Chapter 2

Martingales

Martingales are adapted stochastic processes for which the conditional expected value of the process at some time t, given the past observations upto some earlier time s, is equal to the observation at the earlier time s. Martingales and submartingales can, roughly speaking, be thought to be the stochastic analogs of constants and increasing functions, respectively. We will show that martingales will serve as building blocks for more complicated stochastic processes. Martingales also arises naturally in applications. Originally, the term "martingale" referred to a rein to hold a horse, and was the name given to a class of betting strategies that was popular in 18th century France. The concept of a martingale was introduced into probability theory by Paul Lévy, and Joseph Doob contributed greatly to the early development of martingale theory. One of the early motivations for studying martingales was to show the impossibility of successful betting strategies.

2.1 Basic Definitions and Properties

In what follows, let $\mathcal{T}$ be a subset of $\mathbb{R}_+$.

Definition 2.1.1. *An adapted stochastic process $\{X_t, \mathcal{F}_t \mid t \in \mathcal{T}\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}\,|X_t| < \infty$ for all $t \in \mathcal{T}$, is said to be a*

1. sub-martingale *if $\mathbb{E}[X_t \mid \mathcal{F}_s] \geq X_s$ for all $s < t, s, t \in \mathcal{T}$;*
2. super-martingale *if $\mathbb{E}[X_t \mid \mathcal{F}_s] \leq X_s$ for all $s < t, s, t \in \mathcal{T}$.*
3. martingale *if it is both a sub- and super-martingale.*

When the parameter set $\mathcal{T}$ is countable, it is referred to as a discrete (parameter) martingale.

Remark 2.1.2. Note first that if $\{B_t, \mathcal{F}_t\}$ is a Brownian motion then it is a martingale. Indeed,

$$\mathbb{E}[B_t - B_s + B_s \mid \mathcal{F}_s] = \mathbb{E}[B_t - B_s] + B_s = B_s.$$

We will see later that all continuous martingales share many important properties with $\{B_t, \mathcal{F}_t\}$.

The following theorem summarizes some basic inequalities satisfied by martingales. The proof for discrete martingales can be found in [8]. The results can be extended to continuous martingales using approximation arguments as long as the continuous martingales satisfy some path regularity.

We first define the number of upcrossings of a stochastic process. Suppose $\{X_t, \mathcal{F}_t \mid t \in \mathcal{T}\}$ is a stochastic process. For $a < b$ and $F \subseteq \mathcal{T}$ finite, let

1. $\tau_1 = \min\{t \in F \mid X_t \le a\}$
2. $\sigma_j = \min\{t \in F \mid t \ge \tau_j, X_t > b\}$
3. $\tau_{j+1} = \min\{t \in F \mid t \ge \sigma_j, X_t < a\}$

Given an interval $I \subseteq \mathcal{T}$, let

$$U_I(a, b; X) = \sup\{U_F[a, b] \mid F \subseteq I, F \text{ finite}\}.$$

Theorem 2.1.3. (Basic Inequalities) *Let $\{X_t, \mathcal{F}_t, t \in \mathcal{T}\}$ be a sub-martingale whose every path is right-continuous* [1], *and suppose that $0 \le \sigma < \tau < \infty$, $a < b$, and $\lambda > 0$. Then we have the following results.*

1. $\lambda \, \mathbb{P}\left(\sup_{t\in[\sigma,\tau]} X_t \ge \lambda\right) \le \mathbb{E}[X_\tau^+]$.
2. $\lambda \, \mathbb{P}\left(\inf_{t\in[\sigma,\tau]} X_t \le -\lambda\right) \le \mathbb{E}[X_\tau^+] - \mathbb{E}[X_\sigma]$.
3. *(Up-crossing inequality)* $(b-a)\, \mathbb{E}\left[U_{[\sigma,\tau]}(a, b; X)\right] \le |a| + \mathbb{E}\left[X_\tau^+\right]$.
4. *(Doob's L^p inequality) If X is non-negative and $1 < p < \infty$, then*

$$\mathbb{E}\left[\sup_{t\in[\sigma,\tau]} |X_t|^p\right] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|X_\tau|^p\right].$$

This can be rewritten more succinctly as

$$\|X_\tau^*\|_p \le C_p \, \|X_\tau\|_p \,,$$

where $X_t^* \doteq \sup_{\sigma\le s\le t} X_s$, $C_p \doteq p/(p-1)$ and, as usual, $\|X_t\|_p = (\mathbb{E}[|X_t|^p])^{1/p}$.

We now show how the inequalities above can be applied in practice to establish an important property of Brownian motion.

Theorem 2.1.4. *(***(Law of the Iterated Logarithm)**

$$\mathbb{P}\left(\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{2t\log\log(\frac{1}{t})}}\right) = 1$$

[1] Note that this is automatically satisfied if $\mathcal{T}$ is discrete since we can view the paths as being piecewise constant in between times.

