The Skorokhod problem in a time-dependent interval

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Abstract: We consider the Skorokhod problem in a time-varying interval. We prove existence and uniqueness for the solution. We also express the solution in terms of an explicit formula. Moving boundaries may generate singularities when they touch. We establish two sets of sufficient conditions on the moving boundaries that guarantee that the variation of the local time of the associated reflected Brownian motion is, respectively, finite and infinite. We also apply these results to study the semimartingale property of a class of two-dimensional reflected Brownian motions.

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1. Introduction

We consider the Skorokhod problem with two moving boundaries. Informally speaking, the problem is concerned with reflecting or constraining a given path in a space-time region defined by two moving boundaries. We will address several problems inspired by recent related developments. First, we study the question of existence and uniqueness to a slight generalization of the Skorokhod problem, which we refer to as the extended Skorokhod problem. We show that the solution not only exists and is unique, but can be represented in terms of an explicit and rather simple formula. Second, we prove some monotonicity relations for solutions to the extended Skorokhod problem. Similar monotonicity properties are quite obvious when there is only one reflecting boundary; they are not so obvious in our context. In addition, we study the issue of whether the constraining process associated with reflected Brownian motion has finite or infinite total variation. This issue arises when the two end points of the time-varying interval are allowed to meet and is related to the question of whether the reflected Brownian motion is a semimartingale. Finally, we apply our analysis of one-dimensional reflected Brownian motion in a time-dependent interval to study the behavior of the constraining process and, in particular, the semimartingale property of a class of two-dimensional reflected Brownian motions in a fixed domain that were studied in [4, 11, 17, 19, 24]. Reflecting Brownian motions in time-dependent domains arise in queueing theory [13, 16], statistical physics [5, 22], control theory [10] and finance [9].

The present paper is related to several articles. First, the papers [14] and [15] present an explicit formula for the Skorokhod mapping in the simpler setting of a constant interval

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$[0, a]$. Second, the works [2] and [3] contain an analysis of Brownian motion reflected on one moving boundary. In particular, the second paper presents results on singularities at rough boundary points. In the present paper, we analyze singularities due to the interaction of two moving boundaries. In our context, a "singularity" means the infinite variation of the associated constraining process, which we henceforth refer to as the local time process (see Remark 4.2 for a discussion of this terminology). Finally, the paper [4] (see also [11, 24]) studied a special case of two-dimensional reflected Brownian motion in "valley-shaped" domains, with all reflection vectors parallel to the same straight line. We establish the somewhat surprising result that this two-dimensional reflected Brownian motion is not a semimartingale, irrespective of the particular shape of the domain. In addition, we also provide new proofs of some of the qualitative results of the papers [4, 24].

The rest of the paper is organized as follows. We start with a short section collecting the notation used throughout the paper. Section 2 is devoted to the foundational results—existence, uniqueness and an explicit formula for the so-called extended Skorokhod mapping. Section 3 contains some "comparison" or "monotonicity" results. Finally, Sections 4.1 and 4.2 present theorems on the local time of reflected Brownian motion in a time-dependent interval. These results are applied in Section 4.3 to study the local time of a class of two-dimensional reflected Brownian motions.

1.1. Notation

We use $\mathcal{D}[0,\infty)$ to denote the space of càdlàg functions (i.e., continuous on the right with finite left limits) that are defined on $[0,\infty)$ and take values in $(-\infty,\infty)$. The space of right continuous functions whose left limits (at all points in $(0,\infty)$) and values both lie in $[-\infty,\infty)$ (respectively, $(-\infty,\infty]$) will be denoted $\mathcal{D}^-[0,\infty)$ (respectively, $\mathcal{D}^+[0,\infty)$). Given two functions $f \in \mathcal{D}^-[0,\infty)$, $g \in \mathcal{D}^+[0,\infty)$, we will say $f \le g$ (respectively, $f < g$) if $f(t) \le g(t)$ (respectively, $f(t) < g(t)$) for every $t \in [0,\infty)$. We let $\mathcal{C}[0,\infty)$ represent the subspace of continuous functions in $\mathcal{D}[0,\infty)$. We denote the variation of a function f on $[t_1, t_2]$ by $\mathcal{V}_{[t_1,t_2]}(f)$. We denote by $\ell(\cdot)$ a generic function in $\mathcal{D}^-[0,\infty)$ and by $r(\cdot)$ a generic function in $\mathcal{D}^+[0,\infty)$, and assume that $\ell \le r$.

Moreover, given $a, b \in \mathbb{R}$, denote $a \wedge b \doteq \min\{a, b\}$, $a \vee b \doteq \max\{a, b\}$, and $a^+ \doteq a \vee 0$. We denote by $\mathbb{I}_A$ the indicator function of a set A.

We also use the following abbreviations, whose meaning will be explained later: SP—Skorokhod problem, SM—Skorokhod map, ESP—extended Skorokhod problem, ESM—extended Skorokhod map, BM—Brownian motion, RBM—reflected Brownian motion.

2. Skorokhod and Extended Skorokhod Maps in a Time-Dependent Interval

The so-called Skorokhod Problem (SP) was introduced in [21] as a convenient tool for the construction of reflected Brownian motion (RBM) in the *time-independent* domain $[0,\infty)$. Specifically, given a function $\psi \in \mathcal{D}[0,\infty)$, the SP on $[0,\infty)$ consists of identifying a non-negative function ϕ such that the function $\eta \doteq \phi - \psi$ is non-decreasing and, roughly speaking, increases only at times t when $\phi(t) = 0$. It was shown in [21] that there is a unique mapping that takes any given $\psi \in \mathcal{C}[0,\infty)$ to the corresponding function ϕ (the extension to $\psi \in \mathcal{D}[0,\infty)$ is straightforward). Moreover, this mapping, which we shall refer to as the Skorokhod map (SM) on $[0,\infty)$ and denote by Γ_0, admits the explicit representation

$$\Gamma_0(\psi)(t) = \psi(t) + \sup_{s\in[0,t]} [-\psi(s)]^+, \quad \psi \in \mathcal{D}[0,\infty). \tag{2.1}$$

Given a Brownian motion (BM) B on $\mathbb{R}$ with $B(0) = 0$, and any $x \geq 0$, the process $W = \Gamma_0(x + B)$ defines RBM on $[0,\infty)$, starting at x. More generally, due to the Lipschitz continuity of the map Γ_0, standard Picard iteration techniques can be used to construct solutions to stochastic differential equations with reflection on $[0,\infty)$, under the usual Lipschitz assumptions on the drift and diffusion coefficients.

In a similar fashion, the generalizations of the SP given in Section 2.1 will be the basis for the construction of 1-dimensional RBM in a time-dependent interval. We also establish some basic properties of these generalizations in Section 2.1 and then provide an explicit formula for the ESM in Section 2.2.

2.1. Basic Definitions and Properties

We first describe the SP on a time-varying interval $[\ell(\cdot), r(\cdot)]$ (see, e.g., Appendix A of [13]).

Definition 2.1. (Skorokhod problem on $[\ell(\cdot), r(\cdot)]$**)** *Suppose that* $\ell \in \mathcal{D}^-[0,\infty)$*,* $r \in \mathcal{D}^+[0,\infty)$ *and* $\ell \leq r$*. Given any* $\psi \in \mathcal{D}[0,\infty)$*, a pair of functions* $(\phi,\eta) \in \mathcal{D}[0,\infty)\times\mathcal{D}[0,\infty)$ *is said to solve the SP on* $[\ell(\cdot), r(\cdot)]$ *for* ψ *if and only if it satisfies the following properties:*

1. *For every* $t \in [0,\infty)$*,* $\phi(t) = \psi(t) + \eta(t) \in [\ell(t), r(t)]$*;*
2. $\eta = \eta_\ell - \eta_r$*, where* η_ℓ *and* η_r *are non-decreasing functions such that*

$$\int_0^\infty \mathbb{I}_{\{\phi(s)>\ \ell(s)\}} d\,\eta_\ell(s) = 0, \qquad \int_0^\infty \mathbb{I}_{\{\phi(s)<\ r(s)\}} d\,\eta_r(s) = 0. \tag{2.2}$$

If (ϕ,η) is the unique solution to the SP on $[\ell(\cdot), r(\cdot)]$ for ψ then we will write $\phi = \Gamma_{\ell,r}(\psi)$, and refer to $\Gamma_{\ell,r}$ as the associated SM. Moreover, the pair (η_ℓ, η_r) will be referred to as the constraining processes associated with the SP.

Although Definition 2.1 is a natural extension of the SP to time-dependent domains in $\mathbb{R}$ it is restrictive in that it only allows "constraining terms" η that are of bounded variation. In particular, this implies that any RBM constructed via the associated SM is automatically a semimartingale. For fixed domains in $\mathbb{R}^d$, a generalization of the SP that allows for a pathwise construction of RBMs that are not necessarily semimartingales was introduced in [17] (see also [4] for a formulation in two dimensions). The following is the analog of these generalizations for time-dependent domains in $\mathbb{R}$.

Definition 2.2. (Extended Skorokhod problem on $[\ell(\cdot), r(\cdot)]$**)** *Suppose that* $\ell \in \mathcal{D}^-[0,\infty)$*,* $r \in \mathcal{D}^+[0,\infty)$ *and* $\ell \leq r$*. Given any* $\psi \in \mathcal{D}[0,\infty)$*, a pair of functions* $(\phi,\eta) \in \mathcal{D}[0,\infty) \times \mathcal{D}[0,\infty)$ *is said to solve the ESP on* $[\ell(\cdot), r(\cdot)]$ *for* ψ *if and only if it satisfies the following properties:*

1. *For every* $t \in [0,\infty)$, $\phi(t) = \psi(t) + \eta(t) \in [\ell(t), r(t)]$;
2. *For every* $0 \le s < t < \infty$,

$$\begin{aligned} \eta(t) - \eta(s) \ge 0, &\qquad \text{if } \phi(u) < r(u) \text{ for all } u \in (s,t] \\ \eta(t) - \eta(s) \le 0 &\qquad \text{if } \phi(u) > \ell(u) \text{ for all } u \in (s,t]; \end{aligned}$$

3. *For every* $0 \le t < \infty$,

$$\begin{aligned} \eta(t) - \eta(t-) \ge 0 &\qquad \text{if } \phi(t) < r(t), \\ \eta(t) - \eta(t-) \le 0 &\qquad \text{if } \phi(t) > \ell(t), \end{aligned}$$

where $\eta(0-)$ *is to be interpreted as* 0.

If (ϕ,η) is the unique solution to the ESP on $[\ell(\cdot), r(\cdot)]$ for ψ then we will write $\phi = \overline{\Gamma}_{\ell,r}(\psi)$, and refer to $\overline{\Gamma}_{\ell,r}$ as the associated extended Skorokhod map (ESM).

We conclude this section by establishing certain properties of SPs and ESPs (see Theorem 1.3 of [17] for analogs for time-independent multi-dimensional domains). The first property describes in what sense the ESP is a generalization of the SP.

Proposition 2.3. *Suppose we are given* $\ell \in \mathcal{D}^-[0,\infty)$, $r \in \mathcal{D}^+[0,\infty)$ *with* $\ell \le r$ *and* $\psi \in \mathcal{D}[0,\infty)$. *If* (ϕ,η) *solve the SP on* $[\ell(\cdot), r(\cdot)]$ *for* ψ, *then* (ϕ,η) *solve the ESP on* $[\ell(\cdot), r(\cdot)]$ *for* ψ. *Conversely, if* (ϕ,η) *solve the ESP on* $[\ell(\cdot), r(\cdot)]$ *for* ψ *and* η *has finite variation on every bounded interval, then* (ϕ,η) *solve the SP for* ψ.

Proof. The first statement follows from the easily verifiable fact that property 2 of Definition 2.1 implies properties 2 and 3 of Definition 2.2. For the converse, let (ϕ,η) be a solution to the ESP on $[\ell(\cdot), r(\cdot)]$ for ψ and suppose η has finite variation on every bounded interval. Then the Lebesgue-Stieltjes measure $d\eta$ is absolutely continuous with respect to the corresponding total variation measure $d|\eta|$. Let γ be the Radon-Nikodỳm derivative $d\eta/d|\eta|$ of $d\eta$ with respect to $d|\eta|$. Then γ is $d|\eta|$-measurable, $\gamma(s) \in \{-1,1\}$ for $d|\eta|$ a.e. $s \in [0,\infty)$ and

$$\eta(t) = \int_{[0,t]} \gamma(s) d|\eta|(s).$$

Moreover, it is well-known (see, for example, Section X.4 of [7]) that for $d|\eta|$ a.e. $s \in [0,\infty)$,

$$\gamma(s) = \lim_{n\to\infty} \frac{\eta(s+\varepsilon_n) - \eta(s-)}{|\eta|(s+\varepsilon_n) - |\eta|(s-)}, \tag{2.3}$$

where the sequence ε_n depends on s and is such that $|\eta|(s+\varepsilon_n) - |\eta|(s-) > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$. Now, for each $t \ge 0$, define $\eta_\ell(t) = \int_{[0,t]} \mathbb{I}_{\{\gamma(s)=1\}} d|\eta|(s)$ and $\eta_r(t) = \int_{[0,t]} \mathbb{I}_{\{\gamma(s)=-1\}} d|\eta|(s)$. Since γ only takes the values 1 and -1 ($d|\eta|$ a.e.), it is clear that $\eta = \eta_\ell - \eta_r$. We shall now show that η_ℓ satisfies the first complementary condition in (2.2). It follows from the definition of η_ℓ that

$$\int_0^\infty \mathbb{I}_{\{\phi(s)>\ \ell(s)\}} d\,\eta_\ell(s) = \int_0^\infty \mathbb{I}_{\{\phi(s)>\ \ell(s)\}} \mathbb{I}_{\{\gamma(s)=1\}} d|\eta|(s).$$

