21-880: Advanced Stochastic Calculus I – Fall 2008
Homework Assignment 3
(Distributed Friday, October 3, 2008)
(Due Monday, October 13, 2008)

This homework concentrates on martingale theory, uniform integrability and stochastic integration. Read through your class notes, the posted lecture notes and Chapters 4 and 5 of the text.

1. Prove Theorem 2.1.5 in the class notes (see last page of Lec 7–8) using the following steps.

   (a) Show that $\{X_n^+, \mathcal{F}_n\}$ is a backward submartingale.

   (b) Show that $\lim_{\lambda\to\infty} \sup_n \mathbb{P}(|X_n| > \lambda) = 0$.

   (c) Show that $\{X_n^+\}$ is uniformly integrable.

   (d) Show that $\{X_n^-\}$ is uniformly integrable.

   See how this theorem is used to prove the continuous optional sampling theorem from the discrete optional sampling theorem (refer to Theorem 2.3.4 of Lecture 9-10 notes, which was not covered in class).

2. (Continuous Martingales) This problem provides an example of a continuous martingale that arises naturally in the study of risk, queueing theory and many other fields.
   Let $T : (\Omega, \mathcal{F}, P) \to (0, \infty)$ be a positive random variable. Let F be the distribution function of T (i.e. $F(t) \doteq P(T \leq t)$). Define

   $$A_t(\omega) \doteq \begin{cases} 1 & \text{if } t \geq T(\omega); \\ 0 & \text{if } t < T(\omega) \end{cases}$$

   and let $\{\mathcal{F}_t^A\}$ be the filtration generated by A. Let h be the *hazard function* or *cumulative risk* function, defined by

   $$h(u) \doteq \int_{(0,u)} \frac{dF(v)}{1 - F(v-)}.$$

   Show that the process M, where $M_t \doteq A_t - h(T \wedge t)$, is a right-continuous martingale relative to the filtration $\mathcal{F}_t$.

3. Prove that if X and Y are submartingales then $H = X + Y$ and $Z = \max(X, Y)$ are submartingales. Justify your steps carefully.

4. This problem aims at giving you practice in manipulating martingale inequalities. Let $X = \{X_t, \mathcal{F}_t, t \in [0,\infty)\}$ be a continuous, nonnegative martingale with $X_\infty = \lim_{t\to\infty} X_t = 0$ $\mathbb{P}$ a.s. Then, for every $s \geq 0, b > 0$, show that

$$\mathbb{P}\left(\sup_{t>s} X_t \geq b | \mathcal{F}_s\right) = \frac{1}{b} X_s \qquad \text{a.s. on } \{X_s < b\}.$$

Conclude from this that

$$\mathbb{P}\left(\sup_{t>s} X_t \geq b\right) = \mathbb{P}(X_s \geq b) + \frac{1}{b}\mathbb{E}[X_s \mathbb{I}_{\{X_s<b\}}].$$

5. In class, we showed that the Itô integral $\{I_t^B(X), t \in [0,\infty)\}$ of a right-continuous, $\mathcal{F}_t$-adapted, bounded process X with respect to the Brownian motion B is again a martingale. In this problem, we investigate the properties of stochastic integrals with respect to the Poisson process (which is more straightforward to define). Let $\{N_t, \mathcal{F}_t\}$ be a Poisson process with parameter λ and let $M_t \doteq N_t - \lambda t$, $t \in [0,\infty)$. Let X be a right-continuous, $\mathcal{F}_t$-adapted process. Since X is of bounded variation, we can define

$$I_t^M(X) = \int_0^t X_s \, dM_s, \quad t \in [0,\infty).$$

pathwise as a Riemann-Stieltjes integral. State whether $I_t^M(H)$ is an $\mathcal{F}_t$-martingale if H is right-continuous ? If H is continuous ?