Sometimes, one would like to integrate the process along its paths, i.e., given some interval $I \subset \mathbb{R}$, one would like quantities of the sort

$$\int_I X_t \, dt \qquad \text{and} \qquad \int_I \mathbb{E}[X_t] \, dt$$

to be well-defined. A necessary condition for this is that you need the trajectories or sample paths to be Borel-measurable functions of $t \in [0, \infty)$, which can be obtained only if you impose some additional joint measurability properties on X as a mapping on $[0, \infty) \times \Omega$ (see definition 3 above).

Definition 1.2.6. *A stochastic process is said to be* measurable *iff for all* $A \in \mathcal{B}(S)$, $\{(t, \omega) \mid X(t, \omega) \in A\} \in \mathcal{B}[0, \infty) \times \mathcal{F}$, *i.e., if* X *is an* $(S, \mathcal{B}(S))$*-valued random element on* $([0, \infty) \times \Omega, \mathcal{B}[0, \infty) \times \mathcal{F})$.
A stochastic process is said to be progressively measurable *with respect to the filtration* $\{\mathcal{F}_t\}$ *iff for each* $t \geq 0$ *and* $A \in \mathcal{B}(S)$*, the set*

$$\{(s, \omega) \in [0, t] \times \Omega : X_s(\omega) \in A\} \in \mathcal{B}([0, t]) \otimes \mathcal{F}_t.$$

We will see, when we introduce stopping times subsequently, that the notion of progressive measurability ensures that the stopped process is a well-defined random variable, i.e., that you can evaluate quantities of the form $\mathbb{P}\{X_T \in A\}$, $A \in \mathcal{B}(S)$, where T is a so-called stopping time with respect to the filtration $\{\mathcal{F}_t\}$.

1.2.3 Definition of Brownian Motion

We now state the definition of Brownian motion, which generalizes the BMP.

Definition 1.2.7. *A real-valued stochastic process* $\{B_t, t \in [0, \infty)\}$ *defined on a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ *is said to be a Brownian motion with respect to the filtration* $\{\mathcal{F}_t\}$ *if* $\{B_t\}$ *is adapted to the filtration* $\{\mathcal{F}_t\}$ *and satisfies the following four properties:*

1. *(Normal marginals)* $B_0 = 0$ *and for every* $t \in [0, \infty)$, $B_t \sim \mathcal{N}(0, t)$.

2. *(Time homogeneity or stationarity) Given any* $t \in [0, \infty)$, $B_{t_0+t} - B_{t_0}$ *has the same distribution for all* $t_0 \in [0, \infty)$.

3. *(Independent increments) For* $0 \leq s < t < \infty$, $B_t - B_s$ *is independent of* $\mathcal{F}_s$.

4. *(Continuity of paths) The set*

$$C \doteq \{\omega \in \Omega : \text{ the mapping } t \mapsto B_t(\omega) \text{ is continuous } \}$$

lies in $\mathcal{F}$ *and satisfies* $\mathbb{P}(C) = 1$.

We will sometimes shorten this to saying that $\{B_t\}$ *is an* $\mathcal{F}_t$*-Brownian motion [defined on the filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$*]. Moreover, an* $\mathbb{R}^d$*-valued stochastic process* $\{B_t\}$ *is said to be a d-dimensional (standard) Brownian motion with respect to the filtration* $\{\mathcal{F}_t\}$ *[defined on* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$*] if for each* $i = 1, \ldots, d$, $(B_t)_i$ *is an* $\mathcal{F}_t$*-Brownian motion and the component processes* $(B_t)_i, i = 1, \ldots, d$, *are independent.*

Note that this coincides with the BMP when $\{\mathcal{F}_t\}$ is taken to be the natural filtration $\{\mathcal{F}_t^X\}$ of the process. However, in many applications it is necessary to consider a more general filtration $\{\mathcal{F}_t\}$ or, at a more intuitive level, more general "information structure". Moreover, from the discussion in Section 1.1 and the fact that the limit in the central limit theorem is unique in distribution, it is natural to conjecture that Brownian motion is unique in distribution. In Section 1.4, we will present several different approaches to prove that Brownian motion exists and we will also show that it is unique in distribution.

