**Lemma 1.3.7.** *(Nowhere differentiability) For $\mathbb{P}$-almost every $\omega \in \Omega$, the Brownian path $B.(\omega)$ is nowhere Hölder continuous with exponent $\gamma > 1/2$, i.e., the set*

$$\{\omega \in \Omega : \exists\, t, h_0 \in (0,1] \text{ s.t. } |B_{t+h}(\omega) - B_t(\omega)| \le j h^{\gamma} \text{ for all } h \le h_0\}$$

*is contained in a set $F \in \mathcal{F}$ with $\mathbb{P}(F) = 0$. In particular, the set*

$$\{\omega \in \Omega : \text{ for each } t \in [0,\infty), \text{ either } D^+ B_t(\omega) = \infty \text{ or } D_+ B_t(\omega) = -\infty\}$$

*contains an event $F \in \mathcal{F}$ with $\mathbb{P}(F) = 1$, which shows that the Brownian path is a.s. nowhere differentiable.*

We omit the proof of this result. The interested reader can refer to Theorem 9.18 of Chapter 2 of [4]. Instead, we establish an easier result that is implied by Lemma 1.3.7.

**Lemma 1.3.8.** *For $\mathbb{P}$-almost every $\omega \in \Omega$, the sample path $B.(\omega)$ is monotone in no interval of $[0,\infty)$.*

*Proof.* Let $F$ be the set of $\omega \in \Omega$ with the property that $B.(\omega)$ is monotone in some interval. Since every non-empty interval contains one with rational end-points, we can write

$$F = \bigcup_{s,t \in \mathbb{Q}} \{\omega \mid B(\omega) \text{ is monotone on } [s,t]\}.$$

Since $F$ is a countable union, to show $\mathbb{P}(F) = 0$, it suffices to show that each set in the union as zero measure. We first show that

$$A \doteq \{\omega \mid B(\omega) \text{ is non-decreasing on } [0,1]\} \in \mathcal{F}$$

and $\mathbb{P}(A) = 0$. However, $A = \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$, where

$$A_n = \bigcap_{i=0}^{n-1} \{B_{\frac{i+1}{n}} - B_{\frac{i}{n}} \ge 0\}.$$

Since, by the independent increments property and the symmetry of the distribution of $B_{(i+1)/n} - B_{i/n}$, $\mathbb{P}(A_n) = (\frac{1}{2})^n$, it follows that $\mathbb{P}(A) = \lim_{n\to\infty} \mathbb{P}(A_n) = 0$. ☐

We will often be interested in evaluating integrals of the form "$\int_0^t Y_s dX_s$" for some stochastic processes $X$ and $Y$. For example, if $X$ describes the movement of a stock price and $Y$ is the number of shares that you hold at time $s$, then the integral $\int_0^t Y_s dX_s$ represents your net profit over the interval $[0,t]$. From the theory of Lebesgue-Stieltjes/Riemann-Stieltjes integration, it follows that if $Y$ is continuous and $X$ has finite first variation, then one can define this integral pathwise (i.e., for each fixed $\omega$) as a Riemann-Stieltjes integral. This naturally lead us to investigate the nature of the first variation of stochastic processes and, in particular, Brownian, sample paths.

Given a real-valued function $f : [0, \infty) \to \mathbb{R}$ and partition $\Pi = \{0 = t_0 < t_1 < \ldots < t_{k_\Pi} < \cdots < \infty\}$ of $[0, \infty)$, for each $t \geq 0$, let

$$V_t^{(p)}[\Pi](f) = \sum_{i=1}^{\infty} |f(t_i \wedge t) - f(t_{i-1} \wedge t)|^p$$

and let $\|\Pi\|$ denote the maximum mesh size of $\Pi$:

$$\Pi = \max_{i=1,2,\ldots,} |t_{i+1} - t_i|.$$

The (classical) $p^{\text{th}}$-*variation* (as defined in analysis) is given by

$$\tilde{V}_t^{(p)}(f) = \sup_{\Pi} V_t^{(p)}[\Pi](f).$$

If $f$ is continuous and $p = 1$, one can equivalently write

$$\tilde{V}_t^{(1)}(f) = \lim_{\|\Pi_n\| \downarrow 0} V_t^{(1)}[\Pi](f).$$

Note that if $f$ is not continuous or $p > 1$, this equivalence does not necessarily hold.

We now introduce the notion of the $p^{\text{th}}$ variation of a stochastic process.

**Definition 1.3.9.** *Given a real-valued stochastic process $X$, the $\mathbb{R} \cup \{\infty\}$-valued stochastic process $\{V_t^{(p)}(X)\}$ is said to be the $p^{th}$-variation of the stochastic process $X$ iff for every $t < \infty$,*

$$V_t^{(p)}(X) = \lim_{\|\Pi_n\| \downarrow 0} V_t^{(p)}[\Pi_n](X), \quad t \in [0, \infty),$$

*where $\Pi_n, n \in \mathbb{N}$, is any sequence of partitions of $[0, \infty)$ such that $\|\Pi_n\| \downarrow 0$ and the limit is taken in probability.*

Note that in many cases the $p^{\text{th}}$-variation may be a deterministic process. We will be particularly interested in the case when $p = 2$. In a later section, we will show that for certain stochastic processes $X$ called semimartingales, one can equivalently define the quadratic variation process, denoted $[X]$, to be

$$[X]_t = X_t^2 - X_0^2 - 2\int_0^t X_{s-}\, dX_s, \quad t \in [0, \infty), \tag{1.7}$$

where the last term is a stochastic integral that we are yet to define.

