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CHARACTERIZATION OF STATIONARY DISTRIBUTIONS OF REFLECTED DIFFUSIONS

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Given a domain G, a reflection vector field $d(\cdot)$ on ∂G, the boundary of G, and drift and dispersion coefficients $b(\cdot)$ and $\sigma(\cdot)$, let $\mathcal{L}$ be the usual second-order elliptic operator associated with $b(\cdot)$ and $\sigma(\cdot)$. Under mild assumptions on the coefficients and reflection vector field, it is shown that when the associated submartingale problem is well posed, a probability measure π on $\bar{G}$ with $\pi(\partial G) = 0$ is a stationary distribution for the corresponding reflected diffusion if and only if

$$\int_{\bar{G}} \mathcal{L} f(x) \pi(dx) \leq 0$$

for every f in a certain class of test functions. The assumptions are verified for a large class of obliquely reflected diffusions in piecewise smooth domains, including those that are not semimartingales. In addition, it is shown that any nonnegative solution to a certain adjoint partial differential equation with boundary conditions is an invariant density for the reflected diffusion. As a corollary, for bounded smooth domains and a class of polyhedral domains that satisfy a skew-symmetry condition, it is shown that if a certain skew-transform of the drift is conservative and of class $\mathcal{C}^1$, and the covariance matrix is non-degenerate, then the corresponding reflected diffusion has an invariant density p of Gibbs form, that is, $p(x) = e^{H(x)}$ for some $\mathcal{C}^2$ function H. Finally, under a non-degeneracy condition on the diffusion coefficient, a boundary property is established that implies that the condition $\pi(\partial G) = 0$ is necessary for π to be a stationary distribution. This boundary property is of independent interest.

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1. Introduction.

1.1. *Description of Main Results.* This work establishes a simple characterization of stationary distributions of a broad class of reflected diffusions in piecewise smooth domains with oblique reflection, including those that are not necessarily semimartingales, and uses it to identify classes of reflected diffusions with state-dependent drift for which the stationary density takes an explicit form. Reflected diffusions arise in a variety of applications, ranging from queueing theory and operations research to finance and mathematical physics, and their stationary distributions often serve to characterize or approximate important quantities of interest. Consider a domain $G \subset \mathbb{R}^J$, equipped with a vector field $d(\cdot)$ on the boundary ∂G, and drift and dispersion coefficients $b : \bar{G} \mapsto \mathbb{R}^J$ and $\sigma : \bar{G} \mapsto \mathbb{R}^J \times \mathbb{R}^N$, where $\bar{G}$ is the closure of G. A reflected diffusion associated with $(G, d(\cdot))$, $b(\cdot)$ and $\sigma(\cdot)$ is, roughly speaking, a continuous Markov process that behaves locally, near $x \in G$, like a diffusion with state-dependent drift $b(x)$ and dispersion $\sigma(x)$, and is constrained to stay inside $\bar{G}$ by a pushing term that is only allowed to act when the process is on the boundary, and then only along the directions specified by the vector field $d(\cdot)$ at that point on the boundary. One approach to making this heuristic description precise is a generalization of the martingale problem referred to as the submartingale problem, which was introduced by Stroock and Varadhan [43] to characterize the law of reflected diffusions in smooth domains. Other approaches to constructing reflected diffusions include Dirichlet forms [8, 20], controlled martingale problems [30] and stochastic differential equations with reflection (SDER) defined via the Skorokhod problem [16, 26, 35]. However, Dirichlet forms are more naturally suited to analyzing normally reflected diffusions (which are symmetric Markov processes), and the controlled martingale problem and Skorokhod problem approaches can be used only to construct semimartingale reflected diffusions. While extensions of these approaches have been considered in particular cases [15, 26, 35], the submartingale problem seems most suitable for providing a common framework for the characterization of the distributions of semimartingale and non-semimartingale reflected diffusions with oblique reflection in piecewise smooth domains.

fWe provide a precise formulation of the submartingale problem in piecewise smooth domains in Definition 2.1. Prior to this work, although the submartingale problem framework had been used to study specific examples such as reflected Brownian motion (RBM) in two-dimensional cusps and wedges, conical domains and skew-symmetric RBMs in polyhedral domains (which almost surely do not visit the non-smooth parts of the domain) [12, 13, 46, 31, 49], there was no clear definition for the submartingale problem in general piecewise smooth domains. Indeed, the development of a theory of reflected diffusions that could fail to be semimartingales in dimensions greater than two has long been posed as a challenging open problem (see (iii) in Section 4 of [50]). One of the contributions of this work is the identification of a suitable formulation of the submartingale problem that allows for the unique characterization of both the reflected process and its stationary distribution in some generality (see Remark 2.4 for a discussion of some of the subtleties involved). Further justification for the definition of the submartingale problem that we introduce is provided in [28], where it is shown that well-posedness of the submartingale problem is equivalent to existence and uniqueness in law of a weak solution to the corresponding SDER, thus generalizing a classical result for (unconstrained) diffusions obtained by Stroock and Varadhan (cf. Corollary 3.1 of [42]). If the submartingale problem has a unique solutio, it is said to be well-posed.

For a reflected diffusion in a bounded domain, the family of time-averaged occupation measures is automatically tight, and existence of a stationary distribution can be deduced as a simple consequence. On the other hand, for reflected diffusions in unbounded domains suitable conditions on the drift and reflection vector field need to be imposed to guarantee positive recurrence, and have been identified in various cases (see, e.g., [1, 24]). In either case, when the diffusion coefficient is uniformly elliptic, uniqueness of the stationary distribution follows from standard results in ergodic theory. Explicit expressions for the stationary distribution have been obtained mostly for reflected Brownian motions (RBMs) with constant drift in polyhedral domains, either in two dimensions [14, 48] or when a special skew-symmetry condition is satisfied [49, 23]. The focus of the present paper is on characterization of the stationary distribution for a general class of reflected diffusions and the identification of general classes of reflected diffusions with state-dependent drift whose stationary densities are of exponential form, and therefore strictly positive.

Given continuous drift and dispersion coefficients $b : G \mapsto \mathbb{R}^J$ and σ :

$G \mapsto \mathbb{R}^{J\times N}$, let $a : G \mapsto \mathbb{R}^{J\times J}$ be the associated diffusion coefficient given by $a(\cdot) = \sigma(\cdot)\sigma^T(\cdot)$, where $\sigma^T(x)$ denotes the transpose of the matrix $\sigma(x)$, and let $\mathcal{L}$ be the associated second-order differential operator given by

$$\mathcal{L}f(x) \doteq \sum_{i=1}^{J} b_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{J} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x), \qquad f \in \mathcal{C}_b^2(\bar{G}), \tag{1}$$

where $\mathcal{C}_b^2(\bar{G})$ is the space of twice continuously differentiable functions on $\bar{G}$ that, along with their first and second partial derivatives, are bounded. The first main result of this paper, Theorem 1, shows that under some analytical conditions (see Assumption 1), a probability measure π on $\bar{G}$ is a stationary distribution for a reflected diffusion defined by a well-posed submartingale problem if and only if π satisfies $\pi(\partial G) = 0$ and

$$\int_{\bar{G}} \mathcal{L}f(x)\, d\pi(x) \le 0 \tag{2}$$

for all f belonging to $\mathcal{H}$, a certain class of test functions defined in (3). A subtlety in this result lies in the correct choice of test functions in (2). See Remarks 2.4 and 5.2 for further discussion of this issue. The second result, Theorem 2, shows that the conditions of Theorem 1 are satisfied by a large class of reflected diffusions in piecewise smooth domains described in Definition 3.3 that satisfy a mild condition (Assumption 2), that is satisfied whenever the so-called generalized completely-$\mathcal{S}$ condition holds (see Illustrative examples of reflected diffusions that arise in applications and satisfy our conditions are presented in Section 4.