Suppose that there exists $s \geq 0$ such that $\phi(s) > \ell(s)$, $\gamma(s) = 1$ and (2.3) holds. We will show that this assumption leads to a contradiction. Since $\phi(s) > \ell(s)$, by the right continuity of ϕ and ℓ, there exists $\delta > 0$ such that $\phi(u) > \ell(u)$ for all $u \in [s, s+\delta]$. By properties 2 and 3 of Definition 2.2, we have $\eta(u) - \eta(s-) \leq 0$ for each $u \in [s, s+\delta]$. On the other hand, since $\gamma(s) = 1$ and $|\eta|$ is a non-decreasing function, for all sufficiently large n we have from (2.3) that $\eta(s+\varepsilon_n) - \eta(s-) > 0$. This leads to a contradiction. Hence $\phi(s) > \ell(s)$ and $\gamma(s) = 1$ cannot hold simultaneously for $d|\eta|$ a.e. s, which proves the first complementarity condition in (2.2). The second complementary condition in (2.2) can be established in a similar manner. □

Corollary 2.4. *Suppose that $\ell \in \mathcal{D}^-[0,\infty)$, $r \in \mathcal{D}^+[0,\infty)$ and $\inf_{t\geq 0}(r(t) - \ell(t)) > 0$. If $(\phi, \eta) \in \mathcal{D}[0,\infty) \times \mathcal{D}[0,\infty)$ solve the ESP on $[\ell(\cdot), r(\cdot)]$ for some $\psi \in \mathcal{D}[0,\infty)$, then (ϕ,η) solve the SP on $[\ell(\cdot), r(\cdot)]$ for ψ.*

Proof. By Proposition 2.3, it suffices to show that η has bounded variation on every finite time interval. Let $\tau_0 = 0$ and for $n \in \mathbb{Z}_+$, let $\tau_{2n+1} = \inf\{t \geq \tau_{2n} : \phi(t) = r(t)\}$ and $\tau_{2n+2} = \inf\{t \geq \tau_{2n+1} : \phi(t) = \ell(t)\}$. For each $n \in \mathbb{Z}_+$, on the interval $[\tau_n, \tau_{n+1})$, ϕ will touch exactly one of the boundaries ℓ and r. By properties 2 and 3 of the ESP, this implies that η will be either non-decreasing or non-increasing, and hence in particular of bounded variation, on each interval $[\tau_n, \tau_{n+1})$. Moreover, under the assumption $\inf_{t\geq 0}(r(t) - \ell(t)) > 0$ and the fact that $\phi \in \mathcal{D}[0,\infty)$, it is easy to see that there are finitely many τ_n's in each bounded time interval. Thus η will have finite variation on each bounded time interval. □

The following property is a simple, but extremely useful, closure property of the ESP. Below, the abbreviation u.o.c. stands for uniformly on compacts, i.e., we say $f_n \to f$ u.o.c. if for every $T < \infty$, $\sup_{s\in[0,T]} |f_n(s) - f(s)| \to 0$ as $n \to \infty$.

Proposition 2.5. **(Closure Property)** *For each $n \in \mathbb{N}$, let $\ell_n \in \mathcal{D}^-[0,\infty)$, $r_n \in \mathcal{D}^+[0,\infty)$ be such that $\ell_n \leq r_n$, and let $\psi_n \in \mathcal{D}[0,\infty)$. Suppose there exist $\ell \in \mathcal{D}^-[0,\infty)$, $r \in \mathcal{D}^+[0,\infty)$ and $\psi \in \mathcal{D}[0,\infty)$ such that $\psi_n \to \psi$, $\ell_n \to \ell$ and $r_n \to r$ u.o.c., as $n \to \infty$. Moreover, suppose that for each $n \in \mathbb{N}$, (ϕ_n, η_n) solve the ESP on $[\ell_n(\cdot), r_n(\cdot)]$ for ψ_n. If $\phi_n \to \phi$ u.o.c., as $n \to \infty$, then $(\phi, \phi - \psi)$ solve the ESP on $[\ell(\cdot), r(\cdot)]$ for ψ.*

Proof. Let $\psi_n, \ell_n, r_n, \phi_n, \eta_n, n \in \mathbb{N}$, and ψ, ℓ, r, ϕ be as in the statement of the proposition and let $\eta \doteq \phi - \psi$. By property 1 of Definition 2.2, $\eta_n = \phi_n - \psi_n$ and $\phi_n(t) \in [\ell_n(t), r_n(t)]$ for all $t \in [0,\infty)$. Together with the assumed u.o.c. convergences of ψ_n, ϕ_n, ℓ_n and r_n to ψ, ϕ, ℓ and r, respectively, this implies $\eta_n \to \eta$ u.o.c., as $n \to \infty$, and $\phi(t) \in [\ell(t), r(t)]$ for all $t \in [0,\infty)$. Thus (ϕ,η) satisfy property 1 of Definition 2.2.

Now, suppose that $\eta(t-) > \eta(t)$ for some t. We will show that then $\phi(t) = r(t)$. Since $\eta_n \to \eta$ u.o.c., we have $\eta_n(t-) > \eta_n(t)$ for all sufficiently large n. By property 3 of Definition 2.2, this implies that $\phi_n(t) = r_n(t)$ for all sufficiently large n. The convergences $\phi_n(t) \to \phi(t)$ and $r_n(t) \to r(t)$ then imply that $\phi(t) = r(t)$. An analogous argument shows that if $\eta(t-) < \eta(t)$ then $\phi(t) = \ell(t)$, thus showing that (ϕ,η) satisfy property 3 of Definition 2.2.

In order to show that (ϕ, η) satisfy the remaining property 2 of Definition 2.2, fix $0 \leq s < t < \infty$ and suppose that

$$\phi(u) < r(u) \qquad \text{for } u \in (s, t]. \tag{2.4}$$

We want to show that then $\eta(t) \geq \eta(s)$. By the right continuity of η, it suffices to show that $\eta(t) \geq \eta(\tilde{s})$ for every $\tilde{s} \in (s, t]$. Suppose, to the contrary, that $\eta(t) < \eta(\tilde{s})$ for some $\tilde{s} \in (s, t]$. Since (ϕ, η) satisfy property 3, due to condition (2.4) we must have $\eta(u) \geq \eta(u-)$ for every $u \in [\tilde{s}, t]$. In particular, this implies that $B \doteq \{\eta(u) : u \in [\tilde{s}, t]\} \supseteq [\eta(t), \eta(\tilde{s})]$. Thus the set B is uncountable, while the set A of all discontinuities of all functions $\ell_n, r_n, \psi_n, \phi_n, \ell, r, \psi$ and ϕ is countable. Hence, there exists $\alpha \in [\eta(t), \eta(\tilde{s})) \setminus \{\eta(u) : u \in A\}$. Define $u_1 \doteq \inf\{u \in (\tilde{s}, t) \setminus A : \eta(u) = \alpha\}$ and note that $u_1 > \tilde{s}$ and

$$\eta(u) > \eta(u_1) \qquad \text{for all } u \in [\tilde{s}, u_1). \tag{2.5}$$

In addition, the functions ϕ and r are continuous at u_1. Therefore, by (2.4) we know that there exists $u_- \in [\tilde{s}, u_1)$ such that $\inf_{u \in [u_-, u_1]}(r(u) - \phi(u)) > 2\varepsilon$, where $\varepsilon \doteq (r(u_1) - \phi(u_1))/4 > 0$. Since $\phi_n \to \phi$ and $r_n \to r$ u.o.c., we know that for all sufficiently large n, $\inf_{u \in [u_-, u_1]}(r_n(u) - \phi_n(u)) > \varepsilon$. Then property 2 of Definition 2.2 implies that $\eta_n(u_-) \leq \eta_n(u_1)$ for all sufficiently large n. Passing to the limit, we obtain $\eta(u_-) \leq \eta(u_1)$, which contradicts (2.5). Thus, we must have $\eta(s) \leq \eta(t)$ when (2.4) holds. An analogous argument can be used to show that $\eta(s) \geq \eta(t)$ whenever $\phi(u) > \ell(u)$ for all $u \in [s, t]$. This completes the proof that (ϕ, η) solve the ESP on $[\ell(\cdot), r(\cdot)]$ for ψ. □

2.2. An Explicit Formula for Solutions to the ESP on $[\ell(\cdot), r(\cdot)]$

The following theorem is our main result in this section.

Theorem 2.6. *Suppose that* $\ell \in \mathcal{D}^-[0,\infty)$, $r \in \mathcal{D}^+[0,\infty)$ *and* $\ell \leq r$. *Then for each* $\psi \in \mathcal{D}[0,\infty)$, *there exists a unique pair* $(\phi, \eta) \in \mathcal{D}[0,\infty) \times \mathcal{D}[0,\infty)$ *that solves the ESP on* $[\ell(\cdot), r(\cdot)]$ *for* ψ. *Moreover, the ESM* $\overline{\Gamma}_{\ell,r}$ *admits the following explicit representation:*

$$\overline{\Gamma}_{\ell,r}(\psi) = \psi - \Xi_{\ell,r}(\psi), \tag{2.6}$$

where the mapping $\Xi_{\ell,r} : \mathcal{D}[0,\infty) \mapsto \mathcal{D}[0,\infty)$ *is defined as follows: for each* $t \in [0,\infty)$,

$$\begin{aligned}\Xi_{\ell,r}(\psi)(t) \doteq \max\Bigg(&\left[(\psi(0) - r(0))^+ \wedge \inf_{u \in [0,t]}(\psi(u) - \ell(u))\right], \\ &\sup_{s \in [0,t]}\left[(\psi(s) - r(s)) \wedge \inf_{u \in [s,t]}(\psi(u) - \ell(u))\right]\Bigg).\end{aligned} \tag{2.7}$$

Furthermore, the map $(\ell, r, \psi) \mapsto \overline{\Gamma}_{\ell,r}$ *is a continuous map on* $\mathcal{D}^-[0,\infty) \times \mathcal{D}^+[0,\infty) \times \mathcal{D}[0,\infty)$ *(with respect to the topology of uniform convergence on compact sets). Lastly, if* $\inf_{t \geq 0}(r(t) - \ell(t)) > 0$ *then* $\Gamma_{\ell,r} = \overline{\Gamma}_{\ell,r}$.

Remark 2.7. When $r \equiv \infty$ and $\ell \in \mathcal{D}[0,\infty)$, Definition 2.2 reduces to a one-dimensional SP with time-varying domain $[\ell(\cdot),\infty)$, and the right-hand side of (2.6) reduces to $\Gamma_\ell(\psi)$, where the mapping $\Gamma_\ell : \mathcal{D}[0,\infty) \mapsto \mathcal{D}[0,\infty)$ is given by

$$\Gamma_\ell(\psi)(t) \doteq \psi(t) + \sup_{s\in[0,t]} [\ell(s) - \psi(s)]^+ \text{ for } t \in [0,\infty). \tag{2.8}$$

In this situation, the proof of (2.1) can be extended in a straightforward manner (see, for example, Lemma 3.1 of [2]) to show that Γ_ℓ defines the unique solution to the associated SP.

The rest of this section is devoted to the proof of Theorem 2.6. For the case of time-independent boundaries $\ell \equiv 0$ and $r \equiv a > 0$, this result was established in Theorem 2.1 of [14] using a completely different argument from that used here. The proof of Theorem 2.6 presented in this paper thus provides, in particular, an alternative proof of Theorem 2.1 of [14] (see also [6], Section 14 of [23] and the discussion in [14] of related formulas in the time-independent case).

For the rest of this section, we fix $\ell \in \mathcal{D}^-[0,\infty)$ and $r \in \mathcal{D}^+[0,\infty)$ such that $\ell \le r$. We first establish uniqueness of solutions to the ESP on $[\ell(\cdot), r(\cdot)]$ in Proposition 2.8 —the proof is a relatively straightforward modification of the standard proof for the SP on $[0,\infty)$ (see, for example, Lemma 3.6.14 in [12] and also Lemma 3.1 of [2]).

Proposition 2.8. *Given any $\psi \in \mathcal{D}[0,\infty)$, there exists at most one $\phi \in \mathcal{D}[0,\infty)$ that satisfies the ESP on $[\ell(\cdot), r(\cdot)]$ for ψ.*

Proof. Let (ϕ,η) and (ϕ',η') be two pairs of functions in $\mathcal{D}[0,\infty) \times \mathcal{D}[0,\infty)$ that solve the ESP on $[\ell(\cdot), r(\cdot)]$ for $\psi \in \mathcal{D}[0,\infty)$. It is easy to check that, for property 3 of Definition 2.2 to be satisfied when $t = 0$, we must have $\phi(0) = (\psi(0) \wedge r(0)) \vee \ell(0)$. The same is true for ϕ', and so $\phi(0) = \phi'(0)$.

Suppose that there exists $T \ge 0$ such that $\phi(T) > \phi'(T)$. Let

$$\tau = \sup\{t \in [0,T] : \ \phi(t) \le \phi'(t)\}. \tag{2.9}$$

Note that τ is well defined because $\phi(0) = \phi'(0)$.

We now consider two cases.

Case 1. $\phi(\tau) \le \phi'(\tau)$. In this case, for $t \in (\tau, T]$, by the definition of τ and property 1 of Definition 2.2, we have $\ell(t) \le \phi'(t) < \phi(t) \le r(t)$. Since on $(\tau,T]$, ϕ will not hit ℓ and $\bar{\phi}'$ will not hit r, by property 2 of Definition 2.2, we see that $\eta(T) - \eta(\tau) \le 0$ and $\eta'(T) - \eta'(\tau) \ge 0$. Consequently,

$$0 < \phi(T) - \phi'(T) = \eta(T) - \eta'(T) \le \eta(\tau) - \eta'(\tau) = \phi(\tau) - \phi'(\tau),$$

which contradicts the case assumption.

Case 2. $\phi(\tau) > \phi'(\tau)$. In this case $\tau > 0$ because $\phi(0) = \phi'(0)$. By the definition of τ it follows that

$$\phi(\tau-) \le \phi'(\tau-). \tag{2.10}$$

In addition, the case assumption also implies that $\phi(\tau) > \ell(\tau)$ and $\phi'(\tau) < r(\tau)$. By property 3 of Definition 2.2, this implies that $\eta(\tau)-\eta(\tau-) \leq 0$ and $\eta'(\tau)-\eta'(\tau-) \geq 0$. When combined with property 1 of Definition 2.2, this shows that

$$0 < \phi(\tau) - \phi'(\tau) = \eta(\tau) - \eta'(\tau) \leq \eta(\tau-) - \eta'(\tau-) = \phi(\tau-) - \phi'(\tau-),$$

which contradicts (2.10).