1.3 Basic Properties of Brownian Motion

In this section, we shall first assume the existence of a process that satisfies the Brownian Motion Postulates (BMP), and that it is unique in distribution, and will refer to this process as Brownian motion. Under this assumption, we will deduce certain intriguing properties of this process. In Section 1.4, we will use several different approaches to establish the existence and uniqueness (in distribution) of this process.

1.3.1 Distributional Properties

We start showing that Brownian motion must be a Gaussian process. First, we recall the definition of a Gaussian process.

Definition 1.3.1. *A stochastic process $\{X_t, t \in [0,\infty)\}$ is said to be a Gaussian process iff for every $0 < t_1 < t_2 < \ldots < t_n$, the $\mathbb{R}^n$-valued random vector $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ has a Gaussian distribution, i.e., there exist positive reals σ_{ij} and reals μ_j such that*

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{n} t_j X_{t_j}\right)\right] = \exp\left(-\frac{1}{2}\sum_{j,k=1,\ldots,n} \sigma_{jk} t_j t_k + \sum_j \mu_j t_j\right).$$

Alternatively, a stochastic process $\{X_t, t \in [0,\infty)\}$ is said to be a Gaussian process iff every finite linear combination $\sum_{i=1}^n a_i X_{t_i}$, $t_i \in [0,\infty), a_i \in \mathbb{R}, i = 1,\ldots,n$, is normally distributed.

The covariance function of a Gaussian process is defined as follows:

$$\rho(s,t) \doteq \mathbb{E}[(X_s - \mathbb{E}[X_s])(X_t - \mathbb{E}[X_t])] \quad s,t \geq 0.$$

Lemma 1.3.2. *Brownian motion is a zero-mean Gaussian process with covariance function $\rho(s,t) = s \wedge t$. Specifically, for every $0 < t_1 < \ldots < t_n$, the cumulative distribution function $F_{(t_1,t_2,\ldots,t_n)}$ of $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ is*

$$F_{(t_1,t_2,\ldots,t_n)}(x_1,\ldots,x_n) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_n} p(t_1;0,y_1)p(t_2-t_1;y_1,y_2)\ldots p(t_n-t_{n-1};y_{n-1},y_n)dy_n\ldots dy_2 dy_1$$

for $(x_1,\ldots,x_n)$ in $\mathbb{R}^n$, where $p(t;x,y)$ is the Gaussian kernel

$$p(t;x,y) \doteq \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}, \quad t > 0, x,y \in \mathbb{R}. \tag{1.6}$$

Proof. From property 1 of the BMP, $\mathbb{E}[B_t] = 0$ for every $t \geq 0$. Moreover, fix $0 < s < t < \infty$. Then, using the independent increments property, we have

$$\mathbb{E}[B_s B_t] = \mathbb{E}[B_s(B_t - B_s)] + \mathbb{E}[B_s^2] = \mathbb{E}[B_s]\mathbb{E}[B_t - B_s] + \mathbb{E}[B_s^2] = s,$$

which shows that $\rho(s,t) = s \wedge t$. The rest of the lemma follows from the independent increment property and the fact that the first three conditions of the BMP together show that for any $0 < s < t$, $B_t - B_s \sim \mathcal{N}(0, t-s)$. ☐

We now establish some important scaling properties of Brownian motion.

Proposition 1.3.3. *(Equivalence Transformations) If $\{B_t, \mathcal{F}_t\}$ is a Brownian motion [defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$], then the process X defined in the following six ways is also a Brownian motion with respect to its natural filtration.*

1. $X_t = \frac{1}{\sqrt{c}} B_{ct}$ *for* $c > 0$;
2. $X_t = B_{t+c} - B_c$ *for* $c \geq 0$;
3. $X_t = B_T - B_{T-t}$ *for* $t \in [0, T]$;
4. $X_t = tB_{\frac{1}{t}}$ *for* $t > 0$ *and* $X_0 = 0$;
5. $X_t = -B_t$;
6. $X_t = UB_t$ *for an orthogonal matrix* U *(here,* $\{B_t\}$ *is a d-dimensional Brownian motion).*

