

∞ . When combined with (3.1), this shows that the variation of X over $[0, 1]$ must be finite, and the theorem is proved. \square

Note that in the proof of the above theorem, we constructed a sequence of continuous functions that could “look into the future of X ” since $h_n(t_k)$ depends on the value of $X(t_{k+1})$, where $t_k, t_{k+1} \in \Pi_n$. One of Itô’s key insights was to realize that if we restrict the class of integrands to those that are adapted to (the filtration associated with) the process $\{X_t\}$, then the limit (in probability) of the above Riemann sums does exist for continuous integrands. However, just as in deterministic integration, we would like to go beyond continuous integrands to more general, measurable (and integrable) integrands. To do so, we will first define the stochastic integral with respect to Brownian motion for a more general class of integrands, and then show that when restricted to continuous integrands, it can equivalently be defined as a limit (in probability) of Riemann sums.

3.1 Itô’s Formula

We start by stating Itô’s formula. Recall that $\mathcal{C}^{1,2}([0, \infty) \times \mathbb{R}^d)$ is the space of functions that are once continuously differentiable in $t \in [0, \infty)$ and twice continuously differentiable in $x \in \mathbb{R}^d$. In particular, Itô’s formula shows that the class of semimartingales is closed under the action of $\mathcal{C}^{1,2}$ functions.

Theorem 3.1.1. *Suppose X is a continuous semimartingale with decomposition $X = M + A$, where M is a continuous local martingale and A is a continuous finite variation process (i.e., each path of A has a.s. finite variation on each bounded interval). Then, given any $f \in \mathcal{C}^{1,2}([0, \infty) \times \mathbb{R}^d)$,*

$$\begin{aligned} f(X_t) &= f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, X_s) dX_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) d\langle X^i, X^j \rangle_s \\ &= f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, X_s) dM_s^i + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, X_s) dA_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) d\langle M^i, M^j \rangle_s \end{aligned}$$

Proof. Refer to text book. \square

3.1.1 Lévy’s Characterization of Brownian Motion

Theorem 3.1.2. *Suppose that $X = (X^{(1)}, X^{(2)}, \dots, X^{(d)})$ is a d -dimensional, continuous, \mathcal{F}_t -adapted process in \mathbb{R}^d such that for every component $1 \leq k \leq d$,*

the process

$$M_t^{(k)} \doteq X_t^{(k)} - X_0^{(k)}$$

is a continuous local martingale with respect to $\{\mathcal{F}_t\}$, and

$$\langle M^{(k)}, M^{(j)} \rangle_t = \delta_{jk}t.$$

Then $\{X_t, \mathcal{F}_t\}$ is a d -dimensional Brownian motion.

Proof. Proof given in class – also see page 126 of text book □

3.1.2 Bessel Processes and Recurrence and Transience of Multi-dimensional Brownian Motion

Just as martingales proved to be convenient in the study of mean first passage times of one-dimensional Brownian motion, they will also be useful in the computation of first-passage times of the d -dimensional Bessel process or, equivalently, the absolute value of d -dimensional Brownian motion. As demonstrated by the following result, Itô's formula provides a convenient way of constructing martingales associated with multi-dimensional Brownian motions.

Lemma 3.1.3. *Suppose that $f : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}$ is $C^{1,2}$, and that there exists a constant $K < \infty$ such that, for all $t \geq 0$, $x \in \mathbb{R}^d$,*

$$|f(t, x)| + \sum_{i=1}^d \left| \frac{\partial f}{\partial t}(t, x) \right| + \sum_{i=1}^d \sum_{j=1}^d \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(t, x) \right| \leq K e^{K(t+|x|)}. \quad (3.2)$$

Then the process

$$C_t^f \doteq f(t, B_t) - f(0, B_0) - \int_0^t \mathcal{G}f(s, B_s) ds$$

is a martingale, where

$$\mathcal{G}f(t, x) = \frac{\partial f}{\partial t}(t, x) + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_j^2}(t, x).$$

Proof. From Itô's formula, it is clear that C_t^f equals the stochastic integral

$$\sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, B_s) dB_s^i. \quad (3.3)$$

Thus C_t^f is in general a local martingale (even if the additional exponential growth condition on f is not satisfied). Now, by (3.2) and an application of Fubini's theorem, for every $i = 1, \dots, d$,

$$\begin{aligned} \mathbb{E} \left[\int_0^t \left| \frac{\partial f}{\partial x_i}(s, B_s) \right|^2 ds \right] &\leq K^2 \mathbb{E} \left[\int_0^t e^{2K(t+|B_s|)} ds \right] \\ &= K^2 e^{2Kt} \int_0^t \mathbb{E} \left[e^{2K|B_s|} \right] ds \\ &< \infty, \end{aligned}$$

where the last inequality follows from a standard calculation involving the multi-dimensional Gaussian distribution. By the construction of the stochastic integral, it then follows that C_t^f is in fact a (square-integrable) martingale. \square

Remark 3.1.4. Any function f that satisfies $\Delta f \doteq \sum_{i=1}^d \partial^2 f / \partial x_i^2 = 0$ is said to be harmonic. Lemma 3.1.3 shows that when f is harmonic then $f(B)$ is a local martingale.