We thus conclude that $\phi(T) \leq \phi'(T)$ for all $T \geq 0$. Using an exactly analogous argument we can show that $\phi'(T) \leq \phi(T)$ for all $T \geq 0$. Hence $\phi(T) = \phi'(T)$ and, therefore, $\eta(T) = \eta'(T)$ for all $T \geq 0$. □

Next, in Proposition 2.9, we show that the ESM is given by the formula (2.6) when ℓ, r and ψ are piecewise constant. The proof will make use of the following family of mappings: given $\ell \in \mathcal{D}^- [0,\infty), r \in \mathcal{D}^+ [0,\infty)$ with $\ell \leq r$, for $t \in [0,\infty)$, consider the mapping $\pi_t : \mathbb{R} \to \mathbb{R}$ with the property that $\pi_t(x) = x$ if $x \in [\ell(t), r(t)]$, $\pi_t(x) \in \{\ell(t), r(t)\}$ if $x \notin [\ell(t), r(t)]$ and

$$\begin{array}{ll} \pi_t(x) - x \geq 0 & \text{if } \pi_t(x) = \ell(t), \\ \pi_t(x) - x \leq 0 & \text{if } \pi_t(x) = r(t). \end{array} \tag{2.11}$$

It is straightforward to deduce that, for every $t \geq 0$, there exists a unique mapping with these properties that is given explicitly by

$$\pi_t(x) = x + [\ell(t) - x]^+ - [x - r(t)]^+ = (x \wedge r(t)) \vee \ell(t). \tag{2.12}$$

Using property 3 of the ESP, it is easy to verify (see, e.g., Appendix B of [13]) that the ESM $\overline{\Gamma}_{\ell,r}$ must satisfy

$$\overline{\Gamma}_{\ell,r}(\psi)(0) = \pi_0(\psi(0)), \qquad \overline{\Gamma}_{\ell,r}(\phi)(t) = \pi_t\left(\overline{\Gamma}_{\ell,r}(\psi)(t-) + \psi(t) - \psi(t-)\right) \quad \forall t > 0. \tag{2.13}$$

Proposition 2.9. *Suppose that ℓ, r and ψ are three piecewise constant functions in $\mathcal{D}^- [0,\infty)$, $\mathcal{D}^+ [0,\infty)$ and $\mathcal{D} [0,\infty)$, respectively, each with a finite number of jumps and such that $\ell \leq r$. Then for each $\psi \in \mathcal{D}[0,\infty)$, the pair $(\psi - \Xi_{\ell,r}(\psi), -\Xi_{\ell,r}(\psi))$ is the unique solution to the ESP on $[\ell(\cdot), r(\cdot)]$, i.e., $\overline{\Gamma}_{\ell,r}(\psi) = \psi - \Xi_{\ell,r}(\psi)$.*

Proof. Fix ψ, ℓ, r as in the statement of the proposition, and let $\{\pi_t, t \in [0,\infty)\}$ be the associated family of mappings as defined in (2.12). Now, let $J = \{t_1, t_2, \ldots, t_n\}$ be the union of the times of jumps of ℓ, r and ψ, suppose $0 < t_1 < t_2 < \cdots < t_n < \infty$, and set $t_{n+1} \doteq \infty$. Define $\phi \doteq \psi - \Xi_{\ell,r}(\psi)$ and $\eta \doteq \phi - \psi$. We will use induction to show that (ϕ, η) solve the ESP on $[\ell(\cdot), r(\cdot)]$ for ψ. When $t = 0$, it is straightforward to verify from (2.12) and the definition of $\Xi_{\ell,r}$ that $\phi(0) = (\psi(0) \wedge r(0)) \vee \ell(0) = \pi_0(\psi(0))$. When combined with (2.13), this shows that (ϕ, η) solve the ESP (on $[\ell(\cdot), r(\cdot)]$) for ψ when $t = 0$. Since ℓ, r, ψ are constant on $[0, t_1)$, it immediately follows from the definition (2.7) of $\Xi_{\ell,r}$ that ϕ is also constant on $[0, t_1)$, and so it follows that (ϕ, η) solve the ESP for ψ on $[0, t_1)$.

Now, suppose (ϕ, η) solve the ESP on $[\ell(\cdot), r(\cdot)]$ for ψ over the time interval $[0, t_m)$ for some $m \in \{1, \ldots, n\}$. We first observe that, for any $t \in [0,\infty)$, $\Xi_{\ell,r}(\psi)(t)$ is the maximum of the following three terms:

1. $(\psi(0) - r(0))^+ \wedge \inf_{u\in[0,t)}(\psi(u) - \ell(u)) \wedge (\psi(t) - \ell(t))$
2. $\sup_{s\in[0,t)} \left[(\psi(s) - r(s)) \wedge \inf_{u\in[s,t)}(\psi(u) - \ell(u)) \wedge (\psi(t) - \ell(t))\right]$,
3. $(\psi(t) - r(t)) \wedge (\psi(t) - \ell(t))$

and therefore admits the representation

$$\Xi_{\ell,r}(\psi)(t) = \max\left[\Xi_{\ell,r}(\psi)(t-), (\psi(t) - r(t))\right] \wedge (\psi(t) - \ell(t)).$$

Recalling the description of the map π_t given in (2.12), we see that

$$\begin{aligned} \phi(t) &= \psi(t) - \max\left(\Xi_{\ell,r}(\psi)(t-), (\psi(t) - r(t))\right) \wedge (\psi(t) - \ell(t)) \\ &= \min\left(\psi(t) - \Xi_{\ell,r}(\psi)(t-), r(t)\right) \vee \ell(t) \\ &= \pi_t(\psi(t) - \Xi_{\ell,r}(\psi)(t-)) \\ &= \pi_t(\phi(t-) + \psi(t) - \psi(t-)). \end{aligned}$$

Substituting $t = t_m$, this yields the relation $\phi(t_m) = \pi_{t_m}(\phi(t_m-) + \psi(t_m) - \psi(t_m-))$. By (2.13), this implies (ϕ, η) solve the ESP on $[\ell(\cdot), r(\cdot)]$ for ψ during the interval $[0, t_m]$. Once again, since ψ, ℓ, r, and therefore ϕ, are constant on $[t_m, t_{m+1})$ this implies that (ϕ, η) solve the ESP on $[\ell(\cdot), r(\cdot)]$ for ψ on $[0, t_{m+1})$. By the induction argument and the uniqueness result established in Proposition 2.8, we have the desired result. □

A simple approximation argument can now be used to complete the proof of Theorem 2.6.

Proof of Theorem 2.6. Given $\ell \in \mathcal{D}^-[0,\infty), r \in \mathcal{D}^+[0,\infty)$ such that $\ell \le r$, it is easy to see that there exist sequences of functions $\ell_n \in \mathcal{D}^-[0,\infty)$, $n \in \mathbb{N}$, $r_n \in \mathcal{D}^+[0,\infty)$, $n \in \mathbb{N}$, with $\ell_n \le r_n$, that are piecewise constant with a finite number of jumps and such that $\ell_n \to \ell$, $r_n \to r$ u.o.c. as $n \to \infty$. Likewise, given $\psi_n \in \mathcal{D}[0,\infty)$, there exists a sequence of piecewise constant functions ψ_n with a finite number of jumps such that $\psi_n \to \psi$ u.o.c., as $n \to \infty$. For each $n \in \mathbb{N}$, by Proposition 2.9, we know that $\Gamma_{\ell_n,r_n}(\psi_n) = \phi_n \doteq \psi_n - \Xi_{\ell_n,r_n}(\psi_n)$. Since $\psi_n - \ell_n$ and $\psi_n - r_n$ converge u.o.c., as $n \to \infty$, to $\psi - \ell$ and $\psi - r$, respectively, and u.o.c. convergence is preserved under the operations $\inf$, $\sup$, $\wedge$, $\max$, we then conclude, from (2.7), that $\phi_n = \psi_n - \Xi_{\ell_n,r_n}(\psi_n) \to \psi - \Xi_{\ell,r}(\psi)$ u.o.c, as $n \to \infty$. In particular, it is clear that $\Xi_{\ell,r}$ is a continuous map on $\mathcal{D}[0,\infty)$ (with respect to the topology of u.o.c. convergence). By the closure property (Proposition 2.5), $(\psi - \Xi_{\ell,r}(\psi), -\Xi_{\ell,r}(\psi))$ is a solution to the ESP on $[\ell(\cdot), r(\cdot)]$ for ψ. Uniqueness follows from Proposition 2.8. In particular, this shows that the map $(\ell, r, \psi) \mapsto \overline{\Gamma}_{\ell,r}(\psi)$ is continuous with respect to the topology of u.o.c. convergence. The last assertion of the theorem is a direct consequence of Corollary 2.4. □

3. Comparison results

This section presents some "comparison" or "monotonicity" results. They are quite intuitive but their proofs require some technical arguments. Recall the definition of the pair of constraining processes (η_ℓ, η_r) associated with an SP given in Definition 2.1. Section 3.1 establishes monotonicity of the individual constraining processes with respect to the domain $[\ell(\cdot), r(\cdot)]$ for a fixed ψ, while in Section 3.2, monotonicity of the constraining processes with respect to the input ψ is established for a given time-varying domain $[\ell(\cdot), r(\cdot)]$.

3.1. Monotonicity with respect to the domain

The main result, Proposition 3.3, will be preceded by a few lemmas.

Lemma 3.1. *Assume that $\ell, \tilde{\ell} \in \mathcal{D}^-[0,\infty)$, $r, \tilde{r} \in \mathcal{D}^+[0,\infty)$, and $\tilde{\ell} = \ell$, $r \leq \tilde{r}$ and $\inf_{t\geq 0}(r(t) - \ell(t)) > 0$. Let $\Gamma_{\ell,r}$ and $\Gamma_{\tilde{\ell},\tilde{r}}$ be the associated SMs on $[\ell(\cdot), r(\cdot)]$ and $[\tilde{\ell}(\cdot), \tilde{r}(\cdot)]$, respectively, and given $\psi \in \mathcal{D}[0,\infty)$, let (η_ℓ, η_r) and $(\eta_{\tilde{\ell}}, \eta_{\tilde{r}})$ be the corresponding pairs of constraining processes. Then, for every $t \in [0,\infty)$,*

$$\eta_r(t) \geq \eta_{\tilde{r}}(t) \qquad \textit{and} \qquad \eta_\ell(t) \geq \eta_{\tilde{\ell}}(t). \tag{3.1}$$

Proof. Let $\tilde{\ell}, \ell, r, \tilde{r}$ and ψ be as in the statement of the lemma. First note that by Theorem 2.6, the conditions on $\tilde{\ell}, \tilde{r}, \ell, r$ guarantee that solutions to the SP on both $[\ell, r]$ and $[\tilde{\ell}, \tilde{r}]$ exist for all $\psi \in \mathcal{D}[0,\infty)$ and so the pairs of constraining processes (η_ℓ, η_r) and $(\eta_{\tilde{\ell}}, \eta_{\tilde{r}})$ are well-defined. Moreover, the explicit formula (2.6) of Theorem 2.6, when combined with the decomposition $\eta = \Gamma_{\ell,r}(\psi) - \psi = \eta_\ell - \eta_r$ (see Definition 2.1), shows that

$$\eta_r = \eta_\ell + \Xi_{\ell,r}(\psi) \qquad \text{and} \qquad \eta_{\tilde{r}} = \eta_{\tilde{\ell}} + \Xi_{\tilde{\ell},\tilde{r}}(\psi), \tag{3.2}$$

where Ξ is as defined in (2.7). For a fixed ℓ, it is easily verified from the explicit formula (2.7) that the map $r \mapsto \Xi_{\ell,r}(\psi)$ is monotone non-increasing (with respect to the obvious ordering). Since $\ell = \tilde{\ell}$ and $r \leq \tilde{r}$, this implies

$$\Xi_{\tilde{\ell},\tilde{r}}(\psi) \leq \Xi_{\ell,r}(\psi) \qquad \forall \psi \in \mathcal{D}[0,\infty). \tag{3.3}$$

By (2.6), this is equivalent to the relation

$$\Gamma_{\tilde{\ell},\tilde{r}}(\psi) \geq \Gamma_{\ell,r}(\psi) \qquad \forall \psi \in \mathcal{D}[0,\infty). \tag{3.4}$$

Combining (3.2) and (3.3), it follows that in order to show (3.1), it suffices to show that

$$\eta_\ell(t) \geq \eta_{\tilde{\ell}}(t) \quad \forall\, t \geq 0. \tag{3.5}$$

Since $\Gamma_{\ell,r}(\psi)(0) = \pi_0(0)$ by (2.13) and $\ell(0) < r(0)$, from the complementarity conditions (2.2) it is clear that $\eta_\ell(0) = [\ell(0) - \psi(0)]^+$ with the analogous expressions for $\eta_{\tilde{\ell}}$. Since $\ell = \tilde{\ell}$, this immediately implies (3.5), in fact with equality, for $t = 0$.

