21-880: Advanced Stochastic Calculus I – Fall 2008
Homework Assignment 2
(Distributed Tuesday, September 16, 2008)
(Due Wednesday, September 24, 2008, *before class*)

This homework covers the following topics: Brownian motion, filtrations, stopping times, discrete martingales and their properties, basic definition of a continuous martingale.

1. Let $\xi_0, \xi_1, \xi_2, \ldots$ be i.i.d. standard normal random variables. Show that for any $t > 0$,
$$B_t = \sqrt{2}\sum_{n=1}^{\infty} \xi_n \frac{\sin(n-\frac{1}{2})\pi t)}{(n-\frac{1}{2})\pi}$$
is a.s. well-defined (i.e. the series on the right is convergent) and is a standard Brownian motion (with respect to its natural filtration). Note that this provides a series representation for Brownian motion.

2. Let A be a closed set in $\mathbb{R}$ and suppose that $\{X_t, \mathcal{F}_t\}$ is an adapted ($\mathbb{R}$-valued) stochastic process. Define
$$D_A(\omega) \doteq \inf\{t \geq 0 : X_t(\omega) \in A\},$$
with the usual convention that the infimum of an empty set is ∞. Show that if X is continuous, then D_A is an $\mathcal{F}_t$-stopping time. Does this continue to hold if X is right-continuous or A is open?

3. If S and T are $\mathcal{F}_t$-stopping times, state whether $S \wedge T$ and $S + T$ are also $\mathcal{F}_t$-stopping times. Justify your answers rigorously.

4. Suppose $\{B_t, \mathcal{F}_t\}$ and $\{W_t, \mathcal{F}_t\}$ are independent Brownian motions, and suppose $\{N_t, \mathcal{F}_t\}$ is a Poisson process. State whether the process $\{Z_t\}$ is a martingale with respect to $\{\mathcal{F}_t\}$ in each of the following cases – justify your answers rigorously.

   (a) $Z_t = \exp\left(\theta B_t - \frac{1}{2}\theta^2 t\right)$, $t \in [0, \infty)$, where θ is any complex number

   (b) $Z_t = \sqrt{t} B_1$, $t \in [0, \infty)$.

   (c)
$$Z_t = B_t^2 W_t - \int_0^t W_u \, du, \quad t \in [0, \infty).$$

(d) $Z_t = (N_t - \lambda t)^2 - \lambda t^2$, $t \in [0, \infty)$.

5. Let $\{X_n\}$ be a discrete nonnegative supermartingale. Show that $\lim_{n\to\infty} X_n$ exists a.s.

6. (*) Let $\{\mathcal{F}_t^B\}$ be the natural filtration generated by a Brownian motion B (i.e., for every $t \geq 0$, $\mathcal{F}_t^B = \sigma(B_s, 0 \leq s \leq t)$. Is $\{\mathcal{F}_t^B\}$ right continuous? Is $\{\mathcal{F}_t^B\}$ left continuous? Justify your answer.