Let B be a d -dimensional Brownian motion. In order to study Brownian motion in $d \geq 2$, we need to find some appropriate harmonic functions. Given the rotational invariance of the multi-dimensional standard normal distribution and Remark 3.1.4, it is natural to look for a function $f(x) = \varphi(|x|^2)$ and try to pick $\varphi : \mathbb{R} \mapsto \mathbb{R}$ such that $\Delta f = 0$. (Here we could just as well have used $|x|$, but we use $|x|^2$ since it is easier to differentiate.) Note that then

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= \varphi'(|x|^2) 2x_i \\ \frac{\partial^2 f}{\partial x_i^2} &= \varphi''(|x|^2) 4x_i^2 + 2\varphi'(|x|^2). \end{aligned}$$

Therefore, for $\Delta f = 0$, we need

$$\begin{aligned} 0 = \sum_{i=1}^d (\varphi''(|x|^2) 4x_i^2 + 2\varphi'(|x|^2)) &= 4|x|^2 \varphi''(|x|^2) + 2d\varphi'(|x|^2) \\ &= 4y\varphi''(y) + 2d\varphi'(y), \end{aligned}$$

where $y = |x|^2$. Therefore, for $y > 0$, we need to find a solution to the ordinary differential equation

$$\varphi''(y) = -\frac{d}{2y}\varphi'(y).$$

This shows that taking $\varphi'(y) = Cy^{-d/2}$ guarantees that $\Delta f = 0$ for $x \neq 0$. This implies that the function

$$f(x) = \begin{cases} \log |x| & d = 2, \\ |x|^{2-d} & d \geq 3, \end{cases}$$

is harmonic.

Theorem 3.1.5. For $0 < a < |x| < b$,

$$\mathbb{P}^x(T_a < T_b) = \begin{cases} \frac{\log b - \log |x|}{\log b - \log a} & d = 2 \\ \frac{|x|^{2-d} - b^{2-d}}{a^{2-d} - b^{2-d}} & d \geq 3 \end{cases}$$

Proof. Let $T = T_a \wedge T_b$. From the a.s. unboundedness of one-dimensional Brownian motion, it follows that $T < \infty$ a.s. As shown above, $f = 0$ on

$G = \{x \in \mathbb{R}^d : a < |x| < b\}$ and so by Remark 3.1.4, $f(B_{t \wedge T})$ is a local martingale. Since f is also continuous on G , f is bounded on G . This shows that $\{f(B_{t \wedge T}), t \in [0, \infty)\}$ is a uniformly bounded local martingale, and is therefore in fact a uniformly integrable martingale. Therefore, by a standard application of the optional stopping theorem, sending $t \rightarrow \infty$, interchanging limits and expectations and using the continuity of φ , the fact that $T < \infty$ a.s., it follows that

$$f(x) = \mathbb{E}_x[f(B_T)] = f(a)\mathbb{P}_x(T_a < T_b) + f(b)\mathbb{P}_x(T_b < T_a).$$

Since $\mathbb{P}_x(T_b < T_a) = 1 - \mathbb{P}_x(T_a < T_b)$, a minor rearrangement of the last equation yields the desired result. \square

Corollary 3.1.6. *For any $0 < a < |x|$,*

$$\mathbb{P}^x(T_a < \infty) = \begin{cases} 1 & \text{if } d = 2 \\ \left(\frac{a}{|x|}\right)^{d-2} & \text{if } d \geq 3. \end{cases}$$

Proof. Since $\{T_a < \infty\} = \cup_{b \in \mathbb{N}, b > a} \{T_a < T_b\}$, we have $\mathbb{P}(T_a < \infty) = \lim_{b \rightarrow \infty} \mathbb{P}(T_a < T_b)$ and so the result follows from Theorem 3.1.5. \square

Corollary 3.1.7. *When $d \geq 2$,*

$$\mathbb{P}^x(T_0 < \infty) = 0.$$

Proof. It suffices to prove the result for $d = 2$ – this automatically implies the result for all $d \geq 2$. Note that

$$\{T_0 < \infty\} = \cup_{n \in \mathbb{N}} \{T_0 \leq T_n\} = \bigcup_{n \in \mathbb{N}} \left(\bigcap_{m > |x|^{-1}, m \in \mathbb{N}} \{T_{1/m} \leq T_n\} \right).$$

