1.4 Constructions of Brownian motion

By considering the canonical representation of Brownian motion (see Definition 1.2.6 and Remark 1.2.7), it is clear that to show the existence of Brownian motion, it suffices to construct a measure on the space of continuous paths that has the properties implied by the definition of Brownian motion. Before we embark on this project, it will be useful to make a brief digression to consider the different ways in which measures are constructed on $\mathbb{R}^d$.

1.4.1 Common Approaches to Constructing Measures

We shall discuss several common approaches to defining a measure μ on a measurable space $(\Omega, \mathcal{F})$.

1. **(Extension)** The general philosophy behind the method of extension is to first specify the value of the measure on some simple sets in the σ-algebra, and then extend the measure to all sets in the σ-algebra. More precisely, this entails the following four steps:

   (a) Define a finitely additive set function μ on a collection of relatively "simple sets" in Ω that form an algebra $\mathcal{F}_0$.

   (b) Show that μ is, in fact, countably additive on $\mathcal{F}_0$. This is usually the hardest step.

   (c) Invoke Carathéodory's Extension Theorem to conclude that then μ can be uniquely extended to yield a measure μ on the minimal σ-algebra $\sigma(\mathcal{F})$ generated by this collection of simple sets.

   (d) Verify that $\mathcal{F} \subseteq \sigma(\mathcal{F}_0)$.

   This is, for instance, how Borel measure is constructed on $\mathbb{R}^d$ – by extending the concept of length on the algebra of finite disjoint unions of right open intervals to the Borel σ-algebra it generates.

   Given μ as above, one can also further extend the definition of μ to the *completion* $\mathcal{F}^\mu$ (under μ) of $\mathcal{F}$, defined by

   $$\mathcal{F}^\mu = \{A \in \mathcal{F} : \mu^*(A) = \mu_*(A),$$

   μ^* and μ_* are the outer and inner measures, respectively, which are well-defined for every subset A of $\mathcal{F}$. This is how Lebesgue measure is obtained from Borel measure.

   Given a measure μ on a measurable space $(\Omega, \mathcal{F})$, one can sometimes "restrict" the measure to the smaller space $\tilde{\Omega}$, where $\tilde{\Omega} \subseteq \Omega$. Indeed, this is possible if $\tilde{\Omega} \in \mathcal{F}$ since then $\tilde{\mathcal{F}}$, defined by

   $$\tilde{\mathcal{F}} \doteq \{A \cap \tilde{\Omega} : A \in \mathcal{F}\}. \tag{1.8}$$

   is a sub-σ-algebra on Ω and so the restriction $\tilde{\mu}$ of μ to $(\tilde{\Omega}, \tilde{\mathcal{F}})$ is simply defined by

   $$\tilde{\mu}(\tilde{A}) = \mu(A) \qquad \text{for } \tilde{A} = A \cap \tilde{\Omega}, A \in \mathcal{F}.$$

More generally, if $\tilde{\Omega} \notin \mathcal{F}$ but $\mu^*(\tilde{\Omega}) = \mu_*(\tilde{\Omega})$ then one can still define a restriction by first obtaining the completed measure space $(\Omega, \mathcal{F}^\mu, \mu)$, and then using the fact that $\tilde{\Omega} \in \mathcal{F}^\mu$ to define the restriction $\tilde{\mu}$ of μ to the σ-algebra

$$\tilde{\mathcal{F}^\mu} \doteq \{A \cap \tilde{\Omega} : A \in \mathcal{F}^\mu\},$$

as above. In this construction, if μ is a probability measure on $(\Omega, \mathcal{F}, \mathbb{P})$, one needs to verify that $\mu^*(\Omega) = 1$ in order to ensure that $\tilde{\mu}$ is also a probability measure on $\tilde{\Omega}$.