The third result (Theorem 3) shows that any $\mathcal{C}^2$ nonnegative solution to a certain adjoint partial differential equation (PDE), subject to oblique derivative boundary conditions, is an invariant density for the corresponding reflected diffusion. In Corollary 1, this PDE is used to identify a broad class of reflected diffusions with state-dependent drift that have an invariant density of Gibbs form, that is, $p(x) = e^{H(x)}$ for some $\mathcal{C}^2$ function or "potential" H. In particular, for bounded smooth domains and a class of polyhedral domains that satisfy a skew-symmetry condition, it is shown in Corollary 2 that given a nondegenerate covariance matrix A, if a certain skew-transform of the drift is conservative (i.e., of gradient form) and $\mathcal{C}^1$, then the reflected diffusion has an invariant density p of Gibbs form, that is, $p(x) = e^{H(x)}$ for a suitable "potential" function H. In addition, it is also shown that, under the same skew-symmetry conditions, any RBM with such a drift b and covariance matrix A is dual (with respect to p) to an RBM with certain adjoint directions of reflection, drift $-b + AH$ and the same covariance matrix A. This generalizes both the well known property that an (unconstrained) diffusion with constant covariance and drift of gradient form has an invariant density of Gibbs form, as well as results in [23] for reflected Brownian motions with constant drift. We emphasize that in the case of reflected diffusions, the gradient condition is on the so-called skew-transform (see Definition 3.5) of the drift, and not on the drift itself. Furthermore, several examples are provided when the potential H of the stationary density takes an explicit form, including the case of reflected Ornstein-Uhlenbeck processes, which are of interest in applications [38].

Finally, under a non-degeneracy condition on the diffusion coefficient, the last result of this paper (Proposition 6.1) establishes a certain boundary property which shows that the reflected diffusion spends almost surely zero Lebesgue time on the boundary. This boundary property, which is of independent interest, implies that $\pi(\partial G) = 0$ is a necessary condition for π to be a stationary distribution. It is also used in [28] to establish the equivalence between well-posedness of submartingale problems and well-posedness of weak solutions to corresponding SDERs.

1.2. *Relation to Prior Work.* A criterion for invariant measures analogous to (2) was first obtained by Echeverria for (unconstrained) diffusions on a locally compact separable metric space E [19]. It follows from Echeverria's work that, given drift and dispersion coefficients $b(\cdot)$ and $\sigma(\cdot)$ that are associated with a well-posed martingale problem, a probability measure π is a stationary distribution for the corresponding diffusion if and only if (2) holds with inequality replaced by equality and for test functions $f \in \mathcal{C}_c^2(E)$, the space of twice continuously differentiable functions with compact support on E. Extensions of Echeverria's criterion were obtained by Bhatt and Karandikar [5], who relaxed the local compactness condition on E, and by Stockbridge [41] and Bhatt and Borkar [4], who extended it to controlled processes. An extension of Echeverria's criterion to reflected diffusions in smooth domains defined by well-posed submartingale problems was first obtained by Weiss in his PhD thesis [47]. However, the results of [47] do not apply to reflected diffusions in non-smooth domains in $\mathbb{R}^J$. Kurtz and Stockbridge [29, 30] further extended Weiss' result to obtain abstract sufficient conditions for existence of stationary solutions to more general Markov processes defined in terms of controlled and singular martingale problems. However, the framework of controlled martingale problems in [29, 30] cannot be used to uniquely characterize non-semimartingale reflected diffusions, and hence, is not suitable for the analysis of more general processes of interest under study here, which can be characterized via the submartingale problem. Nevertheless, we use a result from [30] in our proof of Theorem 1, and also clarify the connection between the stationary solutions in [30] and stationary solutions to well-posed submartingale problems, which have been used more widely in the literature to characterize reflected diffusions in curved domains.

For a class of semimartingale RBMs in the non-negative orthant associated with so-called $\mathcal{M}$-reflection matrices, a certain basic adjoint relation (BAR), which is related to the adjoint PDE established in Theorem 3, was established in the seminal work of Harrison and Williams in [24] (see also [10] for an extension). However, there are many RBMs of interest that fall outside the domain of the results of [24, 10] such as, for example, RBMs in polygons in $\mathbb{R}^2$, considered in the work of Harrison, Landau and Shepp [22], which could fail to be semimartingales for some parameter values. In particular, Theorem 3 of the present paper rigorously establishes that the solution to the PDE in two-dimensional polygonal domains obtained in [22] is indeed the stationary density of the associated RBM (see Example 4.3). Indeed, the result of Weiss [47], which was cited in [22] to relate their analytical result to the stationary distribution of the RBM, does not cover the case of non-smooth polygonal domains studied in [22]. As mentioned above, the identification of a large class of reflected diffusions with stationary density of Gibbs form generalizes some of the results obtained for RBMs with constant drift by Harrison and Williams in [23, 49].

Finally, the boundary property established in Proposition 6.1) can be viewed as a generalization of results in [24, 11] (see also [49]) for semimartingale reflecting Brownian motions in the orthant or in more general convex polyhedral domains, to more general reflected diffusions in possibly curved domains.

1.3. *Outline of the Paper.* Section 2 contains a precise definition of the submartingale problem and of a class of reflected diffusions in piecewise domains. In Section 3.1 the first main results of the paper, Theorems 1 and 2, are stated. The proofs of Theorem 1 and Theorem 2 are given in Section 5 and Section 7, respectively, and Section 4 contains illustrative examples of reflected diffusions for which the assumptions of the two theorems are valid. In Section 3.2, several consequences of Theorems 1 and 2 are established, including the adjoint PDE (Theorem 3), identification of strictly positive solutions under suitable assumptions (Corollaries 1 and 2) and illustrative examples. Finally, the boundary property is stated and proved in Section 6. The proofs of some technical lemmas are relegated to the Appendix. In the next section, we summarize some common notation used in the paper.

1.4. *Notation and Terminology.* The following notation is used throughout the paper. $\mathbb{Z}$ is the set of integers, $\mathbb{N}$ is the set of positive integers, $\mathbb{R}$ is the set of real numbers, $\mathbb{Z}_+$ is the set of non-negative integers and $\mathbb{R}_+$ the set of non-negative real numbers. For each $J \in \mathbb{N}$, $\mathbb{R}^J$ is the J-dimensional Euclidean space and $|\cdot|$ and $\langle \cdot, \cdot \rangle$, respectively, denote the Euclidean norm and the inner product on $\mathbb{R}^J$. Vectors will be represented as column vectors, and for each vector $v \in \mathbb{R}^J$ and matrix $\sigma \in \mathbb{R}^J \times \mathbb{R}^N$, v^T and σ^T denote the transpose of v and σ, respectively. Given a square matrix $A \in \mathbb{R}^{J\times J}$, $\text{diag}(A)$ represents the column vector containing the diagonal elements of A and $tr(A)$ denotes the trace of A, equal to $\sum_{i=1}^{J} A_{ii}$. For each set $A \subset \mathbb{R}^J$, A°, ∂A, $\bar{A}$ and A^c denote the interior, boundary, closure and complement of A, respectively. For each $x \in \mathbb{R}^J$ and $A \subset \mathbb{R}^J$, $\text{dist}(x, A)$ is the distance from x to A (that is, $\text{dist}(x, A) = \inf\{y \in A : \ |y - x|\}$). For each $A \subset \mathbb{R}^J$ and $r > 0$, $B_r(A) = \{y \in \mathbb{R}^J : \ \text{dist}(y, A) \leq r\}$, and given $\varepsilon > 0$ let $A^\varepsilon \doteq \{y \in \mathbb{R}^J : \text{dist}(y, A) < \varepsilon\}$ denote the (open) ε-fattening of A. If $A = \{x\}$, we simply denote $B_r(A)$ by $B_r(x)$. We will use $S_1(0)$ to denote the unit sphere in $\mathbb{R}^J$. We also let $\mathbb{I}_A$ denote the indicator function of the set A (that is, $\mathbb{I}_A(x) = 1$ if $x \in A$ and $\mathbb{I}_A(x) = 0$ otherwise).

Given a domain E in $\mathbb{R}^n$, for some $n \in \mathbb{N}$, let $\mathcal{C}(E) = \mathcal{C}^0(E)$ be the space of continuous real-valued functions on E and, for any $m \in \mathbb{Z}_+ \cup \{\infty\}$, let $\mathcal{C}^m(E)$ be the subspace of functions in $\mathcal{C}(E)$ that are m times continuously differentiable on E with continuous partial derivatives of order up to and including m. When E is the closure of a domain, $\mathcal{C}^m(E)$ is to be interpreted as the collection of functions in $\cap_{\varepsilon>0}\mathcal{C}^m(E^\varepsilon)$, where E^ε is an open ε-neighborhood of E, restricted to E. Also, let $\mathcal{C}_b^m(E)$ be the subspace of $\mathcal{C}^m(E)$ consisting of bounded functions whose partial derivatives of order up to and including m are also bounded, let $\mathcal{C}_c^m(E)$ be the subspace of $\mathcal{C}^m(E)$ consisting of functions that vanish outside compact sets. In addition, let $\mathcal{C}_c^m(E) \oplus \mathbb{R}$ be the direct sum of $\mathcal{C}_c^m(E)$ and the space of constant functions, that is, the space of functions that are sums of functions in $\mathcal{C}_c^m(E)$ and constants in $\mathbb{R}$. For definitions of the space of functions or vector fields on E that are of class C^m for some non-integral m, we refer the reader to a standard book on partial differential equations [21]. If $m = 0$, we denote $\mathcal{C}^m(E)$, $\mathcal{C}_b^m(E)$, $\mathcal{C}_c^m(E)$, $\mathcal{C}_c^m(E)\oplus\mathbb{R}$ simply by $\mathcal{C}(E)$, $\mathcal{C}_b(E)$, $\mathcal{C}_c(E)$, $\mathcal{C}_c(E)\oplus\mathbb{R}$, respectively. The support of a function f is denoted by $\text{supp}(f)$, its gradient of f is denoted by ∇f and the Laplacian of f is denoted by Δf. We say a set-valued function $f(\cdot)$ defined on a subset E of $\mathbb{R}^J$ is continuous at $x \in E$ if for every $\varepsilon > 0$, there exists a neighbourhood $O_x \subset E$ of x such that $f(y) \subseteq B_\varepsilon(f(x))$ for each $y \in O_x$ and we say $f(\cdot)$ is continuous on E if it is continuous at each $x \in E$.

