t + ct$ be Brownian motion with drift. We are interested in $H_x = \inf\{t \geq 0 \mid X_t = x\}$. As before, since X is continuous and $\{x\}$ is closed, H_x is a stopping time. We will now calculate the Laplace transform of the distribution of H_x.

Lemma 2.3.6. *If $x > 0$, then for every $\lambda > 0$,*

$$\mathbb{E}[e^{-\lambda H_x}] = \exp(-x(\sqrt{c^2 + 2\lambda} - c)). \tag{2.3}$$

In particular,

$$\mathbb{P}(H_x \in dt) = \frac{x}{\sqrt{2\pi t^3}} \exp\left(-\frac{(x - ct)^2}{2t}\right),$$

and

$$\mathbb{P}[H_x < \infty] = \begin{cases} 1 & \text{if } c \geq 0 \\ e^{-2|c|x} & \text{if } c < 0 \end{cases}$$

Proof. Here, we use the fact that $\exp(\theta B_t - \frac{1}{2}\theta^2 t)$ is a martingale. Fix $\lambda > 0$. Then from the exponential martingale it follows that

$$\exp(\theta X_t - \lambda t) = \exp(\theta B_t - (\lambda - \theta c)t)$$

is a martingale provided that $\lambda - \theta c = \frac{1}{2}\theta^2$. Let $\alpha = -c - \sqrt{c^2 + 2\lambda}$ and $\beta = -c + \sqrt{c^2 + 2\lambda}$ be the roots of this quadratic equation, and note that $\alpha < 0 < \beta$. In order to apply the OST, we should choose the martingale that is bounded on $[0, H_x]$. Since $x > 0$, it is clear that we should choose β since then the martingale $\exp(\beta X_t - \lambda t)$ is bounded by $\exp(\beta x)$ on $[0, H_x]$. Therefore, we can use the OST to conclude that

$$1 = \mathbb{E}[\exp(\beta X_{H_x} - \lambda H_x)] = e^{\beta x}\, \mathbb{E}[e^{-\lambda H_x}],$$

from which (2.3) follows. The Laplace transform can be inverted explicitly to give the probability distribution (we omit the proof of this since it is not probabilistic in content). Finally, the last assertion follows by taking limits as $\lambda \downarrow 0$ in

$$\mathbb{E}[e^{-\lambda H_x}] = \mathbb{E}\left[e^{-\lambda H_x} \mathbb{I}_{\{H_x < \infty\}}\right],$$

and invoking the Bounded Convergence Theorem. □