Proof. Write $h(t) \doteq \sqrt{2t \log\log(\frac{1}{t})}$. The first step is to show that

$$\limsup_{t \searrow 0} \frac{B_t}{h(t)} \le 1, \quad \mathbb{P} - \text{a.s.} \tag{2.1}$$

Apply Doob's maximal inequality to the exponential martingale $Z_t = \exp(\alpha B_t - \frac{\alpha^2}{2} t)$ yielding for $\alpha > 0$

$$\mathbb{P}\left(\sup_{s\in[0,t]} (B_t - \tfrac{1}{2}\alpha s) > \beta\right) = \mathbb{P}\left(\sup_{s\in[0,t]} Z_t > e^{\alpha\beta}\right) \le e^{-\alpha\beta}\,\mathbb{E}[Z_t] = e^{-\alpha\beta}.$$

Now fix $\theta, \delta \in (0,1)$ and apply the inequality with $t = \theta^n$, $\alpha = \theta^{-n}(1+\delta)h(\theta^n)$, and $\beta = \frac{1}{2}h(\theta^n)$. Then

$$\alpha\beta = \frac{1}{2}(1+\delta)\theta^n h^2(\theta^n) = (1+\delta)\log\log\left(\frac{1}{\theta}\right)^n$$

and $e^{\alpha\beta} = \log(n \log(\frac{1}{\theta}))^{1+\delta} = O(n^{1+\delta})$. So

$$\sup_{s\in[0,\theta^n]} \mathbb{P}(B_s - \frac{1}{2}s(1+\delta)\theta^{-n}h(\theta^n)) \ge \frac{1}{2}h(\theta^n)) \le Cn^{-(1+\delta)}.$$

By the Borel-Cantelli Lemma there is $\Theta'_{\theta,\delta} \in \mathcal{F}$ with $\mathbb{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$ there is $N_{\theta,\delta}(\omega)$ such that for all $n \ge N_{\theta,\delta}(\omega)$

$$\max_{x\in[0,\theta^n]} (B_s - \frac{1}{2}s(1+\delta)\theta^{-n}h(\theta^n)) < \frac{1}{2}h(\theta^n)$$

Thus for $\theta^{n+1} < t \le \theta^n$

$$B_t \le \sup_{s\in[0,\theta^n]} B_s \le \frac{1}{2}(2+\delta)\theta^{-n}h(\theta^n) \le \frac{1}{2}(2+\delta)\theta^{-\frac{1}{2}}h(t)$$

where the last inequality uses the fact that $h(\theta^n) \le \theta^{-\frac{1}{2}}h(\theta^{n+1}) \le \theta^{-\frac{1}{2}}h(t)$, so

$$\limsup_{t\downarrow 0} \frac{B_t}{t} \le (1 + \tfrac{\delta}{2})\theta^{-\frac{1}{2}}.$$

Letting $\delta \downarrow 0$ and $\theta \uparrow 1$ along countable sequences the proof of the first step is complete.

We now establish the reverse inequality in (2.1). For this, we would like to use the second Borel-Cantelli lemma to show that $B_t/h(t)$ exceeds 1 (or, rather, exceeds a function that is less than, but arbitrarily close to, 1) infinitely often, along some subsequence t_n (we will choose $t_n = \theta^n$) converging down to 0. However, the second Borel-Cantelli lemma applies only to a sequence of independent events – and so we need to first rewrite this event in terms of the increments of B. As we will see below, we can do this by using (2.1).

Consider the sequence of independent events

$$A_n \doteq \{B_{\theta^n} - B_{\theta^{n+1}} \geq \sqrt{1-\theta}\,h(\theta^n)\}, \quad n = 1, 2, \ldots,$$

for some fixed $\theta \in (0,1)$. By the scaling and translation invariant property of Brownian motion, we know that $(B_{\theta^n} - B_{\theta^{n+1}})/\sqrt{\theta^n - \theta^{n+1}}$ is a standard normal random variable. Therefore, we can use the following inequality (left to the reader to verify) on the tails of the standard normal distribution:

$$\frac{1}{\sqrt{2\pi}} \frac{x}{1+x^2} e^{-x^2/2} \leq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2}\, du$$

with the choice of

$$x = x_n \doteq \frac{\sqrt{1-\theta}\,h(\theta^n)}{\sqrt{\theta^n - \theta^{n+1}}} = \frac{h(\theta^n)}{\sqrt{\theta^n}} = \sqrt{2\log\log(\theta^{-n})} = \sqrt{2\log n + 2\log\log(1/\theta)},$$

to conclude that

$$\mathbb{P}(A_n) = \mathbb{P}\left(\frac{B_{\theta^n} - B_{\theta^{n+1}}}{\sqrt{\theta^n - \theta^{n+1}}} \geq x_n\right) \geq \frac{e^{-x_n^2/2}}{\sqrt{2\pi}(x_n + 1/x_n)} \geq \frac{C}{n}$$

for all $n > |1/log\theta|$, where C is a positive constant. The last expression is the general term of a divergent series, and so the second half of the Borel-Cantelli lemma shows that for a.s. every $\omega \in \Omega$,, given $k \geq 1$, there exists $m = m(k,\omega)$ such that

$$B_{\theta^m}(\omega) - B_{\theta^{m+1}}(\omega) \geq \sqrt{1-\theta}\,h(\theta^m),$$

where the second inequality follows from the form of h. On the other hand, (2.1) applied to the Brownian motion $-B$ shows that there exists an integer-valued random variable N such that for a.s. every ω,

$$-B_{\theta^{n+1}}(\omega) \leq 2h(\theta^{n+1}) \leq 4\sqrt{\theta}\,h(\theta^n), \quad n \geq N^*(\omega).$$

