21-880: Advanced Stochastic Calculus I – Fall 2009
Homework Assignment 4
(Distributed October 31, 2009)
(Due Wednesday, November 11, 2009, *before class*)

1. The following problem demonstrates a connection between partial differential equations and solutions to SDEs:
(a) Let $a(t,x)$, $b(t,x)$ be functions on $[0,T]\times(-\infty,\infty)$ that are continuous and such that $\partial b/\partial x$ is also continuous. Let the function $u(t,y)$, $(t,y) \in [0,T]\times(-\infty,\infty)$ have continuous derivatives $\partial u/\partial t$, $\partial u/\partial y$ and $\partial^2 u/\partial y^2$, and satisfy the partial differential equations:
$$\frac{\partial u}{\partial t}(t,y) = a(t,u) - \frac{1}{2}b(t,u)\frac{\partial b}{\partial x}(t,u) \qquad \frac{\partial u}{\partial y}(t,y) = b(t,u)$$
for $(t,y) \in [0,T]\times(-\infty,\infty)$ with initial condition $u(0,0) = x_0$. If B is Brownian motion, prove that $X_t = u(t,B_t)$ satisfies
$$dX_t = a(t,X_t)dt + b(t,X_t)dB_t.$$
(b) Use the result from part (a) to solve the Itô stochastic differential equation
$$dX_t = \frac{1}{4}dt + \sqrt{X_t}dB_t. \tag{1}$$
(c) Rewrite the Itô SDE (1) as a Stratonovich differential equation.

2. Let $B = (B^{(1)}, B^{(2)}, B^{(3)})$ be a three-dimensional Brownian motion starting at the origin, and define
$$X = \prod_{i=1}^{3} \text{sgn}(B_1^{(i)}),$$
$$M_t^{(1)} = B_t^{(1)}, \quad M_t^{(2)} = B_t^{(2)}, \quad M_t^{(3)} = XB_t^{(3)}.$$
Show that each of the pairs $(M^{(1)}, M^{(2)})$, $(M^{(1)}, M^{(3)})$ and $(M^{(2)}, M^{(3)})$ is a two-dimensional Brownian motion, but $(M^{(1)}, M^{(2)}, M^{(3)})$ is not a three-dimensional Brownian motion.

3. Let B be standard Brownian motion and suppose X is progressively measurable. For $0 \le s < t < \infty$ define $Z \doteq Z^{(X)}$ by
$$Z_t = \exp\left(\int_0^t X_u dB_u - \frac{1}{2}\int_0^t X_u^2 du\right).$$

(a) Show that Z is the *unique* solution to the stochastic integral equation

$$Z_t = 1 + \int_0^t Z_s X_s dB_s.$$

(b) Define $Y_t = 1/Z_t$. State why Y is well-defined. Show that Y satisfies the stochastic differential equation

$$dY_t = Y_t X_t^2 dt - Y_t X_t dB_t \qquad Y_0 = 1. \tag{2}$$

4. Let R be a Bessel process with dimension $d \geq 3$ starting at $r \geq 0$. Show that

$$\mathbb{P}(\lim_{t \to \infty} R_t = \infty) = 1.$$

5. Let R be the d-dimensional Bessel process, $d \geq 2$, starting at $r > 0$, and define $m = \inf_{0 \leq t < \infty} R_t$.
(a) Show that if $d \geq 2$, then $m = 0$ $\mathbb{P}$-a.s.
(b) Show that if $d \geq 3$, then m has the beta distribution

$$\mathbb{P}(m \leq c) = \left(\frac{c}{r}\right)^{d-2}.$$

6. Let R be a Bessel process with dimension $d \geq 3$, starting at $r = 0$. Show that $\{M_t \doteq (1/R_t^{d-2}); 1 \leq t < \infty\}$,

   (a) is a local martingale;

   (b) satisfies $\sup_{1 \leq t < \infty} \mathbb{E}[M_t^p] < \infty$ for every $0 < p < d/(d-2)$ and is thus uniformly integrable;

   (c) is not a martingale