Chapter 1

Brownian Motion

The term "Brownian motion" refers to two things. The first refers to the inherent, incessant motion of small particles suspended in a fluid, named after the botanist Robert Brown. While this motion was observed by others before him, Robert Brown, in 1827, was the first to give this phenomenon serious scientific study – in particular he showed that this phenomenon was not one of biology, but one of physics. The second meaning of Brownian motion is a well-defined mathematical object – the generic example of a continuous time, $\mathbb{R}$-valued stochastic process – which shall be the focus of our study. The study of physical phenomena led scientists such as Bachelier, Einstein and Smoluchowski in the early 1900's (for more details, see the historical note at the end of Section 1.1) to postulate the existence of a stochastic process with certain properties, stated as the Brownian motion postulate (BMP) below.

Brownian Motion Postulate (BMP). There exists a stochastic process $B = \{B_t, t \in [0, \infty)\}$ with the following properties:

1. (Normal marginals) $B_0 = 0$ and for every $t \in [0, \infty)$, $B_t \sim \mathcal{N}(0, t)$.
2. (Time homogeneity or stationarity) Given any $t \in [0, \infty)$, $B_{t_0+t} - B_{t_0}$ has the same distribution for all $t_0 \in [0, \infty)$.
3. (Independent increments) For $0 \leq s < t < \infty$, $B_t - B_s$ is independent of $\{B_u, 0 \leq u \leq s\}$.
4. (Continuity of paths) For almost every realization, $t \to B_t$ is a continuous function.

Here, $\mathcal{N}(0, t)$ refers to the normal distribution with mean zero and variance t.

In Section 1.4, we will show that there does indeed exist a process that satisfies these postulates, which induces a unique measure on the space of continuous functions, referred to as *Wiener measure*. In fact, we will show that there exists a process satisfying a slightly more general version of these postulates, which is phrased in the terminology of stochastic processes that is introduced in Section

1.2. Before entering the purely mathematical realm, however, in Section 1.1, we shall try to discover these postulates for ourselves – in order to try to get some feel for how experimentally observed randomness may be modelled in terms of stochastic processes and also how scaling limits are identified.

1.1 A Thought Experiment

Brownian motion draws much of its importance from the fact that it has a certain "universality" property in that it arises as the scaling limit of various discrete objects. Since the identification of scaling limits is a fundamental and recurring theme in probability theory, let us briefly look at the simple and insightful experiment described in [9] to "discover" what properties we would expect the scaling limit of the simplest possible discrete-time stochastic process, namely a random walk, to possess.

Let $\xi_k, k = 1, 2, \ldots,$ be a collection of independent and identically distributed (i.i.d.) random variables such that

$$\xi_k \sim U[0,1] \qquad k = 1, 2, \ldots,$$

where $U[0,1]$ represents the uniform distribution on $[0,1]$. You may think of ξ_k as the payoff (in units of \$1000 dollars) received after spinning a wheel in some game show. Then, setting the starting position S_0 to be equal to 0, the cumulative payoff after n spins is given by the nth partial sum

$$S_n = \sum_{k=1}^{n} \xi_k, \qquad n = 1, 2, \ldots$$

The successive partial sums form a *random walk*, with ξ_k being the kth step of the walk, and S_n equal to the position after n steps. We set the starting position S_0 to be equal to 0.

In order to see how the cumulative payoff evolves with the number of spins, for a range of n values, let us plot S_i vs. i, for $i = 0, 1, \ldots, n$. (This simulation experiment is easy to perform – for example by using the statistical package S or $S+$ [9, Section 1.1.1].) Fig. 1.1 of the first handout given to you (which was reproduced from [9, Section 1.1.1]) illustrates four such plots for $n = 10^j$, $j = 1, 2, 3, 4$, for a given random seed or realization ω. Repeated experiments with different random seeds suggest that, as $n \to \infty$, the sequence of plots almost always converges to the same linear function that connects the lower left-hand corner to the upper right-hand corner of the box. This suggests that this linear function is an a.s. limit (as $n \to \infty$) of the sequence of random functions obtained by interpolating the points in the nth plot. To make this statement mathematically precise, we need to express the interpolated function in the nth plot explicitly in terms of the primitive random variables ξ_k. As n increases, clearly the range of possible values of S_n also increases and, if the plotter were to use the same units for all plots, it would need to plot on increasingly large pieces of paper! This is clearly infeasible, and so this is not what the plotter does. Instead, the plotter *automatically rescales* units so that the nth plot, for every n, fits into the same rectangular box. As a result, in the nth plot the plotter plots $\tilde{S}_i^n$ vs. i/n, $i = 0, \ldots, n$, where