Combining the last two displays, it follows that for a.s. every ω, there exists $m = m(k,\omega) \geq k \vee N^*(\omega)$ such that

$$\frac{B_{\theta^m}(\omega)}{h(\theta^m)} \geq \sqrt{1-\theta} - 4\sqrt{\theta}.$$

Sending $m \to \infty$, we conclude that

$$\limsup_{t \downarrow 0} \frac{B_t}{h(t)} \geq \sqrt{1-\theta} - \sqrt{4\theta}$$

holds a.s. Letting $\theta \downarrow 0$ through the rationals we obtain

$$\limsup_{t \downarrow 0} \frac{B_t}{h(t)} \geq 1, \quad \mathbb{P} \text{ a.s.}$$

and the proof is complete. □

The upcrossing inequality established in Theorem 2.1.3 also leads naturally to a condition for the convergence of continuous martingales, as $t \to \infty$.

Theorem 2.1.5. (Doob's Forward Convergence Theorem) *Let $\{X_t, \mathcal{F}_t, t \in \mathcal{T}\}$ be a right-continuous sub-martingale that satisfies $\sup_{t\in\mathcal{T}} \mathbb{E}[|X_t^+|] < \infty$. Then a.s. $X_\infty = \lim_{n\to\infty} X_n$ exists, X_∞ is $\mathcal{F}_\infty$-measurable and $\mathbb{E}[|X_\infty|] < \infty$.*

Proof. First, note that by the monotone convergence theorem and Theorem 2.1.3(iii), for any $a < b$,

$$\mathbb{E}[U_{[0,\infty)}(a, b; X)] = \lim_{N\to\infty} \mathbb{E}[U_{[0,N]}(a, b; X)] \leq \lim_{N\to\infty} \frac{|a| + \mathbb{E}[X_N^+]}{(b-a)} < \infty,$$

where the finiteness follows from the L^1-boundedness assumption on X^+. Let

$$\begin{aligned}
\Lambda &\doteq \{X_t \text{ does not converge to a limit in } [-\infty, \infty]\} \\
&= \{\liminf_t X_t \neq \limsup_t X_t\} \\
&= \bigcup_{a<b\in\mathbb{Q}} \{\liminf_t X_t < a < b < \limsup_t X_t\} \\
&=: \bigcup_{a<b\in\mathbb{Q}} \Lambda_{a,b}.
\end{aligned}$$

But

$$\Lambda_{a,b} \subseteq \{U_{[0,\infty)}[a, b] = \infty\} = \{\lim_{N\to\infty} U_N[a, b] = \infty\}.$$

Since $U_{[0,\infty)}[a, b]$ has finite expectation, the probability of $\Lambda_{a,b}$ is zero, and so (since Λ is a countable union of such $\Lambda_{a,b}$) $\mathbb{P}(\Lambda) = 0$. Thus X_∞ exists a.s. and it is immediate from its definition as a limit that X_∞ is $\mathcal{F}_\infty$-measurable. Moreover, for every $t \in [0, \infty)$,

$$\mathbb{E}[|X_t|] = \mathbb{E}[|X_t^+ - X_t^-|] \leq 2\mathbb{E}[X_t^+] - \mathbb{E}[X_t] \leq 2 \sup_t \mathbb{E}[X_t^+] - \mathbb{E}[X_0] < \infty,$$

where we have used the fact that X is a submartingale in the second-last equality. Therefore, by Fatou's Lemma

$$\mathbb{E}[|X_\infty|] = \mathbb{E}[\liminf_t |X_t|] \leq \liminf_t \mathbb{E}[|X_t|] < \infty.$$

□

We will now introduce another convergence theorem that is particularly useful when generalizing discrete time martingale results to continuous time martingales.

Definition 2.1.6. *Given a decreasing sequence $\{\mathcal{F}_n\}_{n=1}^\infty$ of σ-algebras, $\{X_n, \mathcal{F}_n\}$ is a* backward sub-martingale *iff $\mathbb{E}[|X_n|] < \infty$ for every $n \in \mathbb{N}$ and $\mathbb{E}[X_n \mid \mathcal{F}_{n+1}] \geq X_{n+1}$ a.s. for all n.*

Theorem 2.1.7 (BSM-CT: Backward Submartingale Convergence Theorem)**.** *Let $\{\mathcal{F}_n\}_{n=1}^\infty$ be a decreasing sequence of σ-algebras and suppose that $\{X_n, \mathcal{F}_n\}$ is a backward sub-martingale. If $\lim_{n\to\infty} \mathbb{E}[X_n] > -\infty$ then $\{X_n\}$ is u.i.*