21-880: Advanced Stochastic Calculus I – Fall 2009
Homework Assignment 5
(Distributed , 2009)
(Due Wednesday, November 18, 2009, *before class*)

1. Let $T_0 \doteq \inf\{s > 0 : B_s = 0\}$. Let $R = \inf\{t > 1 : B_t = 0\}$. Use the Markov property at time 1 to show *rigorously* that

$$\mathbb{P}_x(R > 1 + t) = \int p_1(x, y)\mathbb{P}_y(T_0 > t)dy.$$

Use this and the probability density of T_0 to show that R has probability density

$$f_R(t) = \begin{cases} \dfrac{1}{\pi\sqrt{t-1}t} & t > 1 \\ 0 & \text{otherwise.} \end{cases}$$

2. Let B be one-dimensional Brownian motion. Use the Markov property of Brownian motion to show that for any $s < \infty$,

$$\mathbb{P}_x \left(B_t = y \quad \text{for some } t \geq s\right) = 1.$$

From this, conclude that a.s. $B_t = y$ infinitely often. By definition, this implies that one-dimensional Brownian motion is recurrent.