1.3.2 Sample Path Properties

Lemma 1.3.4. *(Unboundedness) Brownian motion is a.s. unbounded, i.e.,*

$$\mathbb{P}\left(\sup_{t\geq 0} B_t = \infty \text{ and } \inf_{t\geq 0} B_t = -\infty\right) = 1.$$

Proof. Let $Z = \sup_{t\geq 0} B_t$. For any $c > 0$, we have

$$cZ = \sup_{t\geq 0} cB_t = \sup_{t\geq 0} B_{\frac{t}{c^2}} = Z.$$

Since Z is non-negative (since $B_0 = 0$), this implies that the law of Z is concentrated on $\{0, \infty\}$. Let $p = \mathbb{P}(Z = 0)$. Then

$$\begin{aligned} p &\leq \mathbb{P}(B_1 \leq 0 \text{ and } B_u \leq 0 \text{ for all } u \geq 1) \\ &\leq \mathbb{P}\left(B_1 \leq 0 \text{ and } \sup_{t\geq 0}(B_{1+t} - B_1) \leq 0\right) \\ &= \mathbb{P}(B_1 \leq 0)\,\mathbb{P}\left(\sup_{t\geq 0}(B_{1+t} - B_1) \leq 0\right) \\ &= \mathbb{P}(B_1 \leq 0)\,\mathbb{P}(Z = 0) \\ &= \frac{1}{2}p, \end{aligned}$$

where the third-last equality uses the independent increment property, the second-last equality uses the translation invariance of B, the definition of Z and the fact that $\mathbb{P}(Z \leq 0) = \mathbb{P}(Z = 0)$, and the last equality follows as a simple consequence of the normal marginal property and the symmetry of the normal distribution around 0. Since $p \in [0, 1]$, this implies $p = 0$ and, therefore, that $\mathbb{P}(Z = \infty) = 1$. Now, define $\tilde{Z} \doteq \inf_{t \geq 0} B_t$. By symmetry, it is clear that $\mathbb{P}(\tilde{Z} = -\infty) = 1$, which when combined with $\mathbb{P}(Z = \infty)$ yields the desired result. □

Remark 1.3.5. *As a direct consequence of Lemma 1.3.4 and the translation invariance of Brownian motion, it follows that a.s., for all $a \in \mathbb{R}$, $\{t \mid B_t = a\}$ is not bounded above.*

We now investigate the differentiability of the sample paths of Brownian motion. Recall that for any continuous function $f : [0, \infty) \mapsto \mathbb{R}$, the upper (right and left) Dini derivates at t are defined by

$$D^{\pm} f(t) = \limsup_{h \to 0\pm} \frac{f(t+h) - f(t)}{h},$$

and the lower (right and left) Dini derivates at t are defined by

$$D_{\pm} f(t) = \liminf_{h \to 0\pm} \frac{f(t+h) - f(t)}{h}.$$

Moreover, the function f is said to be right-differentiable at t if $D^+(t)$ and $D_+(t)$ are finite numbers and are equal, and left-differentiablity is defined in the analogous way.

Lemma 1.3.6. *(Non-differentiability) For $\mathbb{P}$-almost every $\omega \in \Omega$, the sample path $B.(\omega)$ is a.s. not differentiable at zero.*

Proof. We start with the claim that

$$\mathbb{P}(\forall \varepsilon > 0, \exists\, s, t \leq \varepsilon \text{ s.t. } B_s < 0 < B_t) = 1.$$

Indeed, if this is not the case, then there must exist a set $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ such that for all $\omega \in A$, there exists $\varepsilon = \varepsilon(\omega)$ such that either $B_s(\omega) > 0$ or $B_s(\omega) < 0$ for all $s \in (0, \varepsilon]$. This, in turn, implies that the process $\tilde{B}$ defined by $\tilde{B}_s \doteq s B_{1/s}$ for $s \in [0, \infty)$, satisfies for $\omega \in A$, $\tilde{B}_s(\omega) > 0$ or $\tilde{B}_s(\omega) < 0$ for all $s \in [1/\varepsilon(\omega), \infty)$. The last assertion contradicts Lemma 1.3.4 and hence establishes the claim.