Now, let

$$t^* \doteq \inf\{s \geq 0 : \eta_\ell(s) < \eta_{\tilde{\ell}}(s)\}.$$

We will argue by contradiction to show that $t^* = \infty$. Indeed, suppose that $t^* < \infty$. Then

$$\eta_\ell(t^*-) \geq \eta_{\tilde{\ell}}(t^*-) \tag{3.6}$$

and for all $\varepsilon_0 > 0$ there exists $\varepsilon \in (0, \varepsilon_0)$ such that

$$\eta_\ell(t^* + \epsilon) < \eta_{\tilde{\ell}}(t^* + \epsilon). \tag{3.7}$$

Invoking (2.13) and the inequality $\Gamma_{\tilde{\ell},\tilde{r}}(\psi)(t^*-) \geq \Gamma_{\ell,r}(\psi)(t^*-)$ from (3.4), we obtain

$$\begin{array}{rcl} \eta_\ell(t^*) - \eta_\ell(t^*-) & = & [\ell(t^*) - \Gamma_{\ell,r}(\psi)(t^*-) - (\psi(t^*) - \psi(t^*-))]^+ \\ & \geq & [\tilde{\ell}(t^*) - \Gamma_{\tilde{\ell},\tilde{r}}(\psi)(t^*-) - (\psi(t^*) - \psi(t^*-))]^+ \\ & = & \eta_{\tilde{\ell}}(t^*) - \eta_{\tilde{\ell}}(t^*-). \end{array}$$

When combined with (3.6), this implies that $\eta_\ell(t^*) \geq \eta_{\tilde{\ell}}(t^*)$. We now consider two cases. If $\eta_\ell(t^*) > \eta_{\tilde{\ell}}(t^*)$, then the right-continuity of η_ℓ and $\eta_{\tilde{\ell}}$ dictates that $\eta_\ell(t^* + \epsilon) > \eta_{\tilde{\ell}}(t^* + \varepsilon)$ for every positive ε small enough, which contradicts (3.7). On the other hand, suppose $\eta_\ell(t^*) = \eta_{\tilde{\ell}}(t^*)$. When combined with (3.7) and the fact that η_ℓ is non-decreasing, this implies that for every $\varepsilon_0 > 0$, there exists $\varepsilon \in (0, \varepsilon_0)$ such that $\eta_{\tilde{\ell}}(t^*) < \eta_{\tilde{\ell}}(t^*+\varepsilon)$. Due to the complementarity condition (2.2) and the right-continuity of $\Gamma_{\tilde{\ell},\tilde{r}}(\psi)$, this, in turn, implies that $\Gamma_{\tilde{\ell},\tilde{r}}(\psi)(t^*) = \tilde{\ell}(t^*) = \ell(t^*)$. Since $\Gamma_{\tilde{\ell},\tilde{r}}(\psi) \geq \Gamma_{\ell,r}(\psi) \geq \ell$, this means that $\Gamma_{\ell,r}(\psi)(t^*) = \Gamma_{\tilde{\ell},\tilde{r}}(\psi)(t^*) = \ell(t^*)$. Along with the relation $\tilde{\ell}(t^*) = \ell(t^*) < r(t^*) \leq \tilde{r}(t^*)$, the right-continuity of ℓ, r and $\tilde{r}$ and the definition of the SP, it is easy to see that this implies that for all sufficiently small ε, $\Gamma_{\tilde{\ell},\tilde{r}}(\psi)(t^* + \varepsilon)$ (respectively, $\Gamma_{\ell,r}(\psi)(t^* + \varepsilon)$) is equal to $\Gamma_{\tilde{\ell}}(\psi^*)(\varepsilon)$ (respectively, $\Gamma_\ell(\psi^*)(\varepsilon)$), where Γ_ℓ is as defined in (2.8) and

$$\psi^*(t) = \ell(t^*) + \psi(t^* + t) - \psi(t^*) \qquad \text{for } t \geq 0.$$

In particular, using (2.8), this shows that for all ε sufficiently small,

$$\begin{array}{rcl} \eta_{\tilde{\ell}}(t^* + \varepsilon) - \eta_{\tilde{\ell}}(t^*) & = & \sup_{s\in[0,\varepsilon]} \left[\tilde{\ell}(s) - \ell(t^*) - \psi(t^* + s) + \psi(t^*)\right] \\ & = & \sup_{s\in[0,\varepsilon]} \left[\ell(s) - \ell(t^*) - \psi(t^* + s) + \psi(t^*)\right] \\ & = & \eta_\ell(t^* + \varepsilon) - \eta_\ell(t^*). \end{array}$$

Since we are considering the case $\eta_\ell(t^*) = \eta_{\tilde{\ell}}(t^*)$, this once again contradicts (3.7). Thus we have shown that $t^* = \infty$, and hence that (3.5) holds. □

Corollary 3.2. *Assume that* $\ell, \tilde{\ell} \in \mathcal{D}^-[0,\infty)$, $r, \tilde{r} \in \mathcal{D}^+[0,\infty)$, $\tilde{\ell} \leq \ell$, $r = \tilde{r}$ *and* $\inf_{t\geq 0}(r(t) - \ell(t)) > 0$. *Let* $\Gamma_{\ell,r}$ *and* $\Gamma_{\tilde{\ell},\tilde{r}}$ *be the associated SMs on* $[\ell(\cdot), r(\cdot)]$ *and* $[\tilde{\ell}(\cdot), \tilde{r}(\cdot)]$, *respectively, and given* $\psi \in \mathcal{D}[0,\infty)$, *let* (η_ℓ, η_r) *and* $(\eta_{\tilde{\ell}}, \eta_{\tilde{r}})$ *be the corresponding pairs of constraining processes. Then, for every* $t \in [0,\infty)$,

$$\eta_r(t) \geq \eta_{\tilde{r}}(t) \qquad \text{and} \qquad \eta_\ell(t) \geq \eta_{\tilde{\ell}}(t).$$

Proof. The corollary follows from Lemma 3.1 by multiplying all functions by -1. □

Proposition 3.3. *Assume that* $\ell, \tilde{\ell} \in \mathcal{D}^-[0,\infty)$, $r, \tilde{r} \in \mathcal{D}^+[0,\infty)$, $\tilde{\ell} \leq \ell$, $r \leq \tilde{r}$ *and* $\inf_{t\geq 0}(r(t) - \ell(t)) > 0$. *Let* $\Gamma_{\ell,r}$ *and* $\Gamma_{\tilde{\ell},\tilde{r}}$ *be the associated SMs on* $[\ell(\cdot), r(\cdot)]$ *and* $[\tilde{\ell}(\cdot), \tilde{r}(\cdot)]$, *respectively, and given* $\psi \in \mathcal{D}[0,\infty)$, *let* (η_ℓ, η_r) *and* $(\eta_{\tilde{\ell}}, \eta_{\tilde{r}})$ *be the corresponding pairs of constraining processes. Then, for every* $t \in [0,\infty)$,

$$\eta_r(t) \geq \eta_{\tilde{r}}(t) \qquad \text{and} \qquad \eta_\ell(t) \geq \eta_{\tilde{\ell}}(t).$$

Proof. Let $\Gamma_{\ell,\tilde{r}}$ be the SM on $[\ell(\cdot),\tilde{r}(\cdot)]$, and given $\psi \in \mathcal{D}[0,\infty)$, let $(\eta^*_\ell,\eta^*_{\tilde{r}})$ be the corresponding vector of constraining processes. Since $r \leq \tilde{r}$ and $\inf_{t\geq 0}(\tilde{r}(t)-\ell(t)) > 0$ by Lemma 3.1, we know that

$$\eta_r(t) \geq \eta^*_{\tilde{r}}(t) \qquad \text{and} \qquad \eta_\ell(t) \geq \eta^*_\ell(t).$$

Similarly, when $\Gamma_{\ell,\tilde{r}}$ and $\Gamma_{\tilde{\ell},\tilde{r}}$ are considered, by Corollary 3.2 we obtain

$$\eta^*_{\tilde{r}}(t) \geq \eta_{\tilde{r}}(t) \qquad \text{and} \qquad \eta^*_\ell(t) \geq \eta_{\tilde{\ell}}(t).$$

When combined, these inequalities yield the desired result. □

3.2. Monotonicity with respect to input trajectories

Given a fixed time-dependent domain $[\ell(\cdot),r(\cdot)]$, in Proposition 3.4 we first establish the monotonicity of $\overline{\Gamma}_{\ell,r}(\psi)$ and the net constraining term $\overline{\Gamma}_{\ell,r}(\psi)-\psi$ with respect to input trajectories ψ. For the case when $[\ell(\cdot),r(\cdot)]=[0,a]$ for some $a>0$, this result was established as Theorem 1.7 of [15]. Here, we use a simpler argument involving approximations to prove the more general result.

Proposition 3.4. *Given $\ell,r \in \mathcal{D}[0,\infty)$ with $\ell \leq r$, $c_0,c_0' \in \mathbb{R}$ and $\psi,\psi' \in \mathcal{D}[0,\infty)$, suppose (ϕ,η) and (ϕ',η') solve the ESP on $[\ell(\cdot),r(\cdot)]$ for $c_0+\psi$ and $c_0'+\psi'$, respectively. If $\psi=\psi'+\nu$ for some non-decreasing function $\nu \in \mathcal{D}[0,\infty)$ with $\nu(0)=0$, then for each $t\geq 0$, the following two relations hold:*

1. $[-[c_0-c_0']^+-\nu(t)] \vee [-(r(t)-\ell(t))] \leq \phi'(t)-\phi(t) \leq [c_0'-c_0]^+ \wedge [r(t)-\ell(t)]$;
2. $\eta(t)-[c_0'-c_0]^+ \leq \eta'(t) \leq \eta(t)+\nu+[c_0-c_0']^+$.

Proof. We first establish property 1 under the additional assumption that the functions ℓ,r,ψ,ψ' and ν stated in the proposition are piecewise constant with a finite number of jumps. Since $\phi(t),\phi'(t)$ lie in $[\ell(t),r(t)]$ for every $t\in[0,\infty)$, in order to show the first property it suffices to show that

$$-[c_0-c_0']^+-\nu(t) \leq \phi'(t)-\phi(t) \leq [c_0'-c_0]^+. \tag{3.8}$$

Let $t_0=0$ and let $t_1<t_2<\ldots<t_m$ be the ordered jump times of all the functions ℓ,r,ψ,ψ' and ν. Recall the family of (time-dependent) projection operators π_t, $t\geq 0$, defined in (2.12). Using the explicit expression for π_t, a simple case-by-case verification shows that for every $t\geq 0$ and $x,y\in\mathbb{R}$,

$$-[y-x]^+ \leq \pi_t(x)-\pi_t(y) \leq [x-y]^+. \tag{3.9}$$

By (2.13) and the piecewise constant nature of the functions, it follows that $\phi(t)=\phi(0)=\pi_0(c_0)$ and $\phi'(t)=\phi'(0)=\pi_0(c_0')$ for $t\in[0,t_1)$. When combined with (3.9), this shows that (3.8) holds for $t\in[0,t_1)$. Now suppose that (3.8) holds for $t\in[0,t_{k-1})$ for some $k\in\{2,\ldots,m\}$. Then, by (2.13) we know that for $t\in[t_{k-1},t_k)$,

$$\phi(t)=\pi_{t_k}\left(\phi\left(t_{k-1}\right)+\psi\left(t_k\right)-\psi\left(t_{k-1}\right)\right), \quad \phi'(t)=\pi_{t_k}\left(\phi'\left(t_{k-1}\right)+\psi'\left(t_k\right)-\psi'\left(t_{k-1}\right)\right). \tag{3.10}$$

Another application of (3.9), along with the relation $-[z]^+ = [-z] \wedge 0$ and the fact that $\psi = \psi' + \nu$, implies that

$$0\wedge[\phi'(t_{k-1})-\phi(t_{k-1})+\nu(t_{k-1})-\nu(t_k)] \leq \phi'(t_k)-\phi(t_k) \leq [\phi'(t_{k-1})-\phi(t_{k-1})+\nu(t_{k-1})-\nu(t_k)]^+.$$

The function ν is non-decreasing and non-negative and by the induction assumption, the first inequality in (3.8) holds for $t = t_{k-1}$. Therefore

$$-[c_0 - c_0']^+ - \nu(t_k) \leq \phi'(t_k) - \phi(t_k) \leq [c_0' - c_0]^+.$$

Since ϕ and η are constant on $[t_k, t_{k+1})$, we have shown that (3.8) holds for $t \in [0, t_{k+1})$ and, by induction, for $t \in [0, \infty)$ when all the relevant functions are piecewise constant.

For the general case, let $\ell_n \in \mathcal{D}^- [0, \infty), r_n \in \mathcal{D}^+ [0, \infty)$, $n \in \mathbb{N}$, be sequences of piecewise constant functions with a finite number of jumps such that $\ell_n \leq r_n$ for every $n \in \mathbb{N}$ and $\ell_n \to \ell$ and $r_n \to r$ u.o.c., as $n \to \infty$. Moreover, let $\psi_n, \psi_n', \nu_n \in \mathcal{D}[0, \infty)$, $n \in \mathbb{N}$, be sequences of piecewise constant functions with a finite number of jumps such that ν_n is non-decreasing and $\psi_n = \psi_n' + \nu_n$ and $\psi_n \to \psi$, $\psi_n' \to \psi'$ u.o.c., as $n \to \infty$ (see the proof of Lemma 3.3 in [19] for an explicit construction that shows such sequences exist). Moreover, let $\phi_n = \overline{\Gamma}_{\ell,r}(c_0 + \psi_n)$ and $\phi_n' = \overline{\Gamma}_{\ell,r}(c_0' + \psi_n')$. Then the continuity of the map $\psi \mapsto \overline{\Gamma}_{\ell,r}(\psi)$ established in Theorem 2.6 shows that $\phi_n \to \phi$ and $\phi_n' \to \phi'$ u.o.c., as $n \to \infty$. Furthermore, the arguments in the previous paragraph show that for every $n \in \mathbb{N}$ and $t \in [0, \infty)$, (3.8) holds with ϕ, η replaced by ϕ_n and ν_n, respectively. Taking limits as $n \to \infty$, we obtain property 1.

The second property can be deduced from the first using the basic relation

$$\eta' - \eta = \phi' - \phi - (c_0' - c_0) - (\psi' - \psi) = \phi' - \phi - (c_0' - c_0) + \nu.$$

□

Next, we establish monotonicity of the individual constraining processes η_ℓ and η_r with respect to input trajectories ψ.