The space of continuous functions on $[0,\infty)$ that take values in $\mathbb{R}^J$ is denoted by $\mathcal{C}\,[0,\infty)$, the Borel σ-algebra of $\mathcal{C}\,[0,\infty)$ is denoted by $\mathcal{M}$, and the natural filtration on $\mathcal{C}\,[0,\infty)$ is denoted by $\{\mathcal{M}_t\}$. The Borel σ-algebra of $\bar{G}$ is denoted by $\mathcal{B}(\bar{G})$.

2. A Class of Reflected Diffusions. In this section we introduce the class of reflected diffusions that we consider. Let G be a nonempty connected domain in $\mathbb{R}^J$, and let $d(\cdot)$ be a set-valued mapping defined on $\bar{G}$, such that each $d(x)$, $x \in \partial G$, is a non-empty closed convex cone in $\mathbb{R}^J$ with vertex at the origin 0, $d(x) = \{0\}$ for each x in G°, and the graph of $d(\cdot)$ is closed, that is, the set $\{(x,v) : x \in \bar{G}, v \in d(x)\}$ is a closed subset of $\mathbb{R}^{2J}$. Let $\mathcal{V}$ be a subset of ∂G. As shown in Section 4, $\mathcal{V}$ will typically be a (possibly empty) subset of the non-smooth parts of the boundary of the domain G where $d(\cdot)$ is not sufficiently well behaved. For each function f defined on $\mathbb{R}^J$, we say f is constant in a neighborhood of $\mathcal{V}$ if for each $x \in \mathcal{V}$, f is constant in some open neighborhood of x. Given measurable drift and dispersion coefficients $b : \mathbb{R}^J \mapsto \mathbb{R}^J$ and $\sigma : \mathbb{R}^J \mapsto \mathbb{R}^J \times \mathbb{R}^N$, and $a = \sigma\sigma^T : \mathbb{R}^J \mapsto \mathbb{R}^J \times \mathbb{R}^J$, let $\mathcal{L}$ be the associated differential operator defined in (1). One way of characterizing a reflected diffusion is through the so-called submartingale problem. The submartingale problem is a generalization of the martingale problem that was first introduced in [44] to characterize the law of reflected diffusions in smooth domains. Extensions of the submartingale problem to characterize RBMs in two-dimensional piecewise smooth domains were considered in various works [12, 13, 46] and multi-dimensional RBMs that satisfy a special skew-symmetry condition was considered in [49]. Definition 2.1 generalizes these formulations further to accommodate a more general class of multi-dimensional reflected diffusions. As mentioned earlier, a suitable formulation of the submartingale problem for multi-dimensional reflected diffusions that need not be semimartingales has long been a challenging problem [50]. Remark 2.4 provides further discussion of this formulation, and in particular, of the role of the set $\mathcal{V}$. In what follows, recall that $\mathcal{C}^2_c(\bar{G}) \oplus \mathbb{R}$ is the space of functions that are sums of functions in $\mathcal{C}^2_c(\bar{G})$ and constants in $\mathbb{R}$, and that ∇f denotes the gradient of a function f on a domain in $\mathbb{R}^J$. Given a subset $\mathcal{V} \subset \partial G$, let $\mathcal{H} = \mathcal{H}_{\mathcal{V}}$ be the set of functions

$$\mathcal{H} \doteq \left\{ \begin{array}{ll} f \in \mathcal{C}^2_c(\bar{G}) \oplus \mathbb{R} : & f \text{ is constant in a neighborhood of } \mathcal{V}, \\ & \langle d, \nabla f(y) \rangle \geq 0 \text{ for } d \in d(y) \text{ and } y \in \partial G \end{array} \right\}. \tag{3}$$

When $\mathcal{V}$ is the empty set, the condition that f be constant in a neighborhood of $\mathcal{V}$ is understood to be void. When $\mathcal{V}$ is a disjoint union of connected subsets, the condition that f be constant in a neighborhood of $\mathcal{V}$ means that f is constant in a neighborhood of each connected subset.

Definition 2.1. *(Submartingale Problem) A family $\{\mathbb{Q}_z, z \in \bar{G}\}$ of probability measures on $(\mathcal{C}[0,\infty), \mathcal{M})$ is a solution to the submartingale problem associated with $(G, d(\cdot))$, $\mathcal{V}$, drift $b(\cdot)$ and dispersion $\sigma(\cdot)$ if for each $A \in \mathcal{M}$, the mapping $z \mapsto \mathbb{Q}_z(A)$ is $\mathcal{B}(\bar{G})$-measurable and for each $z \in \bar{G}$, $\mathbb{Q}_z$ satisfies the following three properties:*

1. *$\mathbb{Q}_z(\omega(0) = z) = 1$;*
2. *For every $t \in [0,\infty)$ and $f \in \mathcal{H}_{\mathcal{V}} \cap \mathcal{C}_c^2(\mathbb{R}^J)$, the process*

$$f(\omega(t)) - \int_0^t \mathcal{L}f(\omega(u))\, du, \quad t \geq 0, \tag{4}$$

   is a $\mathbb{Q}_z$-submartingale on $(\mathcal{C}[0,\infty), \mathcal{M}, \{\mathcal{M}_t\})$;
3. *For every $z \in \bar{G}$,*

$$\mathbb{E}^{\mathbb{Q}_z}\left[\int_0^\infty \mathbb{I}_{\mathcal{V}}(\omega(s))\, ds\right] = 0.$$

In this case, $\mathbb{Q}_z$ is said to be a solution to the submartingale problem starting from z. Moreover, given a probability distribution π on $\bar{G}$, the probability measure $\mathbb{Q}_\pi$, defined by

$$\mathbb{Q}_\pi(A) = \int_{\bar{G}} \mathbb{Q}_z(A)\, \pi(dz), \qquad \text{for every } A \in \mathcal{M}, \tag{5}$$

is said to be a solution to the submartingale problem with initial distribution π.

The first condition in Definition 2.1 simply states that the family of measures is parameterized by the initial condition. The second condition in Definition 2.1 captures the notion of diffusive behavior in the interior, and reflection along the appropriate directions on the boundary. Since the "test functions" in property 2 are constant in a neighbourhood of $\mathcal{V}$, this condition does not provide information on the behavior of the diffusion in a neighborhood of $\mathcal{V}$. The third condition is imposed to ensure instantaneous reflection (precluding the possibility of absorption or partial reflection) on the boundary. A canonical choice for the set $\mathcal{V}$ is given below in (7).

Definition 2.2. *The submartingale problem associated with $(G, d(\cdot))$, $\mathcal{V}$, drift $b(\cdot)$ and dispersion $\sigma(\cdot)$ is said to be well posed if there exists exactly one solution to the submartingale problem.*

We will only consider submartingale problems that are well posed. In addition, we will also assume throughout, without explicit mention, that the drift and diffusion coefficients are continuous. Under this assumption, for every $f \in \mathcal{C}_c^2(\mathbb{R}^J)$, the mapping $x \mapsto \mathcal{L}f(x)$ is continuous, and so the integral in (4) is clearly well defined.

We next consider reflected diffusions associated to the submartingale problem.