2. **(Densities)** This method is useful once one already has a "reference" measure defined on the same measure space. In this case, one can obtain a whole family of other measures that are absolutely continuous with respect to this reference measure by simply defining the density with respect to the reference measure. For instance, if $(\Omega, \mathcal{F}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, the Gaussian measure on the real line is defined to have density function

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) \tag{1.9}$$

with respect to "the reference" Lebesgue measure, where the mean m and standard deviation σ are real parameters. Recall that the density function is the Radon-Nikodým derivative of the Gaussian measure with respect to Lebesgue measure.

3. **(Mappings)** Given one measure space, one can often use a mapping to induce a measure on another given measurable space. For example, given the measurable space $([0,1], \mathcal{B}[0,1])$ and a non-decreasing, right continuous function $F(x)$ with $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$, consider the measurable mapping $F^{-1} : ([0,1], \mathcal{B}[0,1]) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})$, where recall

$$F^{-1}(x) \doteq \inf\{t > 0 : F(t) \geq x\}.$$

If one equips the measurable space $([0,1], \mathcal{B}[0,1])$ with Lebesgue measure (corresponding to the distribution of a uniform random variable), then it is easily verified that the measure induced by the mapping F^{-1} on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ has c.d.f. F (i.e., $\mu([0,x]) = F(x)$ for every $x \in \mathbb{R}$). This is in fact a standard way to generate random samples from a non-uniform distribution. Thus the mapping $G = F^{-1}$ can be used to produce a new measure on the real line that has c.d.f. F. (Note that this new measure need not have a density with respect to Lebesgue measure, and hence may not be generated using the first technique described above). As another example, consider the measure space obtained as the k-product of the measure space $([0,1], \mathcal{B}[0,1], Leb)$ and consider the mapping

$$G(x_1, \ldots, x_k) \mapsto \sum_{i=1}^k x_i.$$

The induced measure is the distribution of the sum of i.i.d. uniform random variables on $[0,1]$, which is the k-convolution of the uniform distribution on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (having support in $[0,k]$).

4. **(Limits)** A fourth way of defining a measure is with the help of passage to a limit. More precisely, it may be possible to construct a sequence of measures μ_n on the measurable space $(\Omega, \mathcal{F})$ that can be described in a comparatively simple manner, and then to consider the limit of μ_n as $n \to \infty$. For example, limit in the sense of weak covergence is often convenient. The functional central limit is an example of where the (continuous) Gaussian distribution can be obtained as an appropriately centered and normalized sum of independent identically distributed random variables with a simpler distribution – for example binomial random variables, which have a discrete distribution.

1.4.2 Construction of Brownian motion via Extension

For simplicity, we will construct Brownian motion on the interval $[0,1]$. This is without loss of generality, since using the homogeneity of Brownian motion, one can extend it to the whole real line. Due to the canonical representation of stochastic processes (see Definition 1.2.6 and Remark 1.2.7, in order to construct Brownian motion on $[0,1]$, it suffices to construct a measure on the space $\mathcal{C}[0,1]$ of continuous functions on $[0,1]$ (with the topology of uniform convergence), with the associated Borel σ-algebra that has the properties implied by the definition of Brownian motion. The goal of this section is to construct such a measure $\mathbb{P}$, which we shall refer to as "Wiener measure", using two different approaches. In what follows, for $n \in \mathbb{N}$, let $\mathcal{T}_n$ be the space of increasing sequences of length n

$$\mathcal{T}_n := \{(t_1, \ldots, t_n) \mid n \in \mathbb{N}, 0 \le t_1 < \cdots < t_n \le 1\}.$$

1. The Direct Construction. This method closely parallels the construction of Lebesgue measure outlined in the previous section.