Proposition 3.5. *Given $\ell \in \mathcal{D}^- [0, \infty)$, $r \in \mathcal{D}^+ [0, \infty)$ satisfying $\inf_{t \geq 0}(r(t) - \ell(t)) > 0$, $c_0, c_0' \in \mathbb{R}$ and $\psi, \psi' \in \mathcal{D}[0, \infty)$ with $\psi(0) = \psi'(0)$, suppose (ϕ, η) and (ϕ', η') solve the SP on $[\ell(\cdot), r(\cdot)]$ for $c_0 + \psi$ and $c_0' + \psi'$, respectively. Moreover, suppose (η_ℓ, η_r) and (η_ℓ', η_r') are the corresponding constraining processes. If there exists a non-decreasing function ν with $\nu(0) = 0$ such that $\psi = \psi' + \nu$, then for each $t \geq 0$, the following two relations hold:*

1. $\eta_\ell(t) - [c_0' - c_0]^+ \leq \eta_\ell'(t) \leq \eta_\ell(t) + \nu(t) + [c_0 - c_0']^+$;
2. $\eta_r'(t) - [c_0' - c_0]^+ \leq \eta_r(t) \leq \eta_r'(t) + \nu(t) + [c_0 - c_0']^+$.

Proof. Fix $t \in [0, \infty)$. Define

$$\alpha \doteq \inf\{t > 0 : \eta_\ell(t) + \nu(t) + [c_0 - c_0']^+ < \eta_\ell'(t) \text{ or } \eta_r(t) + [c_0' - c_0]^+ < \eta_r'(t)\},$$

where $\alpha = \infty$ if the infimum is over the empty set. Then it follows that for each $s \in [0, \alpha)$, the following two inequalities hold:

$$\eta_\ell'(s) \leq \eta_\ell(s) + \nu(s) + [c_0 - c_0']^+, \tag{3.11}$$
$$\eta_r'(s) \leq \eta_r(s) + [c_0' - c_0]^+. \tag{3.12}$$

Suppose $\alpha < \infty$. Then we claim and prove below that the following relations are satisfied:

$$\eta_\ell'(\alpha) \leq \eta_\ell(\alpha) + \nu(\alpha) + [c_0 - c_0']^+, \tag{3.13}$$

$$\eta_r'(\alpha) \leq \eta_r(\alpha) + [c_0' - c_0]^+. \tag{3.14}$$

It is easy to see from (3.11), (3.12) and the non-decreasing property of η_ℓ and η_r that if η_ℓ' (respectively, η_r') is continuous, then (3.13) (respectively, (3.14)) holds. Thus the claim holds if both η_ℓ' and η_r' are continuous. We now prove the claim under the assumption that $\eta_\ell'(\alpha) - \eta_\ell'(\alpha-) > 0$. First note that by the complementarity condition in (2.2) we have $\phi'(\alpha) = \ell(\alpha)$ and η_r' is continuous at α, and hence (3.14) holds. It follows that

$$\begin{aligned}\eta_\ell'(\alpha) &= \eta_\ell'(\alpha-) + \psi'(\alpha-) - \phi'(\alpha-) - \psi'(\alpha) + \ell(\alpha)\\ &= -c_0' + \eta_r'(\alpha-) - \psi'(\alpha) + \ell(\alpha)\\ &= -c_0' + \eta_r'(\alpha-) - \psi(\alpha) + \nu(\alpha) + \ell(\alpha).\end{aligned} \tag{3.15}$$

Since $\eta_r(\alpha-) = c_0 + \psi(\alpha-) + \eta_\ell(\alpha-) - \phi(\alpha-)$, adding and subtracting $\eta_r(\alpha-)$ to the right hand side of (3.15), we obtain

$$\eta_\ell'(\alpha) = -c_0' + c_0 + \eta_r'(\alpha-) - \psi(\alpha) + \psi(\alpha-) + \nu(\alpha) + \ell(\alpha) + \eta_\ell(\alpha-) - \phi(\alpha-) - \eta_r(\alpha-).$$

¿From (3.12) we infer that $\eta_r'(\alpha-) \leq \eta_r(\alpha-) + [c_0' - c_0]^+$, and so

$$\begin{aligned}\eta_\ell'(\alpha) \leq\ & -c_0' + c_0 + [c_0' - c_0]^+ - \psi(\alpha) + \psi(\alpha-) + \nu(\alpha) + \ell(\alpha)\\ & + \eta_\ell(\alpha-) - \phi(\alpha-).\end{aligned} \tag{3.16}$$

On the other hand, using the relations $\phi(\alpha) \geq \ell(\alpha)$ and $\eta_r(\alpha) - \eta_r(\alpha-) \geq 0$, we have

$$\begin{aligned}\eta_\ell(\alpha) &= \eta_\ell(\alpha-) + \phi(\alpha) - \phi(\alpha-) + \psi(\alpha-) - \psi(\alpha)\\ &\quad + \eta_r(\alpha) - \eta_r(\alpha-)\\ &\geq \eta_\ell(\alpha-) + \ell(\alpha) - \phi(\alpha-) + \psi(\alpha-) - \psi(\alpha).\end{aligned} \tag{3.17}$$

By combining (3.16) and (3.17), we see that (3.13) also holds, and the claim follows. A similar argument shows that (3.13) and (3.14) are also satisfied when $\eta_r'(\alpha) - \eta_r'(\alpha-) > 0$.

Next, note from the definition of α that there exists a sequence of constants $\{s_n\}$ with $s_n \downarrow 0$ as $n \to \infty$ such that one of the following statements must be true:

(i) $\eta_\ell'(\alpha + s_n) > \eta_\ell(\alpha + s_n) + \nu(\alpha + s_n) + [c_0 - c_0']^+$ for all $n \in \mathbb{N}$;

(ii) $\eta_r'(\alpha + s_n) > \eta_r(\alpha + s_n) + [c_0' - c_0]^+$ for all $n \in \mathbb{N}$.

First, suppose Case (i) holds. Then, taking the limit as $n \to \infty$, by the right-continuity of η_ℓ, η_ℓ' and ν, we have $\eta_\ell'(\alpha) \geq \eta_\ell(\alpha) + \nu(\alpha) + [c_0 - c_0']^+$. Together with (3.13), this implies

$$\eta_\ell'(\alpha) = \eta_\ell(\alpha) + \nu(\alpha) + [c_0 - c_0']^+. \tag{3.18}$$

Since η_ℓ and ν are non-decreasing, we have from Case (i) and (3.18) that $\eta_\ell'(\alpha + s_n) > \eta_\ell'(\alpha)$ for each $n \in \mathbb{N}$. By the complementarity condition in (2.2), this implies $\phi'(\alpha) = \ell(\alpha)$. Together with (3.14), (3.18) and the relation $\psi = \psi' + \nu$, this implies

$$\begin{aligned}\phi(\alpha) - \ell(\alpha) = \phi(\alpha) - \phi'(\alpha) &= c_0 - c_0' + \nu(\alpha) + \eta_\ell(\alpha) - \eta_\ell'(\alpha) - \eta_r(\alpha) + \eta_r'(\alpha)\\ &\leq c_0 - c_0' - [c_0 - c_0']^+ + [c_0' - c_0]^+ = 0.\end{aligned}$$

Since $\phi(\cdot) \in [\ell(\cdot), r(\cdot)]$, this implies $\phi(\alpha) = \ell(\alpha)$.

Consider the shift operator $T_\alpha : \mathcal{D}[0,\infty) \mapsto \mathcal{D}[0,\infty)$ defined by $T_\alpha f(s) = f(\alpha+s) - f(\alpha)$ for $s \in [0,\infty)$. By uniqueness of solutions to the SP, it is easy to see that $(\phi(\alpha+\cdot), T_\alpha \eta)$ solve the SP for $\phi(\alpha) + T_\alpha \psi$ with the associated pair of constraining processes $(T_\alpha \eta_\ell, T_\alpha \eta_r)$, and likewise for $(\phi'(\alpha+\cdot), T_\alpha \eta')$. Now, by the right continuity of ϕ, ϕ' and r and the fact that $\phi(\alpha) = \phi'(\alpha) = \ell(\alpha) < r(\alpha)$, there exists $\varepsilon > 0$ such that for each $s \in [0,\varepsilon]$, $\phi(\alpha+s) < r(\alpha+s)$ and $\phi'(\alpha+s) < r(\alpha+s)$. The complementarity condition (2.2) implies that $T_\alpha \eta = T_\alpha \eta_\ell$ and $T_\alpha \eta' = T_\alpha \eta'_\ell$ on the interval $[0,\varepsilon]$. An application of property (ii) of Proposition 3.4, with $c_0 = c'_0 = \ell(\alpha)$ and $T_\alpha \psi$, $T_\alpha \psi'$, $\ell(\alpha+\cdot)$, $r(\alpha+\cdot)$ and $T_\alpha \nu$ in place of ψ, ψ', ℓ, r and ν, shows that for each $s \in [0,\varepsilon]$, $T_\alpha \eta'_\ell(s) \le T_\alpha \eta_\ell(s) + T_\alpha \nu(s)$. Together with (3.18), this shows that for each $s \in [0,\varepsilon]$,

$$\eta'_\ell(\alpha+s) \le \eta_\ell(\alpha+s) + \nu(\alpha+s) + [c_0 - c'_0]^+,$$

which contradicts Case (i). Hence Case (ii) should hold. In this case, a similar argument can be used to show that $\phi'(\alpha) = \phi'(\alpha) = r(\alpha)$, and arguments analogous to those used above can then be applied to arrive at a contradiction to Case (ii). Thus $\alpha = \infty$ or, in other words, the second inequality in property 1 and the first inequality in property 2 of the proposition hold. The first inequality in property 1 and the second inequality in property 2 of the proposition can be proved in a similar way with β instead of α, where

$$\beta = \inf\{t > 0 : \eta'_r(t) + \nu(t) + [c_0 - c'_0]^+ < \eta_r(t) \text{ or } \eta'_\ell(t) + [c'_0 - c_0]^+ < \eta_\ell(t)\}.$$

□

4. Variation of the Local Time of RBM

Throughout this section, let B be a one-dimensional standard BM starting from 0 and defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Also, let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$.

Definition 4.1. *Given $\ell \in \mathcal{D}^-[0,\infty)$ and $r \in \mathcal{D}^+[0,\infty)$ with $\ell \le r$, we define RBM W on $[\ell(\cdot), r(\cdot)]$ starting at $x \in \mathbb{R}$ by*

$$W = \overline{\Gamma}_{\ell,r}(x+B).$$

Moreover, let Y be the unique process such that, each $\omega \in \Omega$, $(W(\omega,\cdot), Y(\omega,\cdot))$ solves the ESP on $[\ell(\cdot), r(\cdot)]$ for $x + B(\omega,\cdot)$. We will refer to Y as the local time of W (on the boundary of $[\ell(\cdot), r(\cdot)]$).

Remark 4.2. A natural alternative definition for the local time of W on the boundary of $[\ell(\cdot), r(\cdot)]$ is

$$\widehat{Y}(t) \doteq \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{I}_{[\ell(s),\ell(s)+\varepsilon]\cup[r(s)-\varepsilon,r(s)]}(W(s))\, ds \qquad \forall t \in [0,\infty).$$

When $\ell \equiv 0$ and $r \equiv \infty$, so that W is "classical" reflected Brownian motion, it is well-known (see, e.g., Definition 3.6.13 and Theorem 3.6.17 of [12]) that $Y = \widehat{Y}$. However, this no longer holds in general when ℓ and r are time-varying. Indeed, when ℓ is a Brownian path independent of B (and $r \equiv \infty$) it follows from Section 3 of [5] that $Y = 2\widehat{Y}$. Therefore, as observed in Remark 3.1 of [5], by pieceing together a function that looks Brownian on some intervals and flat on others, it is possible to construct ℓ such that the associated $\widehat{Y}$ and Y are not constant multiples of each other. In this article we will always refer to the constraining process Y as the local time.

Due to uniqueness of solutions to the ESP, it is easy to see that $\overline{\Gamma}_{\ell,r}(\psi)(t)$ depends only on $\{\ell(u), r(u), \psi(u), u \in [0,t]\}$. Thus, W is adapted to the filtration generated by B. Moreover, W admits the unique decomposition $W(t) = x + B(t) + Y(t)$ for $t \geq 0$ for $x + B(\omega, \cdot)$. We will refer to Y as the local time of W on the (time-dependent) boundary of $[\ell(\cdot), r(\cdot)]$. From Corollary 2.4, it immediately follows that Y a.s. has finite variation on every time interval $[t_1, t_2]$ such that $\inf_{t\in[t_1,t_2]}(r(t) - \ell(t)) > 0$.

For the rest of this section, fix $\ell \in \mathcal{D}^- [0,\infty)$, $r \in \mathcal{D}^+ [0,\infty)$ such that $\ell \leq r$, define

$$\tau \doteq \inf\{t > 0 : \; r(t) = \ell(t) \text{ or } r(t-) = \ell(t-)\},$$

and assume that $\tau \in (0,\infty)$. In Sections 4.1 and 4.2 we identify some necessary and some sufficient conditions for Y to have $\mathbb{P}$-a.s. finite variation on $[0,\tau]$. Recall that the variation of a function f on $[t_1,t_2]$ is denoted by $\mathcal{V}_{[t_1,t_2]}(f)$. We apply these results in Section 4.3 to analyze the local time of a class of two-dimensional RBMs in a fixed domain.

4.1. A Lower Bound

We show that the local time of RBM on $[0,\tau]$ has infinite variation for some ℓ and r by comparing the space-time domain $\{(t,x) : \ell(t) < x < r(t)\}$ to a "comb domain."

Let K' denote a subset of $\mathbb{Z}$, for example, K' may be the sequence of all negative integers, or all positive integers. We denote by K the subset of K' consisting of all elements of K' except the largest element of K', assuming one exists.

Theorem 4.3. *Suppose that there exists a set K' and a sequence $\{s_k\}_{k\in K'}$ that is strictly increasing, takes values in $[0,\tau]$ and, for some constant $c_1 \in (-\infty,\infty)$ and all $k \in K$, satisfies*

$$\frac{\min(r(s_{k+1}) - \ell(s_k), -\ell(s_{k+1}) + r(s_k))}{(s_{k+1} - s_k)^{1/2}} \leq c_1. \tag{4.1}$$

If

$$\sum_{k\in K} (s_{k+1} - s_k)^{1/2} = \infty \tag{4.2}$$

then $\mathcal{V}_{[0,\tau]}Y = \infty$, a.s.

Remark 4.4. The constant c_1 in the statement of Theorem 4.3 does not have to be positive. Intuitively speaking, the smaller c_1, the more the variation accumulated by Y. Examples of domains that satisfy the assumptions of the theorem are provided below the proof.