DEFINITION 2.3. *A stochastic process Z defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a reflected diffusion associated with $(G, d(\cdot))$, $\mathcal{V}$, drift $b(\cdot)$ and dispersion $\sigma(\cdot)$ if its family of distribution laws $\{\mathbb{Q}_z,\ z \in \bar{G}\}$ is the unique solution to the submartingale problem, where for $z \in \bar{G}$, $\mathbb{Q}_z$ is the conditional distribution of Z under $\mathbb{P}$, conditioned on $Z(0) = z$.*

REMARK 2.4. When the domain G is smooth, the class of test functions used in the definition of the submartingale problem are the functions in $\mathcal{H}$ with $\mathcal{V} = \emptyset$, so that property 3 of Definition 2.1 is essentially absent [43]. The analysis of reflected diffusions in non-smooth domains via the submartingale problem has to a large extent concentrated on the case when the set of non-smooth points is a singleton or a collection of isolated points [12, 13, 23, 31, 46] (an exception is [49], where the RBM can be shown not to hit the non-smooth parts of the domain). In each of these cases, the submartingale problem has been defined with $\mathcal{V}$ equal to the non-smooth part of the boundary ∂G.

One natural extension of the submartingale problem to higher dimensions would be to continue to set $\mathcal{V}$ in Definition 2.1 to be the subset of non-smooth points of the boundary ∂G. However, the corresponding set of test functions $\mathcal{H} = \mathcal{H}_{\mathcal{V}}$ would then fail to satisfy separability properties (see Assumption 1) that are typically required for natural approaches to the characterization of stationary distributions to succeed.

We take a slightly different approach. In the analysis of reflected diffusions in non-smooth domains, a special role is played by the following set on the boundary:

$$\mathcal{U} \doteq \{x \in \partial G : \exists\ n \in n(x) \text{ such that } \langle n, d\rangle > 0,\ \forall\ d \in d(x) \setminus \{0\}\}. \tag{6}$$

Here, $n(x)$ is the set of interior normal vectors to the domain G at $x \in \partial G$. Indeed, the condition $\partial G = \mathcal{U}$ can be viewed as a generalization of what is known in the literature as the completely-$\mathcal{S}$ condition [39, 45, 35]. The boundary property in Proposition 6.1 shows that (for a large class of domains), any solution to the submartingale problem with $\mathcal{V} \supseteq \partial G \setminus \mathcal{U}$ spends zero Lebesgue time on the boundary of ∂G. This suggests that a canonical choice of $\mathcal{V}$ in Definition 2.1 is to set

$$\mathcal{V} = \partial G \setminus \mathcal{U}. \tag{7}$$

Further justification for this choice arises from the fact that then the resulting submartingale problem is well-posed for a large class of multidimensional semimartingale and non-semimartingale RBMs in both polyhedral and curved domains that arise in a variety of applications. Indeed, this well-posedness follows from a general result proved in [28], which shows that under fairly general conditions, the submartingale problem with $\mathcal{V} = \partial G \setminus \mathcal{U}$ is well-posed if and only if there exists a weak solution to the corresponding SDER that is unique in law, together with results that establish the latter property in quite some generality [45, 26, 36, 37, 25] (also see Examples 4.4 and 4.5).

3. Statement of Results. The primary goal of this work is to provide a useful characterization of the stationary distributions of a broad class of reflected diffusions that includes several families of reflected diffusions that arise in applications. In Section 3.1 we state our assumptions and the main results, and in Section 3.2 we derive some important consequence of the main result.

3.1. *Main Results.* We start with a basic definition.

Definition 3.1. *A probability measure π on $\bar{G}$ is a stationary distribution for the unique solution $\{\mathbb{Q}_z, z \in \bar{G}\}$ to a well posed submartingale problem if π satisfies the property that the law of $\omega(t)$ under $\mathbb{Q}_\pi$ is π for each $t \geq 0$. In this case, π is also said to be a stationary distribution of any reflected diffusion associated with the well posed submartingale problem.*

The main result of this paper is a necessary and sufficient condition for a probability measure π to be a stationary distribution for the well posed submartingale problem. Recall the definition of $\mathcal{H}$ in (3). It is easy to see that if the unique solution $\{\mathbb{Q}_z, z \in \overline{G}\}$ to a well posed submartingale problem associated with $(G, d(\cdot))$ and $\mathcal{V}$ admits a stationary distribution π, then π must satisfy the inequality (2) for all $f \in \mathcal{H}$, where $\mathcal{L}$ is the operator defined in (1). Indeed, it follows from the second property in Definition 2.1 that for each $f \in \mathcal{H}$,

$$\mathbb{E}^{\mathbb{Q}_\pi}\left[f(\omega(t)) - \int_0^t \mathcal{L}f(\omega(u))\,du\right] \geq \mathbb{E}^{\mathbb{Q}_\pi}\left[f(\omega(0))\right].$$

Since $\mathbb{E}^{\mathbb{Q}_\pi}\left[f(\omega(t))\right] = \mathbb{E}^{\mathbb{Q}_\pi}\left[f(\omega(0))\right]$ due to the stationarity of π, this establishes the inequality in (2) for all functions $f \in \mathcal{H}$. We will show that, under the assumption stated below, the later condition is also sufficient for any probability measure π with $\pi(\partial G) = 0$ to be a stationary distribution of $\{\mathbb{Q}_z, z \in \bar{G}\}$.

ASSUMPTION 1. *The set $\mathcal{H}$ has the following two properties:*

1. *$\mathcal{H}$ separates points in the sense that for any two different points $x, y \in \bar{G}$, there exists a function $f \in \mathcal{H}$ such that $f(x) \neq f(y)$;*
2. *For every $r, s > 0$, there exists a function $f_{r,s} \in \mathcal{H} \cap \mathcal{C}_c^2(\bar{G})$ such that for every $x \in \partial G$ with $|x| \leq r$ and $\operatorname{dist}(x, \mathcal{V}) \geq s$ and $d \in d(x) \cap S_1(0)$, $\langle d, \nabla f_{r,s}(x)\rangle \geq 1$.*

REMARK 3.2. If $d(\cdot) \cap S_1(0)$ is continuous as a set-valued function on $\partial G \setminus \mathcal{V}$ (see Section 1.4 for the definition), then property 2 of Assumption 1 is equivalent to the seemingly weaker condition that for each $x \in \partial G \setminus \mathcal{V}$, there exists a function $f \in \mathcal{H}$ such that $\langle d, \nabla f(x)\rangle > 0$ for each $d \in d(x) \cap S_1(0)$. By replacing f by $f - \lim_{|x|\to\infty} f(x)$, we can assume that for any $x \in \partial G$ the function f lies in $\mathcal{H} \cap \mathcal{C}_c^2(\bar{G})$. Moreover, by the continuity of ∇f and the continuity of $d(\cdot) \cap S_1(0)$, for any $x \in \partial G \setminus \mathcal{V}$, there exists an open neighbourhood $\mathcal{O}_x$ of x such that $\langle d, \nabla f(y)\rangle > 0$ for $d \in d(y) \cap S_1(0)$ and $y \in \mathcal{O}_x \cap \partial G$. Then, since $\mathcal{H} \cap \mathcal{C}_c^2(\bar{G})$ is closed under addition, given any compact set $K \subset \partial G \setminus \mathcal{V}$, a standard finite subcovering argument can be used to construct $f \in \mathcal{H} \cap \mathcal{C}_c^2(\bar{G})$ such that $\inf_{d \in d(y) \cap S_1(0), y \in K} \langle d, \nabla f(y)\rangle > 0$. Since $f \in \mathcal{H}$ implies $af \in \mathcal{H}$ for any $a > 0$, one can ensure that the last infimum is greater than 1 (or any given specified value $C < \infty$). In particular, the above argument can be applied to the compact set $K = \{x \in \partial G : |x| \leq r, \operatorname{dist}(x, \mathcal{V}) \geq s\}$ for any $r, s > 0$.

We now state the first main result of this paper. Its proof is given in Section 5. Recall that we assume throughout that the drift and diffusion coefficients are continuous.

THEOREM 1. *Suppose we are given $(G, d(\cdot))$, $b(\cdot)$, $\sigma(\cdot)$ and a finite set $\mathcal{V}$ such that the associated submartingale problem is well posed and Assumption 1 holds. Let π be a probability measure on $(\bar{G}, \mathcal{B}(\bar{G}))$ with $\pi(\partial G) = 0$. Then π satisfies the inequality (2) for all $f \in \mathcal{H}$ if and only if π is a stationary distribution for the unique solution to the associated submartingale problem.*

We now introduce a broad class of data $(G, d(\cdot))$ and $\mathcal{V}$ for which the stationary distribution characterization obtained in Theorem 1 applies.