Step 1. Consider the following collection of sets called cylinder sets:

$$\mathcal{C} \doteq \{\{\omega \in \mathcal{C}[0,1] : (\omega(t_1), \omega(t_2), \ldots, (\omega_{t_n}) \in A\}, A \in \mathcal{B}(\mathbb{R}^n), (t_1, \ldots, t_n) \in \mathcal{T}_n, n \in \mathbb{N}\}.$$

It is straightforward to verify that $\mathcal{C}$ is an algebra. Moreover, from Lemma 1.3.2, one can deduce the value of $\mathbb{P}$ on such sets – specifically, if $C \in \mathcal{C}$ is the cylinder set associated with $n \in \mathbb{N}$, $(t_1, \ldots, t_n) \in \mathcal{T}_n$ and $A \in \mathcal{B}(\mathbb{R}^n)$, then

$$\mathbb{P}(C) = \int\int \cdots \int_A p(t_1; 0, x_1) \cdots \cdots p(t_n - t_{n-1}; x_{n-1}, x_n)\, dx_1, \ldots, dx_n. \tag{1.10}$$

Note that this characterizes the finite-dimensional distributions of Brownian motion. From elementary properties of multi-dimensional (Gaussian) distributions, it is easy to verify that $\mathbb{P}(C)$ does not depend on the representation of the cylinder set, and is thus well-defined and that $\mathbb{P}$ is finitely additive on $\mathcal{C}$.

Step 2. Show that $\mathbb{P}$ is countably additive on $\mathcal{C}$. This involves tedious estimates using the particular exponential form of $p(t; x, y)$ that we shall skip. The interested reader can look at the book by Ito and Mckean [3] for details of this construction.

Step 3. Invoke Carathéodory's Extension Theorem to conclude that there exists a unique extension of $\mathbb{P}$ to a probability measure on $\sigma(\mathcal{C})$.

Step 4. Show that $\sigma(\mathcal{C}) = \mathcal{B}(\mathcal{C}[0,1])$. This can be established using standard measure-theoretic arguments, and is left as an exercise for the reader.

2. The Two-Step Method. The direct construction outlined above is the most natural if one already assumes the continuity of the sample paths of Brownian motion. However, for construction of a larger class of stochastic processes with more general path properties, it is useful to consider the following two-step approach, in which one first constructs a real-valued stochastic process with given finite-dimensional distributions under very general conditions, i.e., one constructs a measure on $(\mathbb{R}^{[0,1]}, \otimes_{[0,1]}\mathcal{B}([0,1]))$ by invoking the so-called *Daniell-Kolmogorov Theorem*, and one then applies the *Kolmogorov-Centsov theorem*, which states that if the finite-dimensional distributions satisfy a certain additional "regularity" property, then one can restrict the measure obtained on $(\mathbb{R}^{[0,\infty)}, \otimes_{[0,\infty)}\mathcal{B}(\mathbb{R})$ to the space of paths with a certain regularity property (such as continuity, in the case of Brownian motion).

A. Construction of Measure on $(\mathbb{R}^{[0,\infty)}, \otimes_{[0,\infty)}\mathcal{B}(\mathbb{R}))$**.** Define $\tilde{C}$ to be the cylinder sets in $\mathbb{R}^{[0,\infty)}$:

$$\tilde{\mathcal{C}} \doteq \{\{\omega \in \mathcal{R}^{[0,1]} : (\omega(t_1), \omega(t_2), \ldots, (\omega_{t_n}) \in A\}, A \in \mathcal{B}(\mathbb{R}^n), (t_1, \ldots, t_n) \in \mathcal{T}_n, n \in \mathbb{N}\}.$$

As in the case of the direct construction, it is easy to verify that $\tilde{\mathcal{C}}$ is an algebra, that $\mathbb{P}$ must be defined as in (1.10) for cylinder sets $C \in \tilde{\mathcal{C}}$ and that $\mathbb{P}$ is finitely-additive on $\tilde{\mathcal{C}}$. The crux of the proof of the Daniell-Kolmogorov theorem is to show that $\mathbb{P}$ is in fact countably additive on $\tilde{\mathcal{C}}$. By Carathéodory's Extension Theorm and the fact that $\otimes_{[0,\infty)}\mathcal{B}(\mathbb{R}) = \sigma(\tilde{\mathcal{C}})$ by definition, it then immediately follows that there exists a unique measure on $(\mathbb{R}^{[0,\infty)}, \otimes_{[0,\infty)}\mathcal{B}(\mathbb{R}))$ with the finite-dimensional distributions specified by (1.10).