$$\tilde{S}_i^n \doteq \frac{S_i - \min_{k=0,1,\ldots,n} S_k}{\max_{k=0,1,\ldots,n} S_k - \min_{k=0,1,\ldots,n} S_k} = \frac{S_i}{S_n}, \tag{1.1}$$

with the last equality following from the fact that $\min_{k=0,1,\ldots,n} S_k = S_0 = 0$ and $\max_{k=0,1,\ldots,n} S_k = S_n$ since $\xi_i \geq 0$ for every $i = 1, \ldots, n$. Note that with this rescaling, the range of values of both $\tilde{S}_i^n$ and i/n, for $0 \leq i \leq n$, is equal to $[0, 1]$, as desired. It is clear that for all n, $\tilde{S}_0^n = 0$ and $\tilde{S}_n^n = 1$. How do the intermediate values behave for large n ? Suppose neighbouring values in the nth plot are connected by lines so as to obtain a piecewise linear function S^n on $[0, 1]$. In other words, let S^n be the function that satisfies $S^n(i/n) = \tilde{S}_i^n$ for $i = 0, \ldots, n$ (or, equivalently, $S^n(t) = \tilde{S}_{nt}^n$ for $t = i/n$, $i = 0, \ldots, n$) and whose intermediate values are obtained by linear interpolation, i.e., for $t \in [0, 1]$,

$$\begin{aligned} S^n(t) &= \tilde{S}_{\lfloor nt \rfloor}^n + \left(t - \frac{\lfloor nt \rfloor}{n}\right)\left(\tilde{S}_{\lfloor nt \rfloor + 1}^n - \tilde{S}_{\lfloor nt \rfloor}^n\right) \\ &= \frac{S_{\lfloor nt \rfloor}}{S_n} + \left(t - \frac{\lfloor nt \rfloor}{n}\right)\frac{\xi_{\lfloor nt \rfloor + 1}}{S_n}. \end{aligned} \tag{1.2}$$

(Here, $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.) What do we expect the function S^n to look like for large n ? Since the ξ_k are independent and identically distributed with mean $E\xi_1 = 1/2 < \infty$, recall that the strong law of large numbers (SLLN) dictates that

$$\lim_{n \to \infty} \frac{S_n}{n} = E[\xi_1] = 1/2 \quad \text{a.s.}$$

Thus the SLLN tells us that for almost every realization of the random walk, the average step size of the random walk converges to the same deterministic quantity $E[\xi_1]$. Consequently, for $t \in [0, 1]$ and almost surely every realization, we have

$$\begin{aligned} \lim_{n \to \infty} \frac{S_{\lfloor nt \rfloor}}{S_n} &= \lim_{n \to \infty} \frac{S_{\lfloor nt \rfloor}}{\lfloor nt \rfloor} \frac{\lfloor nt \rfloor}{n} \frac{n}{S_n} \\ &= E[\xi_1].t.\frac{1}{E[\xi_1]} \\ &= t. \end{aligned}$$

In addition, since each ξ_k is bounded by 1 and $S_n \to \infty$ almost surely as $n \to \infty$, one obtains

$$\lim_{n \to \infty} \sup_{t \in [0,1]} \frac{\xi_{\lfloor nt \rfloor + 1}}{S_n} = 0 \text{ a.s.}$$

The last two displays, when combined with (1.2), show that the sequence of random functions S^n almost surely converges pointwise to the linear function having slope 1, which appears consistent with the fourth plot in Fig. 1.1. (with

$n = 10^4$). Thus we have found some theoretical justification for the experimentally observed phenomenon. (With a slightly stronger estimate on the rate of convergence in the SLLN, it is possible to show that this convergence takes place with respect to the uniform or supremum norm on function space.)