Proof of Theorem 4.3. Let $\Delta B_k = B(s_{k+1}) - B(s_k)$, $c_2 = 1 \vee c_1$. We define an event A_k by

$$A_k = \{\Delta B_k \in (2c_2(s_{k+1} - s_k)^{1/2}, 3c_2(s_{k+1} - s_k)^{1/2})\}$$

if $r(s_{k+1}) - \ell(s_k) \leq -\ell(s_{k+1}) + r(s_k)$, and we let

$$A_k = \{\Delta B_k \in (-3c_2(s_{k+1} - s_k)^{1/2}, -2c_2(s_{k+1} - s_k)^{1/2})\}$$

if $r(s_{k+1}) - \ell(s_k) > -\ell(s_{k+1}) + r(s_k)$. By Brownian scaling, there exists $p_1 > 0$ such that $\mathbb{P}(A_k) > p_1$ for all $k \in K$. This implies that

$$\mathbb{E}\left[|\Delta B_k| \mathbb{I}_{A_k}\right] \geq p_1 2c_2(s_{k+1} - s_k)^{1/2},$$

and so, in view of (4.2), we have

$$\sum_{k \in K} \mathbb{E}\left[|\Delta B_k| \mathbb{I}_{A_k}\right] = \infty.$$

We also have $|\Delta B_k| \mathbb{I}_{A_k} \leq 3c_2(s_{k+1} - s_k)^{1/2} \leq 3c_2 \tau^{1/2}$, a.s., for every k. The random variables $|\Delta B_k| \mathbb{I}_{A_k}$ are independent. Hence, by the "three series theorem" ([8], Ch. 1, (7.4)), we have a.s.,

$$\sum_{k \in K} |\Delta B_k| \mathbb{I}_{A_k} = \infty. \tag{4.3}$$

Suppose that the event A_k occurs and consider the case when $r(s_{k+1}) - \ell(s_k) \leq -\ell(s_{k+1}) + r(s_k)$. We also have $W(s_k) \geq \ell(s_k)$ and $W(s_{k+1}) \leq r(s_{k+1})$. Together with (4.1) and the case assumption, this implies $W(s_{k+1}) - W(s_k) \leq c_1(s_{k+1} - s_k)^{1/2}$. Since $B(s_{k+1}) - B(s_k) \geq 2c_2(s_{k+1} - s_k)^{1/2}$ by the case assumption and the definition of A_k, we must have $Y(s_{k+1}) - Y(s_k) \leq -c_2(s_{k+1} - s_k)^{1/2}$. It follows that $\mathcal{V}_{[s_k, s_{k+1}]} Y \geq c_2(s_{k+1} - s_k)^{1/2} \geq (1/3)|\Delta B_k|$. A completely analogous argument shows that the same bound holds in the case when $r(s_{k+1}) - \ell(s_k) \geq -\ell(s_{k+1}) + r(s_k)$. This estimate and (4.3) imply that, a.s.,

$$\mathcal{V}_{[0,\tau]} Y \geq \sum_{k \in K} \mathcal{V}_{[s_k, s_{k+1}]} Y \geq \sum_{k \in K} (1/3)|\Delta B_k| \mathbb{I}_{A_k} = \infty.$$

□

Example 4.5. Suppose ℓ and r are such that $\ell \leq r$, $\tau > 0$ and, for some $k_0 \leq 1/2\tau$, $\ell(\tau - 1/2k) \geq 0$ and $r(\tau - 1/(2k+1)) \leq 0$ for $k \geq k_0$. Then, with $s_k \doteq \tau - 1/k$ for $k \geq 2k_0$, it is easy to see that assumption (4.1) is satisfied with $c_1 = 0$ and, because $\sum_k 1/(k(k+1))^{1/2} = \infty$, (4.2) is also satisfied. By Theorem 4.3, the variation of the local time on $[0, \tau]$ is infinite a.s.

Example 4.6. Consider ℓ and r such that $\ell(0) < r(0)$ and $f \doteq r - \ell$ is a non-increasing function. Let the sequence $\{s_k\}$ be defined in the following way. We let $s_0 \doteq 0$, and for $k \geq 1$, we let $s_{k+1} \doteq s_k + f^2(s_k)$. Then $(s_{k+1} - s_k)^{1/2} = f(s_k)$ and it is easily shown that (4.1) is satisfied with $c_1 = 1$. Indeed, for any $k \in \mathbb{N}$, if $r(s_{k+1}) \leq r(s_k)$ then

$$r(s_{k+1}) - \ell(s_k) \leq r(s_k) - \ell(s_k) = f(s_k) = (s_{k+1} - s_k)^{1/2};$$

while if $r(s_{k+1}) > r(s_k)$ then since f is nondecreasing,

$$r(s_k) - \ell(s_{k+1}) < r(s_{k+1}) - \ell(s_{k+1}) = f(s_{k+1}) \le f(s_k) = (s_{k+1} - s_k)^{1/2}.$$

When combined, this shows that (4.1) is satisfied with $c_1 = 1$. By Theorem 4.3 it follows that the variation of the local time is infinite on $[0, \tau]$ a.s., provided $\sum_{k\ge 0} f(s_k) = \infty$.

Note that the number of intervals $[s_k, s_{k+1}]$ inside $[\tau - 2^{-j}, \tau - 2^{-j-1}]$ is bounded below by $2^{-j-2}/f^2(\tau - 2^{-j})$. The contribution of each one of these intervals to the sum $\sum_k f(s_k)$ is bounded below by $f(\tau - 2^{-j-1})$. Hence, the contribution from all these intervals is bounded below by $2^{-j-2} f(\tau - 2^{-j-1})/f^2(\tau - 2^{-j})$. It follows that if for some j_0,

$$\sum_{j>j_0} 2^{-j-2} f(\tau - 2^{-j-1})/f^2(\tau - 2^{-j}) = \infty \tag{4.4}$$

then the variation of the local time is infinite on $[0, \tau]$ a.s.

As a specific example, consider the case when $f(\tau - t) = t^\alpha$ for some $\alpha > 0$. If $\alpha \ge 1$ then (4.4) is true and the variation of the local time is infinite on $[0, \tau]$ a.s.

Example 4.7. This is a modification of the previous example. Suppose that ℓ and r are such that $\ell(0) = r(0)$ and $f \doteq r - \ell$ is a non-decreasing function on some interval $[0, \tau_1]$, with $\tau_1 \in (0, \tau)$. Assume that for some $0 < c_3, c_4 < \infty$, we have $c_3 < f(t)/f(2t) < c_4$ for all $t \in (0, \tau_1/2)$. Let $\{s_k\}_{k\in K'}$ be the usual ordering of all points of the form $2^{-j} + mf^2(2^{-j})$, for $m = 0, \dots, \lfloor 2^{-j}/f^2(2^{-j})\rfloor - 1$, and $j > j_1$, where j_1 is chosen so that $s_k < \tau_1$ for all $k \in K'$. Now, fix s_k of the form $2^{-j} + mf^2(2^{-j})$ for some $j > j_1$. Then $(s_{k+1} - s_k)^{1/2} \ge f(2^{-j})$ and if $\ell(s_k) \le \ell(s_{k+1})$ then since f is nondecreasing, then

$$r(s_k) - \ell(s_{k+1}) \le r(s_k) - \ell(s_k) = f(s_k) \le f(2^{-j+1});$$

while if $\ell(s_k) > \ell(s_{k+1})$ then since f is nondecreasing, then

$$r(s_{k+1}) - \ell(s_k) < r(s_{k+1}) - \ell(s_{k+1}) = f(s_{k+1}) \le f(2^{-j+1}).$$

This immediately implies that (4.1) is satisfied with $c_1 \doteq 1/c_3$. Similarly, this also implies that the condition (4.2) is satisfied if $\sum_{j>j_1} 2^{-j}/f(2^{-j}) = \infty$. Hence, if $f(t) = t^\alpha$ for some $\alpha \ge 1$ and $t \in [0, \tau_1]$ then the variation of the local time is infinite on $[0, \tau]$ a.s.

4.2. An Upper Bound

Our upper bound will be based on the comparison of the space-time domain $\dot{D} \doteq \{(t, x) : \ell(t) < x < r(t)\}$ with a family of "parabolic boxes."

Theorem 4.8. *Suppose that there exists a sequence $\{s_k\}_{k\in\mathbb{Z}}$ that is strictly increasing and is such that $\lim_{k\to-\infty} s_k = 0$, and $\lim_{k\to\infty} s_k = \tau$. Suppose that there exist a constant $c_1 < \infty$ and sequences $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$, such that $\{(t, x) : s_k < t < s_{k+1}, a_k < x < b_k\} \subset \dot{D}$, and*

$$(1/c_1)(s_{k+1} - s_k)^{1/2} < b_k - a_k < c_1(s_{k+1} - s_k)^{1/2} \tag{4.5}$$

for every $k \in \mathbb{Z}$*. Further, given* $m_k \doteq (a_k + b_k)/2$*, define*

$$d_k \doteq r(s_k) - \ell(s_k) = |r(s_k) - m_k| + |\ell(s_k) - m_k|,$$

$$d_k' \doteq |\ell(s_{k+1}) - m_k| \vee |\ell(s_{k+1}-) - m_k| + |r(s_{k+1}) - m_k| \vee |r(s_{k+1}-) - m_k|,$$

and suppose that

$$\sum_{k\in\mathbb{Z}} d_k < \infty \qquad \text{and} \qquad \sum_{k\in\mathbb{Z}} d_k' < \infty. \tag{4.6}$$

Then $\mathbb{E}\left[\mathcal{V}_{[0,\tau]}Y\right] < \infty$*.*

Proof. We first observe that, as a consequence of the first inequality in (4.5), the fact that $b_k - a_k \le r(s_k) - \ell(s_k)$ and the first relation in (4.6), we have

$$\sum_{k\in\mathbb{Z}} (s_{k+1} - s_k)^{1/2} < \infty. \tag{4.7}$$

Now, let $\Delta_k \doteq b_k - a_k$. For some fixed $k \in \mathbb{Z}$, we will estimate the expected amount of local time generated on an interval $[s_k, s_{k+1}]$. First, choose $\varepsilon_0 \in (0, s_{k+1} - s_k)$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$|\ell(s_{k+1} - \varepsilon) - m_k| + |r(s_{k+1} - \varepsilon) - m_k| \le 2d_k'. \tag{4.8}$$

Let $T_1 \doteq s_k$, and for $\varepsilon \in (0, \varepsilon_0)$, define

$$\begin{aligned} U_1 &\doteq \inf\{t \ge T_1 : W(t) = m_k\} \wedge (s_{k+1} - \varepsilon), \\ T_j &\doteq \inf\{t \ge U_{j-1} : |W(t) - m_k| \ge \Delta_k/4\} \wedge (s_{k+1} - \varepsilon), \quad j \ge 2, \\ U_j &\doteq \inf\{t \ge T_j : W(t) = m_k\} \wedge (s_{k+1} - \varepsilon), \quad j \ge 2. \end{aligned}$$

For each $j \ge 2$, W is away from the upper and lower boundaries on $[U_{j-1}, T_j]$, and so we have

$$\mathcal{V}_{[U_{j-1},T_j]}Y = 0 \qquad \text{for all } j \ge 2. \tag{4.9}$$

We now consider intervals of the form $[T_j, U_j]$, $j \in \mathbb{N}$. The elementary relation $Y = W - B$ yields the bound

$$|Y(U_j) - Y(T_j)| \le |W(U_j) - W(T_j)| + \sup_{t\in[T_j,U_j]} |B(t) - B(T_j)| \tag{4.10}$$

for every $j \in \mathbb{N}$. Since $|B(t) - B(T_j)|$, $t \ge T_j$, is a submartingale, by Doob's L^2 inequality, there exists $c_2 < \infty$ such that

$$\begin{aligned} \mathbb{E}\left[\sup_{t\in[T_j,U_j]} |B(t) - B(T_j)|\right] &\le c_2 \left(\mathbb{E}\left[|B(U_j) - B(T_j)|^2\right]\right)^{1/2} \\ &\le c_2 (s_{k+1} - s_k)^{1/2}. \end{aligned} \tag{4.11}$$

For every $j \ge 1$ such that $U_j < s_{k+1} - \varepsilon$, the right-continuity of W ensures that $W(U_j) = m_k$. Likewise, for every $j \ge 2$ such that $T_j < s_{k+1} - \varepsilon$, we have $|W(T_j) - m_k| = \Delta/4$ because, as is easy to see, W is continuous at T_j. Indeed, the latter assertion follows because B is continuous, $W(t) \in [m_k - \Delta_k/4, m_k + \Delta_k/4] \subset (\ell(t), r(t))$ for every $t \in [T_j, U_j]$ and, by equations (2.12) and (2.13), at any jump time t of W, either $\ell(t-) = W(t-), \ell(t) = W(t)$ or $r(t-) = W(t-), r(t) = W(t)$. The last two statements, when combined with the triangle inequality, the fact that $W(s) \in [\ell(s), r(s)]$ for every s, and the relation (4.8), show that

$$\begin{aligned} |W(U_1) - W(T_1)| &\leq \mathbb{I}_{\{U_1 = s_{k+1} - \varepsilon\}} |W(U_1) - m_k| + |m_k - W(T_1)| \\ &\leq 2d'_k + d_k \end{aligned} \tag{4.12}$$

and, for $j \geq 2$, since $U_j = T_j$ if $T_j = s_{k+1} - \varepsilon$, we have

$$\begin{aligned} |W(U_j) - W(T_j)| &= \mathbb{I}_{\{T_j < s_{k+1} - \varepsilon\}} |W(U_j) - W(T_j)| \\ &\leq \mathbb{I}_{\{T_j < s_{k+1} - \varepsilon\}} [\mathbb{I}_{\{U_j = s_{k+1} - \varepsilon\}} |W(U_j) - m_k| + |m_k - W(T_j)|] \\ &\leq \mathbb{I}_{\{T_j < s_{k+1} - \varepsilon\}} [2d'_k + \Delta_k/4]. \end{aligned} \tag{4.13}$$

In turn, together with (4.10) and (4.11), this implies that

$$\mathbb{E}[|Y(U_1) - Y(T_1)|] \leq 2d'_k + d_k + c_2(s_{k+1} - s_k)^{1/2} \tag{4.14}$$

and, for $j \geq 2$, again using the fact that $U_j = T_j$ if $T_j = s_{k+1} - \varepsilon$,

$$\begin{aligned} \mathbb{E}[|Y(U_j) - Y(T_j)|] &\leq \left(2d'_k + \Delta_k/4 + c_2(s_{k+1} - s_k)^{1/2}\right) \mathbb{E}[\mathbb{I}_{\{T_j < s_{k+1} - \varepsilon\}}] \\ &\leq \left(2d'_k + c_3(s_{k+1} - s_k)^{1/2}\right) \mathbb{E}[\mathbb{I}_{\{T_j < s_{k+1} - \varepsilon\}}], \end{aligned} \tag{4.15}$$

where the last inequality holds with $c_3 = c_2 + c_1/4$ due to (4.5).