Thus, we see that this part of the construction follows essentially the same steps as in the previous direct construction, but with the important distinction that countable additivity can now be proved without using particular properties of $p(t; x, y)$. In fact, Daniell-Kolmogorov proved the following more general result. Suppose you are given the family

$$\{Q_{\underline{t}}(A), A \in \mathcal{B}(\mathbb{R}^n), \underline{t} \in \mathcal{T}_n, n \in \mathbb{N}\},$$

where $Q_{\underline{t}}(A) \in [0,1]$. (These should be interepreted as candidate finite-dimensional distributions of the process/measure.) And suppose that this family of functions satisfies the following consistent marginal property (which any actual finite-dimensional distribution would satisfy): given $n \in \mathbb{N}$, $\underline{s} = (t_1, \ldots, t_n) \in \mathcal{T}_n$ and $t = (t_1, \ldots, t_n, t_{n+1}) \in \mathcal{T}_{n+1}$,

$$Q_{\underline{t}}(A \times \mathbb{R}) = Q_{\underline{s}}(A), \qquad A \in \mathcal{B}(\mathbb{R}^n).$$

Let $\mathcal{T} = \cup_{n \in \mathbb{N}} \mathcal{T}_n$.

Theorem 1.4.1 (Daniell-Kolmogorov)**.** *Given any family $\{Q_{\underline{t}}, \underline{t} \in \mathcal{T}\}$ that has consistent marginals, there exists a unique measure $\mathbb{P}$ on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$, whose finite-dimensional distributions are given by $\{Q_{\underline{t}}, \underline{t} \in \mathcal{T}\}$.*

B. Restriction of Measure to the Space of Continuous Paths. We have constructed a measure $\mathbb{P}$ on real-valued functions with prescribed finite-dimensional distributions. We would now like to see if the measure concentrates enough mass on the space of continuous functions. Unfortunately, $\mathcal{C}[0,1] \notin \otimes_{[0,1]}\mathcal{B}(\mathbb{R})$, and so it does not make sense to ask if $\mathbb{P}(\mathcal{C}[0,1]) = 1$. However, we will be able to show that the $\mathbb{P}$-outer measure of $\mathcal{C}[0,1]$ is one. Equivalently, since probabilists prefer to think in terms of processes, we could ask the question whether the constructed (canonical) real-valued process can be modified, without altering its finite-dimensional distributions, so as to make almost all sample paths continuous. (Recall that by modifying a process you do not change its finite-dimensional distributions.) The following theorem specifies conditions on the finite-dimensional distributions (in fact, two-dimensional distributions), that guarantees that this can be done.

Theorem 1.4.2 (Kolmogorov-Čentsov)**.** *If $\{X_t, t \in [0,T]\}$ is a real valued stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies*

$$\mathbb{E}[|X_t - X_s|^\alpha] \le C|t-s|^{1+\beta}$$

for all $0 \le s \le t \le T$ and some constants $\alpha > 0$, $\beta > 0$, and $C < \infty$, then there exists a continuous modification of X which is locally Hölder continuous with exponent γ, for all $\gamma \in (0, \frac{\beta}{\alpha})$, i.e., if there exists a constant $\delta > 0$ and an a.s. positive random variable h such that

$$\mathbb{P}\left(\omega : \sup_{\substack{0<t-s<h(\omega)\\ s,t\in[0,T]}} \frac{|X_t(\omega) - X_s(\omega)|}{|t-s|^\gamma} \le \delta\right) = 1.$$