DEFINITION 3.3. *For $0 \leq k$ and $\ell \leq k$, the pair $(G, d(\cdot))$ is said to be piecewise $\mathcal{C}^k$ with $\mathcal{C}^\ell$ reflection if G and $d(\cdot)$ satisfy the following properties:*

1. *The domain G is a nonempty domain with representation $G = \bigcap_{i \in \mathcal{I}} G_i$, where $\mathcal{I}$ is a finite index set and for each $i \in \mathcal{I}$, G_i is a nonempty domain with $\mathcal{C}^k$ boundary, that is, there exists a $\mathcal{C}^k$ function ϕ^i on $\mathbb{R}^J$ such that $\nabla \phi^i(x) \neq 0$ for all $x \in \partial G$,*

$$G_i = \{x : \phi^i(x) > 0\} \quad \text{and} \quad \partial G_i = \{x : \phi^i(x) = 0\}.$$

Let $n^i(x) = \nabla \phi^i(x)/||\phi^i(x)||$ denote the unit inward normal vector to ∂G_i at $x \in \partial G_i$ and define

$$\mathcal{I}(x) \doteq \{i \in \mathcal{I} : \ x \in \partial G_i\}, \tag{8}$$

and note that for each $x \in \partial G$, the set of inward normals to G at the point x is given by

$$n(x) = \left\{ \sum_{i \in \mathcal{I}(x)} s_i n^i(x), s_i \geq 0, i \in \mathcal{I}(x) \right\}. \tag{9}$$

2. *The direction vector field $d(\cdot)$ is given by*

$$d(x) \doteq \left\{ \sum_{i \in \mathcal{I}(x)} s_i \gamma^i(x), \ s_i \geq 0, i \in \mathcal{I}(x) \right\}, \qquad x \in \partial G, \tag{10}$$

where for each $i \in \mathcal{I}$, $\gamma^i(\cdot)$ is a vector field defined on ∂G_i such that

$$\langle n^i(x), \gamma^i(x) \rangle = 1, \qquad \text{for each } x \in \partial G_i, \tag{11}$$

and $\gamma^i(\cdot)/||\gamma^i(\cdot)||$ is of class $\mathcal{C}^\ell$.

Note that in property 2 above, the condition $\langle n^i(x), \gamma^i(x) \rangle = 1$ for each $x \in \partial G_i$ is equivalent to the seemingly weaker condition that $\langle n^i(x), \gamma^i(x) \rangle > 0$, because the vector field $\gamma^i(x)$ can always be renormalized without changing the definition of $d(\cdot)$.

Assumption 2. *$\mathcal{V}$ is a finite set such that $\mathcal{V} \supseteq \partial G \setminus \mathcal{U}$, and if $\mathcal{V}$ contains at least two elements, then for each $x \in \mathcal{V}$, there exist a unit vector v_x and a constant $\rho_x > 0$ such that $\langle v_x, \gamma^i(y) \rangle \geq 0$ for each $i \in \mathcal{I}(y)$ and $y \in B_{\rho_x}(x)$.*

Remark 3.4. Note that the finiteness assumption is reasonable given the canonical choice of $\mathcal{V}$ in (7). Also, note that Assumption 2 is trivially satisfied when $\partial G = \mathcal{U}$, and $\mathcal{V} = \emptyset$. In the context of certain polyhedral domains with piecewise constant reflection fields, the condition $\mathcal{V} = \emptyset$ has been shown to be necessary and sufficient for the associated reflected diffusion to be a semimartingale [35, 39, 45]. However, in this work we also allow for cases when $\partial G \neq \mathcal{U}$, thus providing a characterization of the stationary distribution for reflected diffusions that are not necessarily semimartingales [13, 6, 26, 35].

We now state the second main result of this paper, whose proof is given in Section 7. Recall that the diffusion coefficient $a(\cdot)$ is said to be uniformly elliptic if there exists $\alpha > 0$ such that

$$u^T a(x) u \geq \alpha |u|^2 \qquad \text{for all } u \in \mathbb{R}^J, x \in \bar{G}. \tag{12}$$

We will assume this condition for simplicity when stating the second part of the result, although only partial uniform ellipticity in a certain direction at each $x \in \partial G \setminus \mathcal{V}$ is actually required, as shown in (66).

THEOREM 2. *Suppose that $(G, d(\cdot))$ is piecewise $\mathcal{C}^1$ with continuous reflection. If $\mathcal{V}$ satisfies Assumption 2, then Assumption 1 holds. Moreover, if $(G, d(\cdot))$ is piecewise $\mathcal{C}^2$ with continuous reflection, the diffusion coefficient $a(\cdot)$ is uniformly elliptic, and the submartingale problem associated with $(G, d(\cdot))$, $b(\cdot)$, $\sigma(\cdot)$ and $\mathcal{V}$ is well posed, then a probability measure π on $\bar{G}$ is a stationary distribution for the associated reflected diffusion if and only if $\pi(\partial G) = 0$ and the inequality condition (2) is satisfied.*

As an immediate consequence we see that Theorem 1 can be used to characterize the stationary distributions of reflected diffusions that satisfy the conditions of Theorem 2 and are associated with well-posed submartingale problems. As shown in Section 4, this includes many classes of reflected diffusions that arise in applications.

3.2. *Some Consequences of the Main Results.* We now describe some ramifications of the main results. First, let $\mathcal{L}^*$ be the adjoint operator to $\mathcal{L}$: for $p \in \mathcal{C}^2(\bar{G})$,

$$\mathcal{L}^* p(x) = \frac{1}{2} \sum_{i,j=1}^{J} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x) p(x)) - \sum_{i=1}^{J} \frac{\partial}{\partial x_i} (b_i(x) p(x)).$$

We start by showing that nonnegative and integrable solutions of a certain adjoint partial differential equation (with boundary conditions), are indeed stationary distributions for the submartingale problem. In what follows, let $\mathcal{S}$ denote the smooth parts of the boundary ∂G.

THEOREM 3. *Suppose that the pair $(G, d(\cdot))$ is piecewise $\mathcal{C}^1$ with $\mathcal{C}^1$ reflection, (11) is satisfied, $\mathcal{V} \subset \partial G$ satisfies Assumption 2 $b_i(\cdot) \in \mathcal{C}^1(\bar{G})$, $a_{ij}(\cdot) \in \mathcal{C}^2(\bar{G})$ for $i, j = 1, \ldots, J$, and the submartingale problem associated with $(G, d(\cdot))$ and $\mathcal{V}$ is well posed. Furthermore, suppose there exists a nonnegative function $p \in \mathcal{C}^2(\bar{G} \setminus \mathcal{V})$ with $\int_{\bar{G}} p(x)dx < \infty$ that solves the adjoint PDE defined by the following three relations:*
1. $\mathcal{L}^* p(x) = 0$ for $x \in G$;
2. for each $i \in \mathcal{I}$ and $x \in \partial G_i \cap \mathcal{S}$,

$$\begin{aligned} -2p(x)\left\langle n^i(x), b(x)\right\rangle + (n^i(x))^T a(x)\nabla p(x) & \\ -\nabla \cdot (p(x)q^i(x)) + p(x)K_i(x) &= 0, \end{aligned} \tag{13}$$

where for $i \in \mathcal{I}$,

$$q^i(x) \doteq (n^i(x))^T a(x) n^i(x)\gamma^i(x) - a(x)n^i(x), \tag{14}$$

and

$$K_i(x) \doteq \left\langle n^i(x), \nabla \cdot a(x)\right\rangle = \sum_{k=1}^{J} n_k^i(x) \sum_{j=1}^{J} \frac{\partial a_{kj}}{\partial x_j}(x);$$

3. for each $i, j \in \mathcal{I}$, $i \neq j$, and $x \in \partial G_i \cap \partial G_j \cap \partial G \setminus \mathcal{V}$,

$$p(x)\left(\langle q^i(x), n^j(x)\rangle + \langle q^j(x), n^i(x)\rangle\right) = 0. \tag{15}$$

Then the probability measure on $\bar{G}$ defined by

$$\pi(A) \doteq \frac{\int_A p(x)dx}{\int_{\bar{G}} p(x)dx}, \qquad A \in \mathcal{B}(\bar{G}), \tag{16}$$

is a stationary distribution for the well-posed submartingale problem.

Note that when $a(\cdot)$ is constant and equal to I, the $J \times J$ identity matrix, q^i in (14) represents the component of the reflection vector field γ^i that is tangential to ∂G.