However, for every $n \in \mathbb{N}$, by Theorem 3.1.5,

$$\begin{aligned} \mathbb{P}^x \left(\bigcap_{m > |x|^{-1}, m \in \mathbb{N}} \{T_{1/m} \leq T_n\} \right) &= \lim_{m \rightarrow \infty} \mathbb{P}^x(T_{1/m} \leq T_n) \\ &= \lim_{m \rightarrow \infty} \frac{\log n - \log |x|}{\log n + \log m} = 0, \end{aligned}$$

which proves the result. \square

The Bessel Process

Let B be a d -dimensional standard Brownian motion starting at $x \in \mathbb{R}^d$, with $|x| = r$. For $d \geq 2$, the d -dimensional Bessel process R (or BES (d) process, for short), is defined by

$$R_t = |B_t|, \quad t \in [0, \infty).$$

Bessel processes are arguably the most important one-dimensional diffusions apart from Brownian motion. Due to the rotation invariance of Brownian motion, it is easy to see that R is well-defined, being independent of the point x chosen on the sphere in \mathbb{R}^{d-1} of radius r , and is also a Markov process with respect to its own filtration (see Chapter 4). We now derive the SDE satisfied by the Bessel process which shows, in particular, that it is a diffusion.

Proposition 3.1.8. *For $d \geq 2$, let R be a BES (d) process, starting at $r \geq 0$. Then R satisfies*

$$R_t = r + \int_0^t \frac{d-1}{2R_s} ds + \tilde{B}_t, \quad (3.4)$$

where

$$\tilde{B}_t = \sum_{i=1}^d \tilde{B}^{(i)}, \quad \tilde{B}^{(i)} = \int_0^t \frac{B_s^{(i)}}{R_s} dB_s^{(i)}.$$

is a standard Brownian motion.

Proof. Let $\mathbb{P} = \mathbb{P}^r$ be the measure induced by R . Since $\{R_t\}$ can be at the origin only when $\{B_t\}$ is at the origin, and the Lebesgue measure of the set $\{s \in [0, t] : R_s = 0\}$ is zero \mathbb{P} a.s., the integrand in (3.4) is well-defined. Moreover, each process $\tilde{B}^{(i)} \in \mathcal{M}_2^c$ because

$$\mathbb{E} \left[\int_0^t \left(\frac{B_s^{(i)}}{R_s} \right)^2 ds \right] \leq t.$$

Moreover,

$$\langle \tilde{B}^{(i)}, \tilde{B}^{(j)} \rangle_t = \int_0^t \frac{1}{R_s^2} B_s^{(i)} B_s^{(j)} d\langle B^{(i)}, B^{(j)} \rangle_s = \delta_{ij} \int_0^t \frac{1}{R_s^2} B_s^{(i)} B_s^{(j)} ds,$$

where δ_{ij} is the Kronecker delta function. This implies that

$$\langle \tilde{B} \rangle_t = \sum_{i=1}^d \langle \tilde{B}^{(i)} \rangle_t = t.$$

From Lévy's characterization theorem, it follows that \tilde{B} is indeed a d -dimensional Brownian motion.

To prove (3.4), if $r > 0$ then Corollary 3.1.7 shows that R lives in $\mathbb{R}^d \setminus \{0\}$. Since

$$f(x) = \|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2} : \mathbb{R}^d \mapsto [0, \infty)$$

we have, for $1 \leq i, j \leq d$, and $x \in \mathbb{R}^d \setminus \{0\}$

$$\frac{\partial f}{\partial x} = \frac{x_i}{\|x\|} \quad \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{\|x\|} - \frac{x_i x_j}{\|x\|^3}.$$

Therefore f is twice continuously differentiable on $\mathbb{R}^d \setminus \{0\}$ and so by Itô's formula shows that

$$R_t = f(B_t) = r + \frac{1}{2} \int_0^t \sum_{i=1}^d \left(\frac{1}{\|B_s\|} - \sum_{j=1}^d \frac{B_s^{(i)} B_s^{(j)}}{\|B_s\|^3} \right) ds + \sum_{i=1}^d \int_0^t \frac{B_s^{(i)}}{R_s} dB_s,$$

which proves (3.4).

Note that the above use of Itô's formula can be rigorously justified by considering a suitable sequence $\{f_n\}$ of continuously differentiable functions on \mathbb{R}^d that converge to f as $n \rightarrow \infty$, apply Itô's formula to each of the f_n and then take limits to show that the right-hand side converges to the r.h.s. of (3.4). We omit the details of this technical argument. \square