4.3.5 The Skorokhod Reflection Problem

The one-dimensional reflection problem

The Skorokhod problem (SP) provides a means to construct and analyze constrained processes. Processes that are constrained to live within a domain often exhibit discontinuous dynamics at the boundary of the domain, which precludes the application of classical techniques for their analysis. The basic idea behind the Skorokhod problem is to represent a constrained process as a suitable mapping of an unconstrained process that is easier to analyze. When the associated mapping is sufficiently regular (for example, uniformly continuous on path space), properties of the constrained process can often be deduced from that of the unconstrained process. The original, one-dimensional problem was introduced by Skorokhod in 1961 in order to construct reflected diffusions on $\mathbb{R}_+$. It takes the following form.

Definition 4.3.8. (Skorokhod Problem on $[0, \infty)$) *Given $\psi \in \mathcal{C}[0, \infty)$, does there exist $\varphi \in \mathcal{C}[0, \infty)$ such that the following properties hold for all $t \in [0, \infty)$:*

1. $\varphi(t) = \psi(t) + \eta(t) \geq 0$*;*
2. $\eta(0-) \doteq 0$ *and η is non-decreasing;*
3. *η is "flat" off $\{s : \phi(s) = 0\}$ or, more precisely,*

$$\int_0^\infty \phi(t) d\eta(t) = 0.$$

If such a pair (ϕ, η) exist, then it is said to solve the SP on $[0, \infty)$ for ψ. Moreover, if the solution is unique, then the mapping $\Gamma : \psi \mapsto \phi$ is referred to as the Skorokhod map.

Skorokhod proved the following result (see class notes for the proof).

Theorem 4.3.9. (Skorokhod Problem) *For every $\psi \in \mathcal{C}[0, \infty)$, there exists a unique solution (φ, η) to the Skorokhod problem on $[0, \infty)$. Moreover, the mapping $\Gamma : \psi \mapsto \varphi$ takes the explicit form*

$$\Gamma(\psi)(t) = \psi(t) + \sup_{s \in [0,t]} [-\psi(s)] \wedge 0.$$

In particular, this shows that Γ is Lipschitz continuous with constant 2 on $\mathcal{C}[0, T]$ for every $T < \infty$.

As an illustration of the use of the Skorokhod problem, consider the deterministic dynamical system that describes the storage system sketched in Figure 4.1. Suppose that $\lambda(\cdot)$ and $\mu(\cdot)$ are non-negative continuous functions, with $\lambda(\cdot)$ representing the arrival rate of fluid into the system and $\mu(\cdot)$ representing the drainage rate of fluid (when present) from the system. Let $Q(t)$ represent

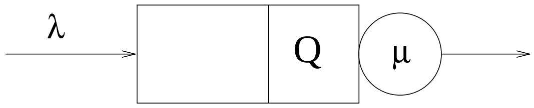

Figure 4.1: A Storage System

the amount of fluid present in the system at time t. The following ordinary differential equation then describes the evolution of Q: $Q(0) = q_0 \geq 0$ and

$$\frac{dQ}{dt}(t) = \begin{cases} \lambda(t) - \mu(t) & \text{if } Q(t) > 0 \\ [\lambda(t) - \mu(t)] \vee 0 & \text{if } Q(t) = 0. \end{cases} \tag{4.4}$$

Thus, even for this simple system, we obtain an ordinary differential equation whose right-hand side could be discontinuous (in the state Q). Thus classical theorems for existence and uniqueness of solutions to ordinary differential equations are not applicable to this situation. Note that Q is constrained to be non-negative or, equivalently, to lie in the domain $[0, \infty)$ and its dynamics exhibits discontinuities on the boundary (i.e., at 0). Recall that the philosophy of the Skorokhod problem is to express Q in terms of an unconstrained process. Let X be the process that satisfies $X(0) = Q(0)$ and the ordinary differential equation

$$\frac{dX}{dt}(t) = \lambda(t) - \mu(t).$$

Then X is clearly well-defined and, moreover, we can write

$$Q = X + Y,$$

where Y satisfies $Y(0) = 0$ and, for $t \in [0, \infty)$,

$$\frac{dY}{dt}(t) = \begin{cases} 0 & \text{if } Q(t) > 0 \\ [\mu(t) - \lambda(t)] \wedge 0 & \text{if } Q(t) = 0. \end{cases}$$

Thus Y is "flat" off the set $\{s : Q(s) = 0\}$ and it is easy to see from Definition 4.3.8 that Q solves the ordinary differential equation (4.4) if and only if (Q, Y) solves the Skorokhod problem for X. Invoking Theorem 4.3.9, we can then conclude that there exists a unique solution to the ordinary differential equation (4.4). The process Y can be interpreted as the cumulative "lost drainage capacity" of the system due to emptiness of the buffer.

Skorokhod's lemma allows a different proof of the following remarkable result by Lévy, which can also be deduced from the results in Section 4.3.3.

Theorem 4.3.10. *Given a BM $\{B_t, \mathcal{F}_t\}$, suppose M_t is the maximum process defined in (4.3) and $L_t(0)$ is the local time of B at 0. Then the process M and $L(0)$ are equal in distribution under $\mathbb{P}^0$.*

Proof. Given the Brownian motion $\{B_t, \mathcal{F}_t\}$. by Tanaka's formula, under $\mathbb{P}^0$,

$$|B_t| = \int_0^t \operatorname{sgn}(B_s)\, dB_s + L_t(0).$$

Let $W_t \doteq \int_0^t \operatorname{sgn}(B_s)\, dB_s$. Then $\{W_t, \mathcal{F}_t\}$ is a local martingale with quadratic variation t. Thus by Lévy's characterization theorem, W is a Brownian motion. Since $|B_t| \geq 0$ and $L_t(0)$ is a non-decreasing process with $L_0(0) = 0$ and $L_t(0)$ satisfies

$$\int_0^t B_s dL_s(0) = 0 \qquad \int_0^t |B_s| dL_s(0) = 0$$

(think about how to prove the above fact rigorously from the definition of local time), from the uniqueness of solutions to the Skorokhod problem and the fact that $W_0 = 0$, it follows that

$$(|B_t|, L_t(0)) = (\Gamma(W)_t, \sup_{s\in[0,t]} (-W(s))).$$

Since W has the same distribution as $-W$ by the symmetry of Brownian motion, $-W$ is also a Brownian motion, and so

$$\sup_{s\in[0,t]} (-W(s)) \stackrel{d}{=} M_t,$$

and the result follows. □