The convergence that we are conjecturing above is an example of a *functional strong law of large numbers* (FSLLN), where a sequence of random functions is shown to converge (in a suitable topology and almost surely) to a deterministic function. Compare this with the SLLN, where a sequence of random variables taking values in $\mathbb{R}$ (or $\mathbb{R}^d$) converges, almost surely, to a constant (or vector). A common theme in the basic theory of stochastic processes will be to study how many of the concepts of probability theory and measure theory on $\mathbb{R}^d$ may be extended to analyze random quantities taking values in function spaces, and also to address interesting new questions that arise in function space, which are not relevant in $\mathbb{R}^d$.

Returning to our thought experiment, if you are allowed to spin for free with the payoffs increasing linearly with the number of spins, the game organizers would go bankrupt ! So, instead suppose a fee of 0.5 units is charged for a spin – this is a fair price since it is equal to the mean payoff expected by the player from a single spin. Thus the net wealth of a player after n spins is given by

$$Z_n = \sum_{k=1}^{n} \left(\xi_k - \frac{1}{2} \right) = S_n - \frac{n}{2}.$$

We now have what is called a *centered random walk*. Note that Z_n measures deviations of the partial sums from their means – in terms of the plots, this means that we are measuring deviations of the points from the diagonal of the rectangle in Figure 1.1. Since the strong law of large numbers tells us that S_n looks like $n/2$ for large n (so that all points tend to lie along or very close to the diagonal in the limit of large n), it is natural to expect that the plots of Z_n would be close to the zero horizontal line. To see if this is indeed the case, let us now plot Z_i vs. i, for $i = 1, \ldots, n$, for different values of n (see Figure 1.2). It turns out that the points are not close to zero (see Figure 1.2), but instead exhibit substantial deviations from zero even for very large n. Furthermore, unlike in Figure 1.1, the sequence of functions obtained by interpolating points in the nth plot do not appear to be converging to any deterministic function for large n. In order to gain a better understanding of this behavior, we need to take a closer look at what the plotter is plotting. Just as in (1.1), the nth plot actually shows $\tilde{Z}_i^n$ vs. i/n, where

$$\tilde{Z}_i^n = \frac{Z_i - \min_{k=1,\ldots,n} Z_k}{R_n}, \tag{1.3}$$

and R_n is the range given by

$$R_n = \max_{k=1,\ldots,n} Z_k - \min_{k=1,\ldots,n} Z_k.$$

Note that, once again, $\tilde{Z}_i^n$ always lies in the range $[0, 1]$ of the box, as desired. As before, let Z^n be the piecewise linear function on $[0, 1]$ obtained by linearly connecting the points in the nth plot, so that $Z^n(i/n) = \tilde{Z}_i^n$, for $i = 0, \ldots, n$, and $Z^n(t)$ is defined by linear interpolation for all other values of $t \in [0,1]$:

$$Z^n(t) = \tilde{Z}^n(\lfloor nt \rfloor) + (\lfloor nt \rfloor - nt)\left(\tilde{Z}^n(\lfloor nt \rfloor + 1) - \tilde{Z}^n(\lfloor nt \rfloor)\right).$$

For any $\alpha > 0$, one can rewrite

$$Z^n(t) = \frac{n^\alpha}{R_n}\left[\frac{S_{\lfloor nt \rfloor} - nt/2}{n^\alpha} - \min_{t \in [0,1]} \frac{S_{\lfloor nt \rfloor} - nt/2}{n^\alpha} + o(n^{-\alpha})\right], \tag{1.4}$$

where $o(n^{-\alpha})$ accounts for the $\xi_{\lfloor nt \rfloor + 1}$ terms that arise due to interpolation. Now recall that while the SLLN tells us that for every t

$$\lim_{n\to\infty} \frac{S_{\lfloor nt \rfloor} - nt/2}{n} = 0 \quad \text{a.s.},$$

the central limit theorem (CLT) says that

$$\frac{S_{\lfloor nt \rfloor} - nt/2}{\sqrt{n}} = \sqrt{n}\left(\frac{S_{\lfloor nt \rfloor} - nt/2}{n}\right) \Rightarrow^{n\to\infty} \mathcal{N}(0, E\xi_1^2),$$

where $\Rightarrow$ represents convergence in distribution and $\mathcal{N}(0,\sigma^2)$ is the Gaussian distribution with zero mean and variance equal to σ^2 (which equals $1/24$ in this particular example). From (1.4) we see that the CLT suggests that the term within the bracket would go to zero if $\alpha > 1/2$ and go to infinity if $\alpha < 1/2$ (note that this has to be rigorously justified because of the second term in the bracket – while this can be done, we shall not do so at this juncture). This suggests that R_n must be of order $n^{1/2}$ (i.e., $R_n/\sqrt{n}$ converges in distribution to some non-zero limit) if $Z^n(t)$ should converge to something finite but non-trivial (as observed). This leads us to conjecture that the plots in Figure 1.2 should show statistical regularity for large n. What properties would we expect the limit random process to satisfy ?