PROOF OF THEOREM 3. By Theorems 1 and 2, it suffices to show that the probability measure π defined in terms of p via (16) satisfies the inequality (2) for all functions $f \in \mathcal{H} \cap \mathcal{C}_c^2(\bar{G})$. For any such function f, straightforward calculations show that for each $x \in \bar{G}$,

$$p(x)\mathcal{L}f(x) - f(x)\mathcal{L}^* p(x) = \frac{1}{2}\nabla \cdot r(x), \tag{17}$$

where $r(\cdot) = r^f(\cdot)$ is the vector field whose ith component is given by

$$\begin{aligned} r_i(x) &= \sum_{j=1}^{J} \left(p(x)a_{ij}(x)\frac{\partial f(x)}{\partial x_j} - f(x)a_{ij}(x)\frac{\partial p(x)}{\partial x_j} - f(x)p(x)\frac{\partial a_{ij}(x)}{\partial x_j} \right) \\ &\qquad +2b_i(x)f(x)p(x). \end{aligned}$$

Since, by assumption, $\mathcal{L}^* p(x) = 0$ for $x \in G$, and f has compact support and vanishes in a neighborhood of $\mathcal{V}$, the Divergence Theorem implies that

$$\begin{aligned} \int_{\bar{G}} p(x)\mathcal{L}f(x)dx &= \frac{1}{2}\int_{\bar{G}} \nabla \cdot r(x)\mu(dx) \\ &= \frac{1}{2}\int_{\partial G} \langle n(x), r(x)\rangle \mu(dx) \\ &= -\frac{1}{2}\sum_{i\in\mathcal{I}} \int_{\partial G_i \cap \partial G} \left\langle n^i(x), r(x)\right\rangle d\mu_i(x), \end{aligned} \tag{18}$$

where $n(\cdot)$ is the outward pointing unit normal field on ∂G, $\mu(dx)$ is the surface measure on ∂G, and $\mu_i(dx)$ is the surface measure on $\partial G \cap \partial G_i$ for each $i \in \mathcal{I}$. Now, for each $i \in \mathcal{I}$ and $x \in \partial G \cap \partial G_i$, we have

$$\begin{aligned} \left\langle n^i(x), r(x)\right\rangle &= p(x)(n^i(x))^T a(x)\nabla f(x) - f(x)(n^i(x))^T a(x)\nabla p(x) \\ &\qquad -f(x)p(x)K_i(x) + 2f(x)p(x)\left\langle n^i(x), b(x)\right\rangle. \end{aligned}$$

Since $f \in \mathcal{H}$, $g\nabla f$, $f\nabla g$ and fg vanish in a neighborhood of $\mathcal{V}$. Thus, combining the above display, (18) and relation (13) of the adjoint PDE, we obtain

$$\begin{aligned} \int_{\bar{G}} \mathcal{L}f(x)p(x)dx &= -\frac{1}{2}\sum_{i\in\mathcal{I}} \int_{\partial G_i\cap\partial G} p(x)(n^i(x))^T a(x)\nabla f(x)d\mu_i(x) \\ &\qquad +\frac{1}{2}\sum_{i\in\mathcal{I}} \int_{\partial G_i\cap\partial G} f(x)\nabla\cdot(p(x)q^i(x))d\mu_i(x). \end{aligned}$$

For each $i \in \mathcal{I}$ and $x \in \partial G_i \cap \partial G$, substituting for q^i from (14), we have

$$\begin{aligned} \nabla\cdot(f(x)p(x)q^i(x)) &= f(x)\nabla\cdot(p(x)q^i(x)) + \langle \gamma^i(x), \nabla f(x)\rangle p(x)(n^i(x))^T a(x)n^i(x) \\ &\qquad -p(x)(\nabla f(x))^T a(x)n^i(x). \end{aligned}$$

In turn, the last two equalities imply that $\int_{\bar{G}} \mathcal{L}f(x)p(x)dx$ is equal to

$$\begin{aligned} &\frac{1}{2}\sum_{i\in\mathcal{I}} \int_{\partial G_i\cap\partial G} \nabla\cdot(f(x)p(x)q^i(x))d\mu_i(x) \\ &\quad -\frac{1}{2}\sum_{i\in\mathcal{I}} \int_{\partial G_i\cap\partial G} \langle\gamma^i(x), \nabla f(x)\rangle p(x)(n^i(x))^T a(x)n^i(x)d\mu_i(x). \end{aligned}$$

The second term above is non-positive since $f \in \mathcal{H}$, $p \geq 0$ and a is positive semidefinite. So, we shall focus on the first term. For each $x \in \partial G_i \cap \partial G$, $\langle n^i(x), q^i(x)\rangle = 0$ because of the assumed normalization $\langle n^i(x), \gamma^i(x)\rangle = 1$. Therefore, the vector $q^i(x)$ is parallel to ∂G_i at x, and the divergence in the first term of the last display is equal to the divergence taken in the $(J-1)$-dimensional manifold $\partial G_i \cap \partial G$. Another application of the Divergence Theorem then yields

$$\begin{aligned}\sum_{i\in\mathcal{I}} \int_{\partial G_i \cap \partial G} & \nabla \cdot (f(x)p(x)q^i(x)) d\mu_i(x) \\ &= - \sum_{i,j\in\mathcal{I}, i\neq j} \int_{F_{ij}\setminus\mathcal{V}} f(x)p(x)q^i(x)\langle n^{ij}(x), q^i(x)\rangle d\mu_{ij}(x),\end{aligned}$$

where $F_{ij} \doteq \partial G_i \cap \partial G_j \cap \partial G$, $n^{ij}(x)$ denotes the unit vector that is normal to both F_{ij} and $n^i(x)$ at x and points into $\partial G_i \cap \mathcal{S}$ from F_{ij}, and $\mu_{ij}(dx)$ is the surface measure on the $(J-2)$-dimensional manifold F_{ij}. To prove the theorem, it suffices to show that the last equality in the above display is zero. To do this, it suffices to show that for each $i, j \in \mathcal{I}$ with $i \neq j$ and $x \in F_{ij} \setminus \mathcal{V}$,

$$p(x)\left(\left\langle n^{ij}(x), q^i(x)\right\rangle + \left\langle n^{ji}(x), q^j(x)\right\rangle\right) \;=\; 0. \tag{19}$$

Since $n^{ij}(x)$ is normal to $\partial G_i \cap \partial G_j$ at $x \in \partial G_i \cap \partial G_j$, it must lie in the two-dimensional space spanned by $n^i(x)$ and $n^j(x)$. In addition, $n^{ij}(x)$ is a unit vector normal to $n^i(x)$ and points into ∂G_i from $\partial G_i \cap \partial G_j$. Therefore, we have

$$n^{ij}(x) = (n^j(x) - \langle n^i(x), n^j(x)\rangle n^i(x))/(1 - \langle n^i(x), n^j(x)\rangle^2)^{1/2},$$

with the analogous expression for $n^{ji}(x)$. Since $\langle n^k(x), q^k(x)\rangle = 0$ for all $k \in \mathcal{I}$, this shows that (19) is equivalent to the third relation (15) of the adjoint PDE. This yields the desired result. $\square$

Solutions to the adjoint PDE have been identified for several classes of two-dimensional RBMs with constant drift (see [34], [22], [48], [40], [14] and also Examples 4.2 and 4.3), and for a class of multi-dimensional RBMs with constant drift satisfying a so-called skew-symmetry condition on the domain and covariance in [23]. Here, we consider the case of RBMs with state-dependent drifts, and investigate the analytical question of when the corresponding adjoint PDE has a strictly positive solution p, thus providing a generalization of some of the results in [23].

Corollary 1. *Given a pair $(G, d(\cdot))$ that is piecewise $\mathcal{C}^1$ with $\mathcal{C}^1$ reflection, a constant covariance matrix $a(\cdot) = A \in \mathbb{R}^{J\times J}$ and a drift vector field $b(\cdot) : \bar{G} \mapsto \mathbb{R}^J$, there exists a strictly positive $\mathcal{C}^2$ solution $p(\cdot)$ to the corresponding adjoint PDE if and only if both the relation*

$$\langle n^i(x), q^j(x)\rangle + \langle n^j(x), q^i(x)\rangle = 0, \quad x \in \partial G_i \cap \partial G_j \cap \partial G \setminus \mathcal{V}, \tag{20}$$

holds for every $i, j \in \mathcal{I}, i \neq j$, and there exists a $\mathcal{C}^2$ function $H : \bar{G} \mapsto \mathbb{R}$ that satisfies the following two properties:

1. *for each $x \in G$,*

$$\begin{aligned}\frac{1}{2}\nabla\cdot(A\nabla H(x)) + \frac{1}{2}\langle\nabla H(x), A\nabla H(x)\rangle - \nabla\cdot b(x)&\\ -\langle\nabla H(x), b(x)\rangle = 0;&\end{aligned} \tag{21}$$

2. *for each $i \in \mathcal{I}$ and $x \in \partial G_i \cap \mathcal{S}$,*

$$-2\langle n^i(x), b(x)\rangle + \langle An^i(x) - q^i(x), \nabla H(x)\rangle - \nabla\cdot(q^i(x)) = 0. \tag{22}$$

In this case, $p(x) = e^{H(x)}$, $x \in \bar{G}$, is a positive solution of the adjoint PDE.

