Example 4.2.5. *This theorem is not true for general random times. For example, consider T to be the last time before 1 that B_t is zero, and show that the assuming that the property holds at T leads to a contradiction.*

4.3 Applications of the Strong Markov Property for Brownian Motion

4.3.1 Occupation Times

Let $D = B(0, r)$, the open ball of radius r centered at 0. In the study of the recurrence and transience of Brownian motion, we saw that d-dimensional Brownian motion will return to D infinitely often when $d \leq 2$, but not when $d > 2$. We now show how the strong Markov property can be used to investigate the "occcupation time" $\int_0^\infty \mathbb{I}_D(B_s)\, ds$ of Brownian motion in two dimensions.

Lemma 4.3.1. *When $d \leq 2$,*

$$\mathbb{P}_x \left(\int_0^\infty \mathbb{I}_D(B_s)\, ds \right) = 1.$$

Proof. Define $G \doteq B(0, 2r)$. Define $T_0 \doteq 0$ and for $k \geq 1$, define

$$S_k \doteq \inf\{t > T_{k-1} : B_t \in \partial D\}$$
$$T_k \doteq \inf\{t > S_k : B_t \notin G\}.$$

It is easy to see that this defines two nested sequences of stopping times.

Let us denote $T_1 \doteq \tau$. For any $k \geq 1$, by the continuity of B, clearly B spends no time in D during the interval $[T_k, S_{k+1})$. To get an estimate of the time spent in D during the interval $[S_k, T_k)$, first note that since Brownian motion starts at 0, by the spherical symmetry it is clear that B_{T_1} is uniformly distributed on ∂G. Moreover, starting from the uniform distribution on ∂G, it is clear that the distribution of Brownian motion when it hits the smaller sphere ∂D is also uniform. By the strong Markov property, this implies that B_{S_k} has a uniform distribution on ∂D for all $k \geq 1$. Next, note that by the strong Markov property, for $k \geq 1$,

$$\mathbb{P}_x \left(\int_{S_k}^{T_k} \mathbb{I}_D(B_t)\, dt \geq s | \mathcal{F}_{S_k} \right) = \mathbb{P}_{B_{S_k}} \left(\int_0^\tau \mathbb{I}_D(B_t)\, dt \geq s \right) \doteq H(s),$$

where the latter quantity does not depend on k due to the fact that B_{S_k} is uniformly distributed on ∂D for every k. This shows that $\{\int_{S_k}^{T_k} \mathbb{I}_D(B_t), k \geq 1\}$ is a collection of identically distributed random variables. These random variables are also independent by Theorem 4.2.2. Moreover, it is easy to see that the random variables have positive mean. Thus, from the strong law or large numbers it follows that

$$\int_0^\infty \mathbb{I}_D(B_t)\, dt \geq \lim_{n\to\infty} \sum_{k=1}^\infty \sum_{S_k}^{T_k} \mathbb{I}_D(B_t)\, dt = \infty \quad a.s.$$

This proves the lemma. □

4.3.2 The reflection principle

Theorem 4.3.2. *Given a Brownian motion $\{B_t, \mathcal{F}_t\}$, let M be the running maximum of Brownian motion. Then for every $t > 0$, $M_t \stackrel{d}{=} |B_t|$. Equivalently, for all $a \geq 0$ and $t \geq 0$,*

$$\mathbb{P}(M_t \geq a) = \sqrt{\frac{2}{\pi t}} \int_a^\infty \exp\left(-\frac{z^2}{2t}\right) dz.$$

Proof. Note that $\{M_t \geq a\} = \{T_a \leq t\} \in \mathcal{F}_t$. Moreover,

$$\begin{aligned} \mathbb{P}(T_a \leq t) &= \mathbb{P}(T_a \leq t, B_t \geq a) + \mathbb{P}(T_a \leq t, B_t < a) \\ &= \mathbb{P}(B_t \geq a) + \mathbb{P}(T_a \leq t, B_{T_a+(t-T_a)} - B_{T_a} < 0\} \\ &= \mathbb{P}(B_t \geq a) + \mathbb{E}\left[\mathbb{P}(B_{T_a+(t-T_a)} - B_{T_a} < 0 | \mathcal{F}_{T_a}) \mathbb{I}_{\{T_a \leq t\}}\right], \end{aligned}$$

since T_a is $\mathcal{F}_{T_a}$-measurable. On the other hand, by the strong Markov property, $\mathbb{P}(B(T_a + (t - T_a)) - B(T_a) < 0 | \mathcal{F}_{T_a})$ is the conditional probability that a Brownian motion independent of $\mathcal{F}_{T_a}$ is below zero at time $t - T_a$, given the value of T_a. By independence and symmetry, we then see that this probability equals a.s. $\mathbb{P}(B(T_a + (t - T_a)) - B(T_a) > 0 | \mathcal{F}_{T_a})$. Therefore, we conclude that