The sequence of processes $\{Z^n\}$ is somewhat complicated by the fact that the numerator of $\tilde{Z}_i^n$ considers the relative position of Z_i with respect to its minimum (rather than Z_i itself), and the denominator R_n is random. We will return to this more complicated process at a later date. For now, to simplify matters, taking a cue from the CLT let us define for $i = 0, 1, \ldots, n$,

$$\tilde{X}_i^n \doteq \frac{S_i - i/2}{\sqrt{n}},$$

and let X^n be the function on $[0,1]$ given by

$$X^n(i/n) = \tilde{X}_i^n = \frac{Z_i}{\sqrt{n}} = \frac{S_i - i/2}{\sqrt{n}} \tag{1.5}$$

for $i = 0, \ldots, n$, with X^n defined by linear interpolation for all other points $t \in [0,1]$. It follows from the central limit theorem that the distribution of $X^n(1)$

would converge weakly to the Gaussian distribution $\mathcal{N}(0, E[\xi^2])$, as $n \to \infty$. Now observe that

$$\begin{aligned} X^n(t) &= \frac{S_{\lfloor nt \rfloor} - nt/2}{\sqrt{n}} + \frac{(t - \lfloor nt \rfloor)\xi_{\lfloor nt+1 \rfloor}}{\sqrt{n}} \\ &= \sqrt{t}\frac{S_{\lfloor nt \rfloor} - \lfloor nt \rfloor/2}{\sqrt{\lfloor nt \rfloor}}\sqrt{\frac{\lfloor nt \rfloor}{nt}} + \frac{(t - \lfloor nt \rfloor)\xi_{\lfloor nt+1 \rfloor} + (\lfloor nt \rfloor - nt)/2}{\sqrt{n}}. \end{aligned}$$

Since the last term clearly converges uniformly to zero as $n \to \infty$, it follows from the FCLT that for every $t \in [0,1]$, $X^n(t) \Rightarrow \mathcal{N}(0,t)$. Thus we see that if the interpolations of the plotted functions converge in any sense to a stochastic process, the limit process should have normally distributed marginals with zero mean and variance t. In other words, it should satisfy the first condition of the BMP. Moreover, since the ξ_k are i.i.d., we would also expect the limit process to be time-homogeneous and have independent increments. From the plots it seems plausible that the limit trajectories are continuous (although, accounting for numerical error, it is not that easy to conclude from the plots whether there is a wild oscillation at a point or whether it is a jump – and this issue can be resolved only by a more rigorous analysis). Thus we see that our "experimental observations" have led us to postulate the existence of a process that satisfies the BMP. We will provide several proofs of the existence of such a process in Section 1.4. The second proof, referred to as Donsker's Invariance Principle, provides a rigorous justification of the conjectured convergence above, of $X^{(n)}$ to Brownian motion. As we will see later, this convergence is in a sense that is stronger than the convergence $X^{(n)}(t) \Rightarrow B(t)$ for every $t \in [0, \infty)$ for which we provided evidence above, and is strong enough to show that

$$\min_{t\in[0,1]} \frac{S_{\lfloor nt \rfloor} - nt/2}{n^{1/2}} \Rightarrow 0 \qquad \text{as } n \to \infty,$$

and to characterize the limit of $\sqrt{n}/R_n$, allowing us to deduce the limit of Z^n from that of X^n.