Proof. First, note that for smooth domains, $|\mathcal{I}| = 1$ and so (20) is trivially satisfied whereas for non-smooth domains, relation (15) shows that (20) is necessary for the existence of a strictly positive solution p of the adjoint PDE. Elementary calculations show that a strictly positive $\mathcal{C}^2$ function p satisfies $\mathcal{L}^*p(x) = 0$ for $x \in G$ if and only if the $\mathcal{C}^2$ function $H = \ln p$ satisfies equation (21). Next, since A is constant implies $K_i(\cdot) = 0$, $i \in \mathcal{I}$, p satisfies relation (13) of the adjoint PDE if and only if $H = \ln p$ satisfies

$$-2\langle n^i(x), b(x)\rangle + \langle n^i(x), A\nabla H(x)\rangle - \langle q^i(x), \nabla H(x)\rangle - \nabla\cdot(q^i(x)) = 0. \tag{23}$$

But, since A is symmetric, this is equivalent to equation (22). □

We now specialize to two classes of domains that were considered in [23]. For simplicity, throughout, we assume that we are given a constant non-degenerate covariance matrix (i.e., positive definite, symmetric matrix) $a(\cdot) = \sigma(\cdot)\sigma^T(\cdot)$, which we denote by A. The first class consists of (possibly unbounded) polyhedral domains $(G, d(\cdot))$, where, for $i \in \mathcal{I}$, $n^i(\cdot)$ and $\gamma^i(\cdot)$, are both constant vector fields, which we denote simply by n^i and γ^i, respectively. Let q^i be defined as in (14), and let N and Q denote the $|\mathcal{I}| \times J$ matrices whose ith rows are $(n^i)^T$ and $(q^i)^T$, respectively. We assume that, for some $c \in \mathbb{R}^J$, $\bar{G} = \{x \in \mathbb{R}^J : Nx \geq c\}$ is the minimal half-space representation of the (closure of the) polyhedral domain, and that G has non-empty interior. Finally, we also assume $|\mathcal{I}| \geq J$ and that N contains an invertible submatrix $\overline{N}$, and let $\overline{Q}$ denote the corresponding submatrix obtained of Q. We further assume that the following global skew-symmetry condition holds:

$$Q^T = -\bar{N}^{-1}\bar{Q}N^T. \tag{24}$$

Since $\bar{N}$ and $\bar{Q}$ are corresponding submatrices of N and Q, (24) immediately implies that $\bar{N}^{-1}\bar{Q}^T$ is skew-symmetric and then by (4.7) of [23], $\bar{N}\bar{Q}^T$ is also skew-symmetric. Furthermore, (24) also shows that $\bar{N}^{-1}\bar{Q}^T$ is independent of the choice of the invertible submatrix $\bar{N}$.

The second class we will consider consists of smooth, bounded domains $(G, d(\cdot))$ that are $\mathcal{C}^{2+\varepsilon}$ with $\mathcal{C}^{1+\varepsilon}$ reflection. In this case, $|\mathcal{I}| = 1$ in Definition 3.3, we let $q(x) = q^1(x)$ be defined as in (14), and we omit the superscript 1 from the vector fields $n(\cdot)$, $\gamma(\cdot)$ and $q(\cdot)$. Choose a set of J points $\bar{x}_1, \ldots, \bar{x}_J$, on ∂G such that the normal vectors $n(\bar{x}_1), \ldots, n(\bar{x}_J)$, are linearly independent (such a set exists because G is bounded), and let $\bar{N}$ (respectively $\bar{Q}$) denote the $J \times J$ matrix whose ith row is the vector $(n(\bar{x}_i))^T$ [respectively $(q(\bar{x}_i))^T$]. We now consider the sub-class of domains for which $\bar{N}^{-1}\bar{Q}^T$ is skew-symmetric. Since q is $\mathcal{C}^{1+\varepsilon}$, it follows from Lemma 3.2 of [23] that this is equivalent to the condition that (24) is satisfied for any pair of corresponding $L \times J$ matrices N and Q formed in an analogous fashion from any set of L other points $x_1, \ldots, x_L \in \partial G$, which in turn is equivalent to the condition that $\langle n(x), q(\tilde{x})\rangle + \langle n(\tilde{x}), q(x)\rangle = 0$ for any $x, \tilde{x} \in G$. In order to present both classes of domains in a common framework, we phrase the global skew-symmetry condition as in (24) in terms of the pair $(\bar{N}, \bar{Q})$, noting that once again $\bar{N}^{-1}\bar{Q}^T$ does not depend on the particular choice of points $\bar{x}_1, \ldots, \bar{x}_J$ (as long as the normals are linearly independent). We refer the reader to [23] for further discussion of these classes of domains. We will use $b(\cdot)$ and the data $(\bar{N}, \bar{Q}), (N, Q)$, A, instead of $(G, d(\cdot))$, A to represent members of either of the two classes above, and always assume that the data satisfy all the stated conditions. We introduce the notion of a skew-transform, which plays an important role in the analysis.

Definition 3.5 (Skew-transform). *Given data $(\bar{N}, \bar{Q})$, (N, Q), A, the skew-transform (with respect to $(\bar{N}, \bar{Q})$, A) of a vector field $v(\cdot)$ on $\bar{G}$, is the vector field $u(\cdot)$ defined by*

$$u(x) = [A - \bar{N}^{-1}\bar{Q}]^{-1}v(x), \quad x \in \bar{G}. \tag{25}$$

Note that the matrix $A - \bar{N}^{-1}\bar{Q}$ is invertible and positive definite because A is positive definite and $\bar{N}^{-1}\bar{Q}$ is skew-symmetric. Hence, the skew-transform vector field $u(\cdot)$ is well defined. Let D be the "reflection" matrix whose ith row is given by γ^i, and let $\bar{D}$ be the corresponding submatrix of D corresponding to $\bar{N}$ (and $\bar{Q}$). From (14) it follows that

$$\bar{Q}^T = \mathrm{diag}(\bar{N}A\bar{N}^T)\bar{D}^T - A\bar{N}^T. \tag{26}$$

REMARK 3.6. We claim that the matrix $\bar{N}\bar{D}^T$ is positive definite; then $\bar{D}$ is invertible since $\bar{N}$ is invertible. In fact, it follows from (26) that

$$\bar{N}\bar{D}^T = \mathrm{diag}(\bar{N}A\bar{N}^T)^{-1}[\bar{N}\bar{Q}^T + \bar{N}A\bar{N}^T].$$

Since A is positive definite and $\bar{N}\bar{Q}^T$ is skew-symmetric, the claim follows.

If $u(\cdot)$ is the skew-transform of $v(\cdot)$, using the skew-symmetry of $\bar{N}^{-1}\bar{Q}$ and invertibility of $\bar{D}$, we have

$$u(\cdot) = [A + \bar{Q}^T(\bar{N}^T)^{-1}]^{-1}v(\cdot) = \bar{N}^T(\bar{D}^T)^{-1}\mathrm{diag}(\bar{N}A\bar{N}^T)^{-1}v(\cdot). \tag{27}$$

When $A = I$, this reduces to the simple form

$$u(\cdot) = \bar{N}^T(\bar{D}^T)^{-1}v(\cdot). \tag{28}$$

In what follows, recall that a vector field $u(\cdot)$ on $\bar{G}$ is said to be conservative if there exists a $\mathcal{C}^1$ function H on $\bar{G}$ such that $u(\cdot) = \nabla H$. In this case, H is said to be the potential of $u(\cdot)$.

