

## 2.4 Stochastic Integration

From the theory of Riemann-Stieltjes Integration, we know that if a stochastic process  $X$  has bounded (first) variation paths, and  $Y$  is another continuous process, then the process  $\int Y dX$  can be defined pathwise as a Riemann-Stieltjes integral. However, we know that Brownian motion has a.s. infinite variation on finite intervals. It is natural to ask if, nevertheless, the integral of a continuous process with respect to Brownian motion can also be defined pathwise as a limit of Riemann sums. This motivates the following question. Let  $X$  be a right continuous function on  $[0, 1]$  and let  $\{\Pi_n\}$  be a refining sequence of dyadic rationals such that  $\|\Pi_n\| \rightarrow 0$ . What conditions are needed on  $X$  so that the sums  $S_n = \sum_{\Pi_n} h(t_k)(x(t_{k+1}) - x(t_k))$  converge to a finite limit for all continuous  $h$ ?

**Theorem 2.4.1.**  *$X$  must be of finite variation for the above question to be answered in the affirmative.*

*Proof.* Let  $\Pi_n$  be a sequence of partitions in  $[0, 1]$  with  $\|\Pi_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $\mathcal{C} = C[0, 1]$  and  $Y = \mathbb{R}$ , and for  $h \in \mathcal{C}$ , let

$$T_n(h) = \sum_{t_k \in \Pi_n} h(t_k)(X(t_{k+1}) - X(t_k)).$$

Then for each  $n \in \mathbb{N}$ ,  $T_n$  is a linear operator from  $\mathcal{C}$  to  $\mathbb{R}_+$  and if  $h$  is uniformly bounded by  $C$ , then clearly  $\|T_n(h)\| \leq C|X(1) - X(0)|$  and so  $\{T_n, n \in \mathbb{N}\}$  is a collection of continuous, linear operators from  $\mathcal{C}$  to  $\mathbb{R}_+$ . For each  $n \in \mathbb{N}$ , it is clearly possible to construct  $h_n$  in  $\mathcal{C}$  such that  $h_n(t_k) = \text{sgn}(X(t_{k+1}) - X(t_k))$  for every  $t_k \in \Pi_n$  and  $\|h_n\| = 1$ . For such an  $h_n$ , we have  $T_n(h_n) = \sum_{t_k \in \Pi_n} |X(t_{k+1}) - X(t_k)|$ , and so

$$\|T_n\| \geq \text{Var}_{[0,1]}[\Pi_n](X), \quad \text{for every } n \in \mathbb{N}. \quad (2.4)$$

Now, suppose that for all  $h \in \mathcal{C}$ ,  $\lim_{n \rightarrow \infty} T_n(h)$  exists, so that  $\sup_n \|T_n(h)\| < \infty$ . Then, by the Banach-Steinhaus theorem (see Theorem A.6.5) applied to the family of linear operators  $\{T_n, n \in \mathbb{N}\}$  from  $\mathcal{C}$  to  $\mathbb{R}$ , it follows that  $\sup_n \|T_n\| < \infty$ . When combined with (2.4), this shows that the variation of  $X$  over  $[0, 1]$  must be finite, and the theorem is proved.  $\square$

Note that in the proof of the above theorem, we constructed a sequence of continuous functions that could “look into the future of  $X$ ” since  $h_n(t_k)$  depended on the value of  $X(t_{k+1})$ , where  $t_k, t_{k+1} \in \Pi_n$ . One of Itô’s key insights was to realize that if we restrict the class of integrands to those that are adapted to (the filtration associated with) the process  $\{X_t\}$ , then the limit (in probability) of the above Riemann sums does exist for continuous integrands. However, just as in deterministic integration, we would like to go beyond continuous integrands to more general, measurable (and integrable) integrands. To do so, we will first define the stochastic integral with respect to Brownian motion for a more general class of integrands, and then show that when restricted to continuous integrands, it can equivalently be defined as a limit (in probability) of Riemann sums.