Historical Context. In 1827 the Irish botanist Robert Brown noticed the erratic motion of pollen grains, but his empirical observations went largely unnoticed since he had no scientific explanation for them. Much later, in 1900, the dissertation of Bachelier, under the supervision of H. Poincaré, modelled the stock market by a stochastic process that corresponds to what we call Brownian motion today. However, since the measure theoretic foundations of probability had not been fully developed at the time, Bachelier's ideas were not appreciated. It was also only in the early 1900's that attention returned to the observations of Brown. The (then controversial) idea was floated by several people that the erratic motion observed by Brown was caused because the pollen grains were being kicked around by randomly moving molecules of the surrounding liquid. This idea had to be repeatedly discarded because calculable properties of the (possibly hypothetical) molecular motion could not seemingly be reconciled with the few known experimental facts about Brownian motion. This apparent contradiction was satisfactorily resolved in two remarkable theoretical papers in 1905 – by Marian Smoluchowski and Albert Einstein. (Von Smoluchowski adopted a completely different approach from Einstein, and he suffered from having to share his first great discovery with a luminary like Einstein !) Einstein (apparently unaware and independent of Bachelier's results) proposed a stochastic process to model the displacement of the pollen grains, which he assumed would satisfy a set of properties akin to the BMP listed above. Assuming the existence of such a stochastic process, he went on to develop a molecular kinetic theory of heat, which was not only consistent with previous observations, but went further to make new theoretical predictions of Avogadro's number, which were capable of being tested experimentally. These (apparently brilliant) experiments, which established the atomistic theory of matter, were carried out by the French scientist Jean Perrin, who won the Nobel Prize in Physics for them in 1926. The mathematical question of the consistency of the properties of the stochastic process assumed by Einstein was not resolved till 18 years later in 1923 by Norbert Wiener (who also credits Paul Lévy for some enlightening conversations). Wiener was apparently motivated by wanting to justify a remark in Jean Perrin's book *"Les Atomes"* which said something to the effect that the irregular curves followed by particles in Brownian motion led one to think of the supposed continuous nondifferentiable curves of the mathetmaticians.

1.2 Stochastic processes

1.2.1 Basic Definitions

Let (S, ρ) be a metric space (we will mostly deal with the case when (S, ρ) is $\mathbb{R}^d$, often with $d = 1$, equipped with the Euclidean metric). Let $\mathcal{B}(S)$ be the associated Borel σ-algebra, generated by the open sets in S. A stochastic process can be viewed in several different ways.

Definition 1.2.1. *A* stochastic process *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* state space $(S, \mathcal{B}(S))$ *is (equivalently) a*

1. *(one-parameter) family of random variables,*
   $\{X_t : (\Omega, \mathcal{F}) \to (S, \mathcal{B}(S)) \mid t \in [0, \infty)\}$*;*

2. *a random element taking values in* $(S^{[0,\infty)}, \otimes_{[0,\infty)}\mathcal{B}(S))$*,*
   $X = \{X_{(\cdot)}(\omega) : ([0, \infty), \mathcal{B}[0, \infty)) \to (S^{[0,\infty)}, \otimes_{[0,\infty)}\mathcal{B}(S)) \mid \omega \in \Omega\}$

3. *a mapping of two variables* $X : (t, \omega) \to X_t(\omega)$ *from* $[0, \infty) \times \Omega$ *to* $(S, \mathcal{B}(S))$*, where for every* $t \in [0, \infty)$ *and* $A \in \mathcal{B}(S)$*,* $\{\omega : X(t, \omega) \in A\} \in \mathcal{F}$*.*

In Definition 1.2.1(2) above, $\otimes_{[0,\infty)}\mathcal{B}(S)$ is the product σ-algebra, which is generated by sets of the form

$$\{w \in \mathbb{R}^{[0,\infty)} : w_t \in A\}, \quad A \in \mathcal{B}(S), t \in [0, \infty).$$

Moreover, when adopting this perspective, the functions $X_{(\cdot)}(\omega)$, $\omega \in \Omega$, are referred to as sample paths of the stochastic process (here, the function associated with the "sample" or "realization" $\omega \in \Omega$ maps each $t \in [0, \infty)$ to $X_t(\omega)$). It is not too hard to verify that these three definitions above are indeed equivalent.

The different perspectives of a stochastic process taken above lead immediately to different ways in which stochastic processes can be compared.