COROLLARY 2. *Given data $(\bar{N}, \bar{Q}), (N, Q)$ and A, the following properties hold:*

1. *If $b(\cdot)$ is a $\mathcal{C}^1(\bar{G})$ vector field whose skew-transform is conservative with potential $H/2$, the function $p = e^H \in \mathcal{C}^2(\bar{G})$ is a strictly positive solution to the corresponding adjoint PDE;*
2. *Given $b(\cdot)$, H and p as in 1. above, define the following dual quantities:*

$$b_*(x) \doteq -b(x) + A\nabla H(x), \quad x \in G, \tag{29}$$

$$\bar{Q}_* \doteq -\bar{Q}, \quad \text{and} \quad Q_* = -\bar{N}^{-1}\bar{Q}_* N^T,$$

and define γ_*^i in terms of $q_*^i = Q_*^T e_i$ and n^i via (14). Then p is also a solution to the adjoint PDE associated with $(\bar{N}, \bar{Q}_*), (N, Q_*)$, A and $b_*(\cdot)$.

PROOF. Given the data, the global skew-symmetric condition (24) implies that (i) the pointwise skew symmetric condition (20) holds, (ii) $\nabla \cdot q^i = 0$ (this is trivially true for the polyhedral case and follows from Lemma 3.1 of [23] in the smooth case), and (iii) H and $b(\cdot)$ satisfy equation (22) if $-2\bar{N}b(x) + [\bar{N}A - \bar{Q}]\nabla H(x) = 0$ for $x \in \partial G$. (Note that the latter matrix equation is need not hold since b and ∇H are not constant, but it is sufficient). When $H/2$ is the potential of the skew-transform of $b(\cdot)$, using the identities $\nabla \cdot (C\nabla F) = 0$ and $\langle \nabla F, C\nabla F\rangle = 0$ for any $\mathcal{C}^2$ function F and skew-symmetric matrix C, it is easily verified that (22) and (21) are satisfied.

On the other hand, note that for $x \in G$,

$$b_*(x) = -\frac{1}{2}[A - \bar{N}^{-1}\bar{Q}]\nabla H(x) + A\nabla H(x) = \frac{1}{2}[A - \bar{N}^{-1}\bar{Q}_*]\nabla H(x).$$

Since $\bar{Q}_* = -\bar{Q}$, it follows that $\bar{N}^{-1}\bar{Q}_*$ is also skew-symmetric. Thus, $H/2$ is also the potential of the skew-transform of b_* [with respect to $(\bar{N}, \bar{Q}_*)$, A], and the argument used in 1) shows that b_* and H also satisfy the corresponding equations (22) and (21). The result then follows from Corollary 1. $\square$

REMARK 3.7. We now address well-posedness of the submartingale problem. In smooth bounded domains, the normalization (11) implies $\inf_{x\in\partial G}\langle n(x), \gamma(x)\rangle > 0$, and thus well-posedness follows from the discussion in Example 4.1. For polyhedral domains we claim that the submartingale problem is well posed under the skew-symmetry condition (24), Indeed, by Lemma 3.1.3 of [9], the positive definiteness of $\bar{N}\bar{D}^T$ established in Remark 3.6 implies that there exists $v > 0$ such that $\bar{N}\bar{D}^T v > 0$. This shows that the so-called completely-$\mathcal{S}$ condition is satisfied, and it follows from [45] and [11] that there exists a weak solution that is unique in law for a large class of polyhedral domains including, in particular, simple polyhedra. When combined with the results of [28], it follows that the submartingale problem is well posed. For a domain in this class, by Theorem 3 the solution p to the adjoint PDE identified in Corollary 2 is in fact an invariant density for the associated reflected diffusion and, when $C = \int_{\bar{G}} p(x)dx$ is finite (which is always true when G is smooth and bounded), $C^{-1}p(x)dx$ is in fact the stationary distribution. Given data associated with smooth and bounded domains that satisfy the conditions of Corollary 2, let X be the associated reflected diffusion, and let X_* be the reflected diffusion associated with the dual data. Since Corollary 2 shows that the (common) stationary distribution $C^{-1}p(x)dx$, is strictly positive, it follows from [32, 33] that X and X_* are dual to each other with respect to the stationary distribution. When the data is associated with a simple polyhedron and $C < \infty$, the duality property in the case of constant drifts was established in Corollary 1.1 of [49]. Similar arguments can be used to extend to the case of state-dependent drift, but we do not provide the details here.

We conclude this section with illustrative examples of domains and drifts when H (and therefore p) takes an explicitly computable form. Here, we will repeatedly use the well known property that (since G is a simply connected domain) a necessary and sufficient condition for a $\mathcal{C}^1$ vector field to be conservative is that its Jacobian is symmetric. We will assume throughout that the data satisfies the global skew-symmetry condition (24), unless explicitly stated otherwise.

Example 3.8. When $\bar{Q} = 0$ (i.e., normal reflection when $A = I$), it follows from Corollary 2 that if the vector field $A^{-1}b(\cdot)$ is conservative with potential $H \in \mathcal{C}^2(\bar{G})$, then $p = e^H$ satisfies the adjoint PDE.

Example 3.9. The simplest generalization of the constant drift vector fields considered in [23] is the case when the drift points along a constant direction, but has varying magnitude. In other words, $b(x) = F(x)v$ for some $v \in \mathbb{R}^{J\times J}$ and $\mathcal{C}^1$ function $F : \bar{G} \mapsto \mathbb{R}$. Let u be the skew-transform of v. By Corollary 2, for the adjoint PDE to have a strictly positive solution p, it suffices for $F(\cdot)u$ to be conservative, which holds if and only if the Jacobian of $F(\cdot)u$ is symmetric. The latter implies that ∇F is parallel to u, and so there exists a $\mathcal{C}^1$ function $r : \mathbb{R} \mapsto \mathbb{R}$ such that $F(x) = r(\langle x, u\rangle)$; for example, one can fix $x_0 \in \bar{G}$ and use a path integration argument to define $r(t) = F(x_0) + ||u||(t - \langle x_0, u\rangle) \int_0^1 ||\nabla F(x_0 + \lambda(t - \langle x_0, u\rangle)u)||\, d\lambda$. If we define $R(t) = \int_0^t r(s)ds$, then clearly $H(x) = R(\langle x, u\rangle)$ is a potential for $F(x)u$. Note that the case $R(x) = x$ corresponds to the case of constant drift. When combined with Corollary 2, this recovers the statements "(ii) implies (i)" in Theorems 2.1 and 6.1 of [23].

Example 3.10. We now study the case of a linear drift vector field, which includes reflected Ornstein-Uhlenbeck. We establish two claims.
Claim 1: If $b(x) = Cx$ for some $C \in \mathbb{R}^{J\times J}$ such that $C_* = [A - \bar{N}^{-1}\bar{Q}]^{-1}C$ is symmetric, then $p = e^{x^T C_* x}$ solves the corresponding adjoint PDE.
Proof of Claim 1: Suppose $b(x) = Cx$, for $C \in \mathbb{R}^{J\times J}$ as in the claim. Then the skew-transform of $b(\cdot)$ is the vector field $u(x) = C_* x$, whose Jacobian is C_*, and hence symmetric. Thus, $u(\cdot)$ is conservative with potential $H/2$, where $H(x) = x^T C_* x$. The claim follows from Corollary 2. In particular, when $A = \bar{N} = I$, it follows from (28) that any drift of the form $b(x) =$

$\bar{D}^T Bx$ for some symmetric matrix $B \in \mathbb{R}^{J\times J}$ has an invariant density of Gibbs form.

We now prove a converse to claim 1. For simplicity, we consider the non-negative orthant (it also holds for any simple polyhedral domain by a change of coordinates). In this setting $\bar{N} = N = I$ and $\bar{Q} = Q$, but we continue to use the bar notations for convenience. Here we do *not* assume *a priori* that the data satisfies the skew-symmetry condition.

Claim 2: Given $\bar{Q} \in \mathbb{R}^{J\times J}$ that has zero on the diagonal, and a nondegenerate covariance matrix $A \in \mathbb{R}^{J\times J}$, suppose there exists an invertible symmetric matrix C_* such that the $\mathcal{C}^2$ function $p(x) = e^{x^T C_* x}$ solves the adjoint PDE associated with $(I, \bar{Q})$, A, and $b(x) = [A - \bar{Q}]C_* x$. Then $\bar{Q}$ must be skew-symmetric.

Proof of Claim 2: Indeed, suppose that such an invertible symmetric matrix C_* exists. Then, by Corollary 1, equation (21) should be satisfied by $H(x) = x^T C_* x$ and $b(x) = [A - \bar{Q}]C_* x$. In other words, it follows that for each $x \in G$,

$$\nabla \cdot (AC_* x) + \langle C_* x, AC_* x\rangle - \nabla \cdot ([A - \bar{Q}]C_* x) - \langle C_* x, [A - \bar{Q}]C_* x\rangle = 0,$$

which, since $\nabla \cdot (\bar{Q}C_* x) