21-880: Advanced Stochastic Calculus I – Fall 2008
Homework Assignment 1
(Distributed Monday, September 1, 2008)
(Due Wednesday, September 10, 2008, *before class*)
Note: On Tuesday, September 9, 2008, **office hours will be from 12:00–1:00 p.m.**, and not at the usual time 3:30–4:30 p.m.

1. (Sample Path Properties of Brownian motion) Suppose $\{B_t, \mathcal{F}_t\}$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Show that the process $\tilde{B}$ defined by $\tilde{B}_0 = 0$,

$$\tilde{B}_t = tB_{\frac{1}{t}}, \quad t \in [0, \infty),$$

is also a Brownian motion, with respect to its natural filtration.

2. (Sample Path Properties of Brownian motion) Show that Brownian motion does not have a finite first variation. (*Hint:* First calculate the quadratic variation process associated with Brownian motion.)

3. (Review of conditional expectations) Consider a random variable X that equals 2, 1, -1 with probability 1/3 each, and let $Y = \operatorname{sgn}(X)$ be the sign of X, and let $Z = |X|$ be the modulus of X.

   (i) Calculate $\mathbb{E}[X|Z]$.

   (ii) Calculate $\mathbb{E}[X|Y]$

   (iii) Consider the statement that on a set of positive probability,

   $$\mathbb{E}\left[\mathbb{E}[X|Y]|Z\right] = \mathbb{E}\left[\mathbb{E}[X|Z]|Y\right].$$

   State whether this statement is true or false.

4. (a) Suppose $\{B_t, \mathcal{F}_t\}$ is a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Given $0 < t_1 < t_2 < t_3 < \infty$ and $x_1, x_3 \in \mathbb{R}$, find the conditional distribution of B_{t_2} given $B_{t_1} = x_1$ and $B_{t_3} = x_3$.

   (b) (*) Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined a countable family $\{\xi_{k2^{-n}}, n \in \mathbb{N} \cup \{0\}, 1 \leq k \leq 2^n\}$ of $\mathcal{N}(0,1)$ random variables, which are mutually independent for distinct values of $k2^{-n}$. We will use these random variables to recursively construct a sequence of piecewise linear stochastic processes on $[0,1]$, with the paths of the $(n+1)$th process linear on intervals of the form

$[(k-1)2^{-n}, k2^{-n}]$, $k = 1, \ldots, 2^n$. More precisely, define a sequence of stochastic processes on $[0,1]$ as follows:

$$B^{(1)}(t) = t\xi_1, \qquad t \in [0,1].$$

and, recursively, for $n \in \mathbb{N}$, let

$$B^{(n+1)}(k2^{-n}) = \begin{cases} B^{(n)}(\ell 2^{-(n-1)}) & \text{if } k = 2\ell \text{ is even,} \\ B^{(n)}(k2^{-n}) + \dfrac{\xi_{k2^{-n}}}{2^{(n+1)/2}} & \text{if } k \text{ is odd,} \end{cases}$$

and by linear interpolation in between. For $n \in \mathbb{N}$ and any $k_1, k_2, k_3, k_4 \in \{1, \ldots, 2^{-n}\}$, with $k_1 < k_2 < k_3 < k_4$, find the distribution of the increment $B^{(n+1)}_{k_2 2^{-n}} - B^{(n+1)}_{k_1 2^{-n}}$ and show that it is independent of the increment $B^{(n+1)}_{k_3 2^{-n}} - B^{(n+1)}_{k_4 2^{-n}}$.

(c) Think about whether for almost surely every $\omega \in \Omega$, you can show that the sequence $B^{(n)}(\omega)$ converges (in the sup norm on continuous functions on $[0,1]$) to some continuous function $B(\omega)$.