From the claim, it follows that the only possible (right) derivative of Brownian motion at zero is 0. If this were the case, then we must have $|B_t| \leq t$ for all small enough t. In turn, this implies that the process $\tilde{B}$ defined by $\tilde{B}_s = s B_{1/s}$ for $s \in [0, \infty)$ satisfies $|\tilde{B}_s| \leq 1$ for all sufficiently large s. Since $\tilde{B}$ is again a Brownian motion by the time inversion property, this contradicts the a.s. unboundedness of Brownian motion proved in Lemma 1.3.4. □

Combining the last result with the translation invariance of Brownian motion, it is clear that for any fixed $t \in [0, \infty)$, Brownian motion is a.s. not differentiable at t. In fact, the following stronger result holds.

Lemma 1.3.7. *(Nowhere differentiability) For $\mathbb{P}$-almost every $\omega \in \Omega$, the Brownian path $B.(\omega)$ is nowhere Hölder continuous with exponent $\gamma > 1/2$, i.e., the set*

$$\{\omega \in \Omega : \exists\, t, h_0 \in (0,1] \text{ s.t. } |B_{t+h}(\omega) - B_t(\omega)| \le jh^\gamma \text{ for all } h \le h_0\}$$

is contained in a set $F \in \mathcal{F}$ with $\mathbb{P}(F) = 0$. In particular, the set

$$\{\omega \in \Omega : \text{ for each } t \in [0,\infty), \text{ either } D^+B_t(\omega) = \infty \text{ or } D_+B_t(\omega) = -\infty\}$$

contains an event $F \in \mathcal{F}$ with $\mathbb{P}(F) = 1$, which shows that the Brownian path is a.s. nowhere differentiable.

We omit the proof of this result. The interested reader can refer to Theorem 9.18 of Chapter 2 of [4]. Instead, we establish an easier result that is implied by Lemma 1.3.7.

Lemma 1.3.8. *For $\mathbb{P}$-almost every $\omega \in \Omega$, the sample path $B.(\omega)$ is monotone in no interval of $[0,\infty)$.*

Proof. Let F be the set of $\omega \in \Omega$ with the property that $B.(\omega)$ is monotone in some interval. Since every non-empty interval contains one with rational end-points, we can write

$$F = \bigcup_{s,t \in \mathbb{Q}} \{\omega \mid B(\omega) \text{ is monotone on } [s,t]\}.$$

Since F is a countable union, to show $\mathbb{P}(F) = 0$, it suffices to show that each set in the union as zero measure. We first show that

$$A \doteq \{\omega \mid B(\omega) \text{ is non-decreasing on } [0,1]\} \in \mathcal{F}$$

and $\mathbb{P}(A) = 0$. However, $A = \bigcap_{n=1}^\infty A_n \in \mathcal{F}$, where

$$A_n = \bigcap_{i=0}^{n-1} \{B_{\frac{i+1}{n}} - B_{\frac{i}{n}} \ge 0\}.$$

Since, by the independent increments property and the symmetry of the distribution of $B_{(i+1)/n} - B_{i/n}$, $\mathbb{P}(A_n) = (\frac{1}{2})^n$, it follows that $\mathbb{P}(A) = \lim_{n\to\infty} \mathbb{P}(A_n) = 0$. □

We will often be interested in evaluating integrals of the form "$\int_0^t Y_s dX_s$" for some stochastic processes X and Y. For example, if X describes the movement of a stock price and Y is the number of shares that you hold at time s, then the integral $\int_0^t Y_s dX_s$ represents your net profit over the interval $[0,t]$. From the theory of Lebesgue-Stieltjes/Riemann-Stieltjes integration, it follows that if Y is continuous and X has finite first variation, then one can define this integral pathwise (i.e., for each fixed ω) as a Riemann-Stieltjes integral. This naturally