$$\begin{aligned} \mathbb{P}(B_t \geq a) &= \mathbb{P}(T_a \leq t) \\ &= \mathbb{P}(B_t \geq a) + \mathbb{E}\left[\mathbb{P}(B(T_a + (t - T_a)) - B(T_a) > 0 | \mathcal{F}_{T_a}) \mathbb{I}_{\{T_a \leq t\}}\right] \\ &= \mathbb{P}(B_t \geq a) + \mathbb{P}(T_a \leq t, B(T_a + (t - T_a)) - B(T_a) > 0) \\ &= \mathbb{P}(B_t \geq a) + \mathbb{P}(T_a \leq t, B_t > a) \\ &= 2\mathbb{P}(B_t \geq a), \end{aligned}$$

since $\mathbb{P}(B_t = a) = 0$. On the other hand, thanks to symmetry,

$$2\mathbb{P}(B_t \geq a) = \mathbb{P}(B_t \geq a) + \mathbb{P}(-B_t \geq a) = \mathbb{P}(|B_t| \geq a).$$

The theorem then follows on using the fact that $B_t \sim \mathcal{N}(0, t)$, and using the explicit form of the normal density. □

4.3.3 Brownian motion and its current maximum

Given a Brownian motion $\{B_t, \mathcal{F}_t\}$, define

$$M_t \doteq \sup_{s \in [0,t]} B_s. \tag{4.3}$$

It is not hard to see that $\{M_t, \mathcal{F}_t\}$ is not a Markov process, since the future evolution of M after t clearly depends on the position of Brownian motion at t, in addition to M_t. We now show that $\{(B_t, M_t), \mathcal{F}_t\}$ is a Markov process.

Theorem 4.3.3. *$\{(B_t, M_t), \mathcal{F}_t\}$ is a Markov process under $\mathbb{P}^0$. Moreover, for $t > 0$ and $b \geq 0$, $a \leq b$,*

$$\mathbb{P}^0(B_t \in da, M_t \in db) = \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2b-a)^2}{2t}\right) dadb.$$

Proof. For $s > 0$, $t \geq 0$, $b \geq a$, $b \geq 0$,

$$\begin{aligned}
\mathbb{P}^0\left(B_{t+s} \geq a, M_{t+s} \leq b \mid \mathcal{F}_t\right) &= \mathbb{P}^0\left(B_{t+s} \geq a, M_t \leq b, \sup_{u\in[0,s]} B_{t+u} \leq b \mid \mathcal{F}_t\right) \\
&= \mathbb{I}_{\{M_t \leq b\}} \mathbb{P}^0(B_{t+s} \geq a, \sup_{u\in[0,s]} B_{t+u}) \leq b \mid \mathcal{F}_t) \\
&= \mathbb{I}_{\{M_t \leq b\}} \mathbb{P}^0(B_{t+s} \geq a, \sup_{u\in[0,s]} B_{t+u} \leq b \mid B_t),
\end{aligned}$$

where the last equality follows from the Markov property of Brownian motion $\{B_t, \mathcal{F}_t\}$. Since the right-hand side is only a function of M_t and B_t, this shows that $\{(M_t, B_t), \mathcal{F}_t\}$ is a Markov process under P^0.

Next, note that for $a \leq b$, $b \geq 0$, the symmetry of Brownian motion shows that for $0 \leq s \leq t$,

$$\mathbb{P}^b(B_{t-s} \leq a) = \mathbb{P}^b(B_{t-s} \geq 2b - a).$$

Combining this with the strong Markov property for Brownian motion, it follows that $\mathbb{P}^0$ a.s. on the event $\{T_b \leq t\}$,

$$\begin{aligned}
\mathbb{P}^0\left(B_t \leq a | \mathcal{F}_{T_b+}\right) = \mathbb{E}^0\left[\mathbb{I}_{(-\infty,a]}(B_t)\right] &= \mathbb{E}^b\left[\mathbb{I}_{(-\infty,a]}(B_{t-T_b})\right] \\
&= \mathbb{E}^b\left[\mathbb{I}_{[2b-a,\infty)}(B_{t-T_b})\right] \\
&= \mathbb{P}^0\left(B_t \geq 2b - a | \mathcal{F}_{T_b+}\right).
\end{aligned}$$

Multiplying both sides of the last equation by $\mathbb{I}_{\{T_b \leq t\}}$ and noting that $\{T_b \leq t\} = \{M_t \geq b\}$, we obtain

$$\mathbb{P}^0(B_t \leq a, M_t \geq b) = \mathbb{P}^0(B_t \geq 2b - a, M_t \geq b) = \mathbb{P}^0(B_t \geq 2b - a) = \frac{1}{\sqrt{2\pi t}} \int_{2b-a}^{\infty} e^{-x^2/2t}\, dx.$$

The second assertion of the theorem follows on differentiating the above expression with respect to a and b. □

If $\{X_t, \mathcal{F}_t\}$ is a time-homogeneous Markov process, and for every $s, t \in [0, \infty)$ and $x \in \mathbb{R}^d$, $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathbb{P}(X_{s+t} \in A | X_s = x) = \int_A p(t; x, y) dy,$$

then $p(t; x, y)$ is said to be the transition density of the Markov process $\{X_t, \mathcal{F}_t\}$. (Here, note that the time-homogeneity of the Markov process refers to the fact that $p(t;x,y)$ does not depend on s). For example, the transition density of Brownian motion is the Gaussian kernel:

$$p(t;x,y) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-(x-y)^2/2t}\, dx.$$

Define $Y_t \doteq M_t - B_t$, $t \in [0,\infty)$. We show that $\{Y_t, \mathcal{F}_t\}$ is a Markov process under $\mathbb{P}^0$.

Lemma 4.3.4. *Under P^0, the process $\{Y_t, \mathcal{F}_t\}$ is a Markov process and has transition density*

$$\mathbb{P}^0(Y_{t+s} \in dy \mid Y_t = z) = (p(s;z,y) + p(s;z,-y)).dy$$

Proof. Since Y_t is a function of M_t and B_t and (M_t, B_t) was shown to be a Markov process in Theorem 4.3.3, it follows that for every $\Gamma \in \mathcal{B}(\mathbb{R})$,

$$P^0(Y_{t+s} \in \Gamma \mid \mathcal{F}_t) = P^0(Y_{t+s} \in \Gamma \mid B_t, M_t).$$

Therefore, to prove the lemma it suffices to show that

$$\mathbb{P}^0(Y_{t+s} \in dy \mid B_t = x, M_t = m) = (p(s; m-x, y) + p(s; m-x, -y))\, dy.$$

Fix $s, t \geq 0$. For $x \in \mathbb{R}$, $m \geq 0$, $b > m > x$, and $a \leq a$, we have

$$\begin{aligned}
\mathbb{P}^0(B_{t+s} \in da, &M_{t+s} \in db \mid B_t = x, M_t = m) \\
&= \mathbb{P}^0(B_{t+s} \in da, \max_{0 \leq u \leq s} B_{t+u} \in db \mid B_t = x, M_t = m) \\
&= \mathbb{P}^x(B_s \in da, M_s \in db) \\
&= \mathbb{P}^0(B_s \in da - x, M_s \in db - x) \\
&= \frac{2(2b-a-x)}{\sqrt{2\pi s^3}} \exp\left(-\frac{(2b-a-x)^2}{2s}\right) da\, db,
\end{aligned}$$

where the second equality above follows from the Markov property for Brownian motion, and the third equality follows from the homegeneity of Brownian motion, and the last equality follows from Theorem 4.3.3. A similar argument shows that when $b = m$,

$$\begin{aligned}
\mathbb{P}^0(B_{t+s} \in da, &M_{t+s} = m | B_t = x, M_t = m) \\
&= \mathbb{P}^0(B_{t+s} \in da, \max_{0 \leq u \leq s} B_{t+u} \leq m | B_t = x, M_t = m) \\
&= \mathbb{P}^x\left(B_s \in da, M_s \leq m\right) \\
&= \mathbb{P}^0(B_s \in da - x, M_s \leq m - x) \\
&= \frac{1}{\sqrt{2\pi s}} \left[\exp\left(-\frac{(a-x)^2}{2s}\right) - \exp\left(-\frac{(2m-a-x)^2}{2s}\right)\right] da.
\end{aligned}$$

Therefore

$$
\begin{aligned}
&\mathbb{P}^0(Y_{t+s} \in dy \mid B_t = x, M_t = m) \\
&\qquad = \int_{(m,\infty)} \mathbb{P}^0(B_{t+s} \in b - dy, M_{t+s} \in db \mid B_t = x, M_t = m) db \\
&\qquad\qquad + \mathbb{P}^0(B_{t+s} \in m - dy, M_{t+s} = m \mid B_t = x, M_t = m),
\end{aligned}
$$

which after some algebraic manipulations can be shown to be equal to $p(s; m - x, y) + p(s; m - x, -y)$. □