**Lemma 1.3.10.** *Given a Brownian motion $\{B_t, \mathcal{F}_t\}$, its quadratic variation process equals the deterministic function $f$ given by $f(t) = t$. Moreover, Brownian motion a.s. does not have finite first variation on any finite interval $[0, t]$.*

*Proof.* Fix a partition $\Pi = \{0 = t_0 < t_1 < t_2 < \ldots\}$ of $[0, \infty)$ and $t \in [0, \infty)$. Then, expanding both $V_t^{(2)}[\Pi](B)$ and $t$ as sums over the partition, it is easy to see that

$$\mathbb{E}[(V_t^{(2)}[\Pi](B) - t)^2] \quad = \quad \mathbb{E}\left[\left(\sum_{j=1}^{\infty} \Delta_j\right)^2\right],$$

where

$$\Delta_j \doteq \left(B_{t_{j+1}\wedge t} - B_{t_j \wedge t}\right)^2 - \left(t_{j+1}\wedge t - t_j \wedge t\right), \qquad j = 1, 2, \ldots$$

Since $B_t - B_s \sim \mathcal{N}(0, t-s)$, we have $\mathbb{E}[|B_t - B_s|^2] = t - s$ and $\mathbb{E}[|B_t - B_s|^4] = 3(t-s)^2$ for every $0 \leq s \leq t$, and hence, for each $j$,

$$\begin{aligned}
\mathbb{E}[\Delta_j^2] \quad &= \quad \mathbb{E}\left[\left(B_{t_{j+1}\wedge t} - B_{t_j\wedge t}\right)^4\right] - 2(t_{j+1}\wedge t - t_j \wedge t)\mathbb{E}\left[\left(B_{t_{j+1}\wedge t} - B_{t_j\wedge t}\right)^2\right] \\
&\qquad + (t_{j+1}\wedge t - t_j \wedge t)^2 \\
&= \quad 2(t_{j+1}\wedge t - t_j \wedge t)^2.
\end{aligned}$$

Moreover, for every $j, k$ with $j < k$, since $\Delta_j$ is $\mathcal{F}_{t_{j+1}}$-measurable and, by the independent increments property, $\mathbb{E}[\Delta_k | \mathcal{F}_{t_{j+1}}] = \mathbb{E}[\Delta_k] = 0$, we have

$$\mathbb{E}[\Delta_j \Delta_k] = \mathbb{E}[\mathbb{E}[\Delta_k \Delta_j | \mathcal{F}_{t_{j+1}}]] = \mathbb{E}[\Delta_j \mathbb{E}[\Delta_k | \mathcal{F}_{t_{j+1}}]] = \mathbb{E}[\Delta_j \mathbb{E}[\Delta_k]] = 0.$$

Combining all these facts, we obtain

$$\mathbb{E}[(V_t^{(2)}[\Pi](B) - t)^2] = \mathbb{E}\left[\left(\sum_{j=1}^{\infty} \Delta_j^2\right)\right] \quad = \quad 2\sum_{j=1}^{\infty}(t_{j+1}\wedge t - t_j \wedge t)^2 \leq 2||\pi||t.$$

Therefore, given a sequence $\Pi_n$ such that $||\Pi_n|| \to 0$, we have

$$\mathbb{E}[(V_t^{(2)}[\Pi_n](B.) - t)^2] \leq 2||\pi_n||t \to 0 \quad \text{ as } n \to \infty,$$

which shows that $V_t^{(2)}[\Pi_n]$ converges to $t$ in $L^2$ as $n \to \infty$. Since convergence in $L^2$ implies convergence in probability, we conclude $V_t^{(2)}[\Pi_n]$ converges to $t$ in probability, and the first assertion of the lemma is established.

In turn, since convergence in probability implies convergence a.s. along a subsequence, we infer that there exists a subsequence $\{\Pi_{n_k}\}_{k\in\mathbb{N}}$ of partitions such that $V_t^{(2)}[\Pi_{n_k}](B.(\omega)) \to t$ a.s. Now, for a.s. every $\omega$, by the definition of $V^{(p)}$, $p = 1, 2$, we have

$$V_t^{(1)}[\Pi_{n_k}](B.(\omega)) \geq \frac{V_t^{(2)}[\Pi_{n_k}](B.(\omega))}{\sup_{t_i \in \Pi_{n_k}} |B_{t_{i+1}} - B_{t_i}|}. \tag{1.8}$$

Since $B$ is a.s. continuous and the interval $[0, t]$ is compact, $B$ is a.s. uniformly continuous on $[0, t]$. Therefore, as $k \to \infty$, the fact that $\|\Pi_{n_k}\| \to 0$ implies

$$\sup_{t_i \in \Pi_{n_k}} |B_{t_{i+1}} - B_{t_i})| \to 0.$$

Therefore, taking limits as $k \to \infty$ in (1.8), and using the fact that $V_t^{(2)}[\Pi_{n_k}](B_\cdot(\omega)) \to t$ a.s., we conclude that for a.s. every $\omega$, $V_t^{(1)}(B_\cdot(\omega)) = \lim_{k\to\infty} V_t^{(1)}[\Pi_{n_k}](B_\cdot(\omega)) = \infty$. This shows that the first variation is almost surely not finite, and thus completes the proof of the lemma. $\square$