Proof. The first step is to choose a countable dense subset of $[0,T]$. We use the *dyadic rationals*

$$D = \{k2^{-n} \mid k = 0, \ldots, 2^{n-1}, n \in \mathbb{N}\}.$$

Define $\tilde{\Omega}^* = \{\omega \mid t \mapsto X_t(\omega)$ is uniformly continuous$\}$, where X is the canonical process on $\mathbb{R}^{[0,\infty)}$. We would like to show that $\mathbb{P}(\tilde{\Omega}^*) = 1$. We will in fact show that $\mathbb{P}(\Omega^*) = 1$, where

$$\Omega^* = \{\omega \mid t \mapsto X_t(\omega) \text{ is locally Hölder continuous on } D \text{ with coefficient } \gamma\}.$$

By definition, $\omega \in \Omega^*$ iff

$$\begin{aligned} &\exists n^*(\omega) < \infty \text{ s.t. for all } n \geq n^*(\omega), \\ &\max_{1 \leq k \leq 2^n} |X_{k2^{-n}}(\omega) - X_{(k-1)2^{-n}}(\omega)| < 2^{-n\gamma}. \end{aligned} \tag{1.11}$$

Let

$$E_n = \{\omega \mid \max_{1 \leq k \leq 2^n} |X_{k2^{-n}}(\omega) - X_{(k-1)2^{-n}}(\omega)| \geq 2^{-n\gamma}\}.$$

The set of ω for which (1.11) does not hold is the set of ω for which E_n occurs infinitely often. Whence

$$\Omega \setminus \Omega^* = \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} E_n.$$

Now

$$\begin{aligned} \mathbb{P}(E_n) &= \mathbb{P}\left(\bigcup_{k=1}^{2^n} |X_{k2^{-n}} - X_{(k-1)2^{-n}}| \geq 2^{-n\gamma} \right) \\ &\leq \sum_{k=1}^{2^n} \mathbb{P}(|X_{k2^{-n}} - X_{(k-1)2^{-n}}|^\alpha \geq 2^{-n\gamma\alpha}) \\ &\leq 2^{n\gamma\alpha} \sum_{k=1}^{2^n} \mathbb{E}(|X_{k2^{-n}} - X_{(k-1)2^{-n}}|^\alpha) \\ &\leq C 2^{n\gamma\alpha} \sum_{k=1}^{2^n} \left(\frac{1}{2^n} \right)^{1+\beta} \\ &\leq 2^{(\gamma\alpha - \beta)n}. \end{aligned}$$

Therefore, $\sum_{n=1}^\infty \mathbb{P}(E_n) < \infty$ if $\gamma \in (0, \frac{\beta}{\alpha})$ and, by the Borel-Cantelli Lemma we have $\mathbb{P}(\Omega^*) = 1$.

The next step is to define the modification. We define

$$\tilde{X}(\omega) = \begin{cases} X_t(\omega) & t \in D, \omega \in \Omega^* \\ \lim_{\substack{s_n \to t \\ \{s_n\} \subseteq D}} X_{s_n}(\omega) & t \notin D, \omega \in \Omega^* \\ 0 & \omega \notin \Omega^* \end{cases}$$

and we must show that it is truly a modification of X. For $t \in D$, $\mathbb{P}(X_t = \tilde{X}_t) = \mathbb{P}(\Omega^*) = 1$. For $t \notin D$, we know by construction that $\tilde{X}_{s_n} \to \tilde{X}_t$ a.s. for any $\{s_n\} \subseteq D$ converging to t. We know by the Kolmogorov-Čentsov inequality that $X_{s_n} \to X_t$ in $\mathcal{L}^1$ and hence in probability. Therefore $\tilde{X}_{s_n} \to X_t$ in probability, and so $\mathbb{P}(X_t = \tilde{X}_t) = 1$. □

To complete the two-step construction of Brownian motion, we must find constants $\alpha, \beta > 0, C < \infty$ such that the Kolmogorov-Čentsov inequality holds for Brownian motion. This is left as an exercise for the reader.