Definition 1.2.2. *(Comparison of stochastic processes) Let* X *and* X' *be stochastic processes that are defined on the probability spacess* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $(\Omega', \mathcal{F}', \mathbb{P}')$*, respectively.*

1. X *and* X' *are said to be* equal in distribution *iff* $\mathbb{P}(X \in A) = \mathbb{P}'(X' \in A)$ *for all* $A \in \mathcal{B}(\mathbb{R}^{[0,\infty)})$ *then* X *and* X'*.*

2. X *and* X' *are said to have the same* finite dimensional marginal distributions *(abbreviated to* fi.di. distributions*) iff for every* $t_1, \ldots, t_n$*, we have*

$$\mathbb{P}((X_{t_1}, \ldots, X_{t_n}) \in A) = \mathbb{P}'((X'_{t_1}, \ldots, X'_{t_n}) \in A)$$

   for all $A \in \mathcal{B}(S^n)$*.*

In addition, if $(\Omega', \mathcal{F}', \mathbb{P}') = (\Omega, \mathcal{F}, \mathbb{P})$*, so that* X *and* X' *are both defined on the same probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, then*

3 *X and X' are said to be* indistinguishable *iff* $\mathbb{P}(X_t = X'_t$ *for every* $t \in [0,\infty)) = 1$.

4. *X is said to be a* modification *of X' iff* $\mathbb{P}(X_t = X'_t) = 1$ *for all* t.

In many situations (for example, see statements 3 and 4 of Definition 1.2.2), one is only interested in the distributions (or image measures) of a random element. In such cases, it is often convenient to represent the random element "canonically".

Definition 1.2.3. *(Canonical Representation) Given an $(S, \mathcal{S})$-valued random element X defined on $(\Omega, \mathcal{F}, \mathbb{P})$, the canonical representation of X is the random element $\tilde{X}$ defined on $(S, \mathcal{S}, \mathbb{P}X^{-1})$ by*

$$\tilde{X}(\omega) = \omega, \qquad \omega \in S,$$

where recall that $\mathbb{P}X^{-1}$ is the image measure

$$\mathbb{P}X^{-1}(A) = \mathbb{P}(\omega : X(\omega) \in A), \qquad A \in \mathcal{S}.$$

Remark 1.2.4. The canonical representation of a stochastic process is obtained by considering $\{X_t\}$ as an $\mathbb{R}^{[0,\infty)}$-valued random element. More precisely, the canonical representation of an $\mathbb{R}$-valued stochastic process $\{X_t\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is the stochastic process $\{\tilde{X}_t\}$ defined on $(\mathbb{R}^{[0,\infty)}, \otimes_{[0,\infty)}\mathcal{B}(\mathbb{R}), \mathbb{P}X^{-1})$ by

$$\tilde{X}_t(\omega) = \omega_t, \qquad \omega \in \mathbb{R}^{[0,\infty)}.$$

1.2.2 Filtrations, Measurability and Adaptedness.

A filtration $\{\mathcal{F}_t\} = \{\mathcal{F}_t\}_{t\in[0,\infty)}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a nested family of σ-algebras: $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ if $s \leq t$. A probability space, along with a filtration, is said to be a filtered probability space. Intuitively, it captures the notion of information flow in the stochastic process – loosely speaking, you can think of the sets contained in $\mathcal{F}_t$ as being those events about which you have knowledge, and to which you can assign a probability, at time t. A filtration is said to be right-continuous if $\mathcal{F}_{t+} \doteq \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$ and left-continuous if $\mathcal{F}_{t-} \doteq \sigma\left(\cup_{\varepsilon>0}\mathcal{F}_{t-\varepsilon}\right) = \mathcal{F}_t$.

Definition 1.2.5. *A stochastic process $\{X_t\}$ is said to be adapted to the filtration $\{\mathcal{F}_t\}$ iff X_t is $\mathcal{F}_t$-measurable for every $t \in [0,\infty)$; in other words, for every $t \in [0,\infty)$ and $A \in \mathcal{B}(S)$,*

$$\{\omega \in \Omega : X_t(\omega) \in A\} \in \mathcal{F}_t.$$

Given a stochastic process $\{X_t\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the natural filtration, $\{\mathcal{F}_t^X\}$, is defined by

$$\mathcal{F}_t^X \doteq \sigma\{X_s, s \in [0,t]\}, \qquad t \in [0,\infty).$$

$\{X_t\}$ is clearly adapted to $\{\mathcal{F}_t^X\}$.