Now, if $T_j < s_{k+1} - \varepsilon$ then the process B must have had an oscillation of size $\Delta_k/4$ or larger inside the interval $[U_{j-1}, T_j]$. However, $\Delta_k/4 \geq (s_{k+1} - s_k)^{1/2}/4c_1$ by the inequality (4.5), and so it can be deduced from Brownian scaling and the Kolmogorov-Čentsov theorem that the expected number of oscillations of B of size $(s_{k+1} - s_k)^{1/2}/4c_1$ on the time interval $[s_k, s_{k+1}]$ is bounded by a constant $c_4 < \infty$. In other words,

$$\sum_{j \geq 2} \mathbb{E}\left[\mathbb{I}_{\{T_j < s_{k+1} - \varepsilon\}}\right] = \mathbb{E}\left[\sum_{j \geq 2} \mathbb{I}_{\{T_j < s_{k+1} - \varepsilon\}}\right] \leq c_4. \tag{4.16}$$

Summing (4.15) over $j \geq 2$, adding (4.14) and using (4.16), we obtain

$$\sum_{j \geq 1} \mathbb{E}\left[|Y(U_j) - Y(T_j)|\right] \leq d_k + c_5 d'_k + c_5(s_{k+1} - s_k)^{1/2}, \tag{4.17}$$

where $c_5 \doteq (c_3 c_4 + c_2) \vee (2 + 2c_4)$.

Since W touches at most one boundary on each interval $[T_j, U_j]$, $j \in \mathbb{N}$, Y is monotone on each such interval. Thus

$$\mathbb{E}\left[\mathcal{V}_{[s_k, s_{k+1} - \varepsilon]} Y\right] \leq \sum_{j \geq 1} \mathbb{E}\left[|Y(U_j) - Y(T_j)|\right] \leq d_k + c_5 d'_k + c_5(s_{k+1} - s_k)^{1/2}.$$

By taking the limit as $\varepsilon \to 0$ and using the fact that variation is monotone, we conclude that

$$\mathbb{E}\left[\mathcal{V}_{[s_k,s_{k+1})}Y\right] \le d_k + c_5 d'_k + c_5(s_{k+1}-s_k)^{1/2}. \tag{4.18}$$

The process Y may have a jump at time s_{k+1}, whose size can be bounded, using the relation $Y = W - B$, the triangle inequality, the continuity of the paths of B and the fact that $W(s) \in [\ell(s), r(s)]$ for all s, as follows:

$$\begin{aligned} |Y(s_{k+1}) - Y(s_{k+1}-)| &\le |W(s_{k+1}) - W(s_{k+1}-)| \\ &\le |W(s_{k+1}) - m_k| + |W(s_{k+1}-) - m_k| \le 2d'_k. \end{aligned}$$

Together with (4.18), this implies that

$$\mathbb{E}\left[\mathcal{V}_{[s_k,s_{k+1}]}Y\right] \le d_k + (c_5+2)d'_k + c_5(s_{k+1}-s_k)^{1/2}.$$

Summing over k and using (4.7) and (4.6), we obtain

$$\mathbb{E}\left[\mathcal{V}_{[0,\tau]}Y\right] \le \sum_{k\in\mathbb{Z}} \left(d_k + (c_5+2)d'_k + c_5(s_{k+1}-s_k)^{1/2}\right) < \infty.$$

□

Example 4.9. Our first example is elementary. Let $-\ell(t) = r(t) = t^\alpha$ for $t \in [0, \tau/4]$ and $-\ell(t) = r(t) = (\tau - t)^\alpha$ for $t \in [3\tau/4, \tau]$, where $\alpha > 0$. We assume that ℓ and r are continuous on $[0, \tau]$ and $r(t) > \ell(t)$ for $t \in (0, \tau)$. Let $f \doteq r - \ell$. Let $\{s_k\}_{k\in\mathbb{Z}}$ be the usual ordering of all points belonging to two families: (i) all points of the form $2^{-j} + mf^2(2^{-j})$, for $m = 0, \dots, \lfloor 2^{-j}/f^2(2^{-j})\rfloor - 1$, and $j > j_1$, where j_1 is the smallest integer such that $2^{-j_1} < \tau/4$, and (ii) all points of the form $\tau - 2^{-j} - mf^2(\tau - 2^{-j})$, for $m = 0, \dots, [2^{-j}/f^2(\tau - 2^{-j})] - 1$, and $j > j_1$. We let a_k be the smallest real number and b_k the largest real number such that $\{(t, x) : s_k < t < s_{k+1}, a_k < x < b_k\} \subset \dot{D}$. To verify that (4.5) holds, it suffices to consider the case when $s_k,\ s_{k+1}$ are in the family (i). For $j > j_1$ and s_k of the form $2^{-j} + mf^2(2^{-j})$ for some $m = 0, \dots \lfloor 2^{-j}/f^2(2^{-j}\rfloor - 1$, we have

$$f^2(2^{-j}) \le s_{k+1} - s_k \le 2f^2(2^{-j}).$$

On the other hand, since f is nondecreasing,

$$b_k - a_k = \inf_{t\in(s_k,s_{k+1})} f(t) = f(s_k) = \frac{f(s_k)}{f(2^{-j})} f(2^{-j}),$$

and

$$1 = \frac{f(2^{-j})}{f(2^{-j})} \le \frac{f(s_k)}{f(2^{-j})} \le \frac{f(2^{-j+1})}{f(2^{-j})} = 2^\alpha.$$

Thus (4.5) holds with $c_1 \doteq \max\{2^\alpha, \sqrt{2}\}$.

Recall the notation from the proof of Theorem 4.8. Consider an interval $[s_k, s_{k+1}] \subset [2^{-j}, 2^{-j+2}]$. Then $d_k \vee d'_k \le c_1 f(2^{-j})$. The sum over all k in the indicated range gives us

$$\sum_{k:[s_k,s_{k+1}]\subset[2^{-j},2^{-j+2}]} (d_k \vee d'_k) \le c_2 2^{-j} f(2^{-j})/f^2(2^{-j}).$$

Summing over $j > j_1$ yields a finite number provided $\alpha < 1$. A similar analysis applies in the interval $[3\tau/4, \tau]$, and so assumption (4.6) of Theorem 4.8 is satisfied if $\alpha < 1$. We conclude that if $\alpha < 1$ then $\mathbb{E}\left[\mathcal{V}_{[0,\tau]} Y\right] < \infty$.

Example 4.10. We present a stronger version of the last example, in which we relax the assumption of symmetry between ℓ and r, but impose a little more regularity of the paths ℓ and r. Let $f \doteq r - \ell$, $f(t) = t^\alpha$ for $t \in [0, \tau/4]$ and $f(t) = (\tau - t)^\alpha$ for $t \in [3\tau/4, \tau]$, where $\alpha > 0$. We assume that $r(t) > \ell(t)$ for $t \in (0, \tau)$. The crucial assumption in this example is that both ℓ and r are Hölder continuous with some exponent $\beta > 1/2$, i.e., for some $c_1 < \infty$ and all $t_1, t_2 \in [0, \tau]$, we have $|\ell(t_1) - \ell(t_2)| \le c_1 |t_1 - t_2|^\beta$, and a similar formula holds for r.

We proceed as in the previous example. Let $\{s_k\}_{k\in\mathbb{Z}}$ be the usual ordering of all points belonging to two families: (i) all points of the form $2^{-j} + m f^2(2^{-j})$, for $m = 0, \dots, [2^{-j}/f^2(2^{-j})] - 1$, and $j > j_1$, where j_1 is the smallest integer such that $2^{-j_1} < \tau/4$, and (ii) all points of the form $\tau - 2^{-j} - m f^2(\tau - 2^{-j})$, for $m = 0, \dots, [2^{-j}/f^2(\tau - 2^{-j})] - 1$, and $j > j_1$. We let a_k be the smallest real number, and we let b_k be the largest real number such that $\{(t,x) : s_k < t < s_{k+1}, a_k < x < b_k\} \subset \dot{D}$.

We will verify (4.5). We will consider only the case when s_k, s_{k+1} are in the family (i). For $j > j_1$ and s_k of the form $2^{-j} + m f^2(2^{-j})$ for some $m = 0, \dots \lfloor 2^{-j}/f^2(2^{-j} \rfloor - 1$, we have

$$f^2(2^{-j}) \le s_{k+1} - s_k \le 2 f^2(2^{-j}).$$

It follows from this and the Hölder continuity of ℓ and r with exponent $\beta > 1/2$ that

$$\begin{aligned} b_k - a_k &= \inf_{t\in(s_k,s_{k+1})} r(t) - \sup_{t\in(s_k,s_{k+1})} \ell(t) \\ &\le r(s_k) + \sup_{t\in(s_k,s_{k+1})} |r(t) - r(s_k)| - \ell(s_k) + \sup_{t\in(s_k,s_{k+1})} |\ell(t) - \ell(s_k)| \\ &\le f(s_k) + 2c_1 |s_k - s_{k+1}|^\beta \\ &\le c_2 f(2^{-j}) + 2c_1 |s_k - s_{k+1}|^\beta \\ &\le c_2 |s_k - s_{k+1}|^{1/2} + 2c_1 |s_k - s_{k+1}|^\beta \\ &\le c_3 |s_k - s_{k+1}|^{1/2}. \end{aligned}$$

Similarly, for large k,

$$\begin{aligned} b_k - a_k &= \inf_{t\in(s_k,s_{k+1})} r(t) - \sup_{t\in(s_k,s_{k+1})} \ell(t) \\ &\geq r(s_k) - \sup_{t\in(s_k,s_{k+1})} |r(t) - r(s_k)| - \ell(s_k) - \sup_{t\in(s_k,s_{k+1})} |\ell(t) - \ell(s_k)| \\ &\geq f(s_k) - 2c_1|s_k - s_{k+1}|^\beta \\ &\geq f(2^{-j}) - 2c_1|s_k - s_{k+1}|^\beta \\ &\geq \sqrt{2}|s_k - s_{k+1}|^{1/2} - 2c_1|s_k - s_{k+1}|^\beta \\ &\geq c_4|s_k - s_{k+1}|^{1/2}. \end{aligned}$$

A similar calculation shows that $d_k \vee d'_k \leq cf(2^{-j})$ for k and j such that $[s_k, s_{k+1}] \subset [2^{-j}, 2^{-j+2}]$.

The rest of the analysis proceeds as in the previous example. The sum over all k in the indicated range gives us

$$\sum_{k:[s_k,s_{k+1}]\subset[2^{-j},2^{-j+2}]} (d_k \vee d'_k) \leq c_2 2^{-j} f(2^{-j})/f^2(2^{-j}).$$

Summing over $j > j_1$ yields a finite number provided $\alpha < 1$. A similar analysis applies in the interval $[3\tau/4, \tau]$, and so the assumption (4.6) of Theorem 4.8 is satisfied if $\alpha < 1$. We conclude that if $\alpha < 1$ then $\mathbb{E}\left[\mathcal{V}_{[0,\tau]}Y\right] < \infty$.

We see that within the family of functions $f(\cdot)$ that decay towards the endpoints of $[0, \tau]$ as t^α, our results are sharp, by comparing the present example with Examples 4.6 and 4.7.

Remark 4.11. Note that the parameters α and β in Example 4.10 can be such that $1/2 < \beta < \alpha < 1$. Consider a function ℓ that is Hölder continuous with exponent β but it is not Hölder continuous with exponent $\beta + \varepsilon$ on any interval $[0, s]$, for any $\varepsilon > 0$ and any $s > 0$. A typical trajectory of a fractional Brownian motion with appropriate exponent provides an example of such a function. By making a linear transformation, we may assume that $\ell(0) = \ell(\tau) = 0$. Let $f(t) = t^\alpha$ for $t \in [0, \tau/4]$ and $f(t) = (\tau - t)^\alpha$ for $t \in [3\tau/4, \tau]$, $f(t)$ is continuous on $[0, \tau]$ and $f(t) > 0$ for $t \in (0, \tau)$. Let $r \doteq \ell + f$. Then ℓ and r satisfy the assumptions of the present example, and so $\mathbb{E}\left[\mathcal{V}_{[0,\tau]}Y\right] < \infty$. Note that neither ℓ nor r need be monotone, and both functions can oscillate between positive and negative values. Their local oscillations near 0 may be comparable in absolute value to t^β, a function much larger than $f(t) = t^\alpha$.

4.3. Analysis of a class of 2-dimensional RBMs

We now apply the results obtained in the last two sections to analyze a class of "valley-shaped" two-dimensional RBMs studied in [4], [11] and [24] (see also [18, 19] to see how RBMs in this class arise as diffusion approximations of a class of queueing networks), which provided one of the sources of motivation for the current work.

We begin by recalling the setup from [4] and rephrase some of the results from that paper in our terminology. The domain $D \subset \mathbb{R}^2$ is described by two continuous, real-valued functions L and R defined on $[0,\infty)$ that satisfy $L(0) = R(0) = 0$ and $L(y) < R(y)$ for all $y > 0$ Let D be given by

$$D \equiv \{(x,y) \in \mathbb{R}^2 : y \geq 0, L(y) \leq x \leq R(y)\}.$$

Let $\partial D^1 = \{(x,y) \in \partial D : x = L(y)\}$ and $\partial D^2 = \{(x,y) \in \partial D : x = R(y)\}$. The paper [4] was concerned with the two-dimensional RBM $Z = (Z^1, Z^2)$ in D, with the vectors of reflection horizontal on ∂D^1 and ∂D^2, and an additional vertical direction of reflection at 0 that ensures the RBM stays within D (see [4] for details). From the Skorokhod-type lemma proved in [4], it follows that Z^2 is a one-dimensional RBM on $[0,\infty)$ and Z^1 is the ESM applied to a standard 1-dimensional BM B in the domain with time-dependent boundaries $\ell(t) = L(Z^2(t))$ and $r(t) = R(Z^2(t))$. By Definition 2.2, Z^1 admits the decomposition $Z^1 = B + Y$ where Y is (pathwise) the local time or the constraining process associated with the ESP on $[\ell(\cdot), r(\cdot)]$.

We first study the total variation of the local time Y on a single excursion of the RBM (Z^1, Z^2) from the origin. Let $[\tau_1, \tau_2]$ be an excursion interval for Z^2, i.e., $\tau_1 < \tau_2$, $Z^2(\tau_1) = Z^2(\tau_2) = 0$, and $Z^2(t) > 0$ for $t \in (\tau_1, \tau_2)$. The following result was established in Theorem 3 of [4] – we provide an alternative proof of this result.

Proposition 4.12. *Suppose that there exist $\varepsilon > 0$ and $\gamma > 2$ such that $R(y) - L(y) \leq y^\gamma$ for $y \in [0,\varepsilon]$. Then*

$$\mathcal{V}_{[\tau_1,\tau_2]} Y = \infty \quad a.s.$$

On the other hand, suppose that there exist $\varepsilon > 0$ and $\gamma < 2$ such that $R(y) - L(y) \geq y^\gamma$ for $y \in [0,\varepsilon]$, and R and L are Lipschitz. Then

$$\mathcal{V}_{[\tau_1,\tau_2]} Y < \infty \quad a.s.$$

Proof. We start with the first case. Let $\gamma_1 < 1/2$ be such that $\gamma \cdot \gamma_1 > 1$. Path properties at endpoints of an excursion of (1-dimensional reflected) Brownian motion from 0 are well known to be the same as those of the 3-dimensional Bessel process, see, e.g., [1]. Hence, it follows from Theorem 3.3 (i) of [20] that

$$\limsup_{t \downarrow \tau_1} \frac{Z^2(t-\tau_1)}{(t-\tau_1)^{\gamma_1}} = 0.$$

By the case assumption, this implies that

$$\limsup_{t \downarrow \tau_1} \frac{R(Z^2(t-\tau_1)) - L(Z^2(t-\tau_1))}{(t-\tau_1)^{\gamma_1\gamma}} \leq \limsup_{t \downarrow \tau_1} \left(\frac{Z^2(t-\tau_1)}{(t-\tau_1)^{\gamma_1}} \right)^\gamma = 0.$$

If we set $\tau \doteq \tau_2 - \tau_1$, $\ell(t) \doteq L(Z^2(t-\tau_1))$, $r(t) \doteq R(Z^2(t-\tau_1))$ and $f(t) \doteq r(t) - \ell(t)$, then we see that $f(t) \leq t^{\gamma_1\gamma}$ for t sufficiently close to 0, where $\gamma_1\gamma > 1$. Arguing as in Example 4.7, it is possible to verify the assumptions of Theorem 4.3 in this case, and so it follows that the variation of local time accumulated by Z^1 on the interval $[\tau_1, \tau_2]$ is infinite a.s.

Next, suppose that $\gamma < 2$, there exists $\varepsilon > 0$ such that $R(y) - L(y) \geq y^\gamma$ for $y \in [0, \varepsilon]$, and the functions L and R are Lipschitz. Let $\gamma_2 > 1/2$ be such that $\gamma_2 \gamma < 1$. We use Theorem 3.3 (ii) of [20] to see that

$$\liminf_{t \downarrow \tau_1} \frac{Z^2(t - \tau_1)}{(t - \tau_1)^{\gamma_2}} = \infty.$$

It follows that

$$\liminf_{t \downarrow \tau_1} \frac{R(Z^2(t - \tau_1)) - L(Z^2(t - \tau_1))}{(t - \tau_1)^{\gamma_2 \gamma}} \geq \liminf_{t \downarrow \tau_1} \left(\frac{Z^2(t - \tau_1)}{(t - \tau_1)^{\gamma_2}} \right)^\gamma = \infty.$$

Using the notation introduced above, $f(t) \geq t^{\gamma_2 \gamma}$ for t sufficiently close to 0, where $\gamma_2 \gamma < 1$. We would like to apply Theorem 4.8. We proceed with the construction of boxes as in Example 4.10. The only new subtle point in the argument is the verification of (4.6). In Example 4.10, we used the fact that ℓ and r were Hölder continuous with exponent $\beta > 1/2$. Now, we use the fact that for any $\beta_1 \in (0, 1/2)$, Brownian motion is Hölder continuous with exponent β_1, a.s., and that the same applies to the trajectories of the 3-dimensional Bessel process (because Brownian excursions from 0 have the same local path properties). Since L and R are assumed to be Lipschitz, we conclude that ℓ and r are Hölder continuous with some exponent $\beta_1 > \gamma_2 \gamma / 2$. This suffices to prove that the inequalities in (4.6) hold. A similar analysis applies at the other endpoint of the excursion, i.e., close to τ_2. We conclude that the variation of the local time accumulated by Z^1 on the interval $[\tau_1, \tau_2]$ is finite a.s. □

We now consider a somewhat different, and perhaps more natural, question of whether Z is a semimartingale. A surprising fact is that for any functions R and L such that $R(0) = L(0)$, the amount of local time accumulated on the boundary is infinite. The rate of growth of $R - L$ in a neighborhood of 0 turns out to be irrelevant for this question. Our argument is based exclusively on the scaling properties of Brownian motion. The proof will use excursion theory; see [1] for a review of the relevant definitions and facts.

Proposition 4.13. *Suppose $Z(0) = 0$. Then for every $T > 0$,*

$$\mathcal{V}_{[0,T]} Y = \infty, \quad a.s.$$

Consequently, the process Z starting from 0 is not a semimartingale.

Proof. Let $[s_k, t_k]$, $k \geq 1$, be the collection of all excursion intervals of Z^2 from 0. In other words, we have $Z^2(s_k) = Z^2(t_k) = 0$ and $Z^2(t) > 0$ for $t \in (s_k, t_k)$. Let $\{\sigma(t), t \geq 0\}$ be the local time of Z^2 at 0. Then the family $\{(\sigma(s_k), \{Z^2(t), t \in [s_k, t_k]\})\}_{k \geq 1}$ is a Poisson point process on the space $[0, \infty) \times \mathcal{U}$, where $\mathcal{U}$ is the space of excursions. The intensity of the Poisson point process is the product of Lebesgue measure and an excursion law H.

Note that we have $Z^1 = B + Y$ and B is independent of Z^2. Let $u_k \in [s_k, t_k]$ be the time when Z^2 attains its maximum on the interval $[s_k, t_k]$. Given s_k and t_k, the process $\{(B(t) - B(u_k), Z^2(t) - Z^2(u_k)), t \in [u_k, t_k]\}$ is independent of all processes $\{(B(t) - B(u_j), Z^2(t) - Z^2(u_j)), t \in [u_j, t_j]\}$, $j \neq k$.

Given $t_k - s_k = a$, the distribution of $a^{-1}(t_k - u_k)$ is independent of a, by scaling. Given $t_k - s_k = a$, the distribution of $a^{-1/2}(B(t_k) - B(u_k))$ is otherwise independent of t_k and s_k, and of $Z^1(u_k)$. By Brownian scaling, there exists $c_0 > 0$ such that

$$\mathbb{P}(B(t_k) - B(u_k) > c_0 a^{1/2} \mid t_k - s_k = a) \geq 1/4,$$

and

$$\mathbb{P}(B(t_k) - B(u_k) < -c_0 a^{1/2} \mid t_k - s_k = a) \geq 1/4.$$

Using independence from $Z^1(u_k)$,

$$\mathbb{P}(Z^1(u_k) + B(t_k) - B(u_k) > c_0 a^{1/2} \mid t_k - s_k = a, Z^1(u_k) > 0) \geq 1/4,$$

and

$$\mathbb{P}(Z^1(u_k) + B(t_k) - B(u_k) < -c_0 a^{1/2} \mid t_k - s_k = a, Z^1(u_k) < 0) \geq 1/4.$$

Combining the two cases, we obtain

$$\mathbb{P}(|Z^1(u_k) + B(t_k) - B(u_k)| > c_0 a^{1/2} \mid t_k - s_k = a) \geq 1/4.$$

Note that $Y(u_k) - Y(t_k) = Z^1(u_k) + B(t_k) - B(u_k)$ since $Z(t_k) = 0$. Thus,

$$\mathbb{P}(|Y(u_k) - Y(t_k)| > c_0 a^{1/2} \mid t_k - s_k = a) \geq 1/4,$$

which implies that

$$\mathbb{P}(\mathcal{V}_{[u_k,t_k]} Y > c_0 a^{1/2} \mid t_k - s_k = a) \geq 1/4. \tag{4.19}$$

Recall that H is the excursion law for excursions of Z^2 from 0 and let ζ be the lifetime of an excursion. Then $H(\zeta \in da) = c_1 a^{-3/2}$ (see [1]). It follows from excursion theory that the number of excursions starting at a point $s_k \leq \sigma^{-1}(1)$ and such that $t_k - s_k \in (2^{-j-1}, 2^{-j}]$ has the Poisson distribution with average $c_1 \int_{2^{-j-1}}^{2^{-j}} a^{-3/2} da = c_2 2^{j/2}$. By (4.19), the number of such excursions with the property that $\mathcal{V}_{[u_k,t_k]} Y > c_0 2^{-(j+1)/2}$ is minorized by the Poisson distribution with average $(1/4) c_2 2^{j/2}$. Hence $\sum_{s_k \leq \sigma^{-1}(1), t_k - s_k \in (2^{-j-1}, 2^{-j}]} \mathcal{V}_{[u_k,t_k]} Y$ is minorized by a random variable which is the product of $c_0 2^{-(j+1)/2}$ and a Poisson random variable with average $(1/4) c_2 2^{j/2}$. By excursion theory, the sums $\sum_{s_k \leq \sigma^{-1}(1), t_k - s_k \in (2^{-j-1}, 2^{-j}]} \mathcal{V}_{[u_k,t_k]} Y$ are independent for different j. Now it is elementary to check that, a.s.,

$$\mathcal{V}_{[0,\sigma^{-1}(1)]} Y \geq \sum_{j \geq 1} \sum_{k: s_k \leq \sigma^{-1}(1), t_k - s_k \in (2^{-j-1}, 2^{-j}]} \mathcal{V}_{[u_k,t_k]} Y = \infty.$$

The same argument shows that $\mathcal{V}_{[0,\sigma^{-1}(t)]} Y = \infty$, a.s., for every $t > 0$. Thus $\mathcal{V}_{[0,s]} Y = \infty$, a.s., for every $s > 0$.

It is intuitively clear from the first part of the proof that Z^1 is not a semimartingale. A subtle technical difficulty is that it is not obvious that the Doob decomposition of Z^1 and the Skorokhod representation have to be identical. We shall show that this is indeed the case in the following paragraph.

Suppose that Z^1 is a semimartingale with the decomposition $Z^1 = \tilde{B} + \tilde{Y}$. Fix $i, m \in \mathbb{N}$ and define a sequence of stopping times as follows. Let $T_1^{i,m} \doteq \inf\{t \geq 0 : Z^2 = 2^{-i}\}$, $S_1^{i,m} \doteq \inf\{t \geq T_1^{i,m} : Z^2 = 2^{-m}\}$, and recursively define, for each $n \geq 2$, $T_n^{i,m} \doteq \inf\{t \geq S_{n-1}^{i,m} : Z^2 = 2^{-i}\}$ and $S_n^{i,m} = \inf\{t \geq T_n^{i,m} : Z^2 = 2^{-m}\}$. On each interval $[T_n^{i,m}, S_n^{i,m}]$, Z^1 is a semimartingale with decomposition $\tilde{B}+\tilde{Y}$. On the other hand, since $\inf_{t\in[T_n^{i,m},S_n^{i,m}]} R(Z^2(t)) - L(Z^2(t)) > 0$, by the last assertion of Theorem 2.6 and uniqueness of the Doob decomposition for Z^1, it follows that a.s. $\tilde{Y}(t) - \tilde{Y}(s) = Y(t) - Y(s)$ for all $s, t \in [T_n^{i,m}, S_n^{i,m}]$. In particular, this implies that, a.s., $\sum_{n=1}^{\infty} \mathcal{V}_{[T_n^{i,m},S_n^{i,m}]} Y = \sum_{n=1}^{\infty} \mathcal{V}_{[T_n^{i,m},S_n^{i,m}]} \tilde{Y}$. Letting $m \to \infty$, and then $i \to \infty$, we have almost surely, for each $T > 0$,

$$\begin{aligned}
\mathcal{V}_{[0,\sigma^{-1}(T)]}\tilde{Y} &\geq \lim_{i\to\infty} \lim_{m\to\infty} \sum_{n=1}^{\infty} \mathcal{V}_{[T_n^{i,m}\wedge\sigma^{-1}(T),S_n^{i,m}\wedge\sigma^{-1}(T)]}\tilde{Y} \\
&= \lim_{i\to\infty} \lim_{m\to\infty} \sum_{n=1}^{\infty} \mathcal{V}_{[T_n^{i,m}\wedge\sigma^{-1}(T),S_n^{i,m}\wedge\sigma^{-1}(T)]}Y \\
&\geq \sum_{j\geq 1} \sum_{k:s_k\leq\sigma^{-1}(T),t_k-s_k\in(2^{-j-1},2^{-j}]} \mathcal{V}_{[u_k,t_k]}Y.
\end{aligned}$$

The first part of the proof now shows that the last term equals infinity. We conclude that Z^1, and therefore Z, is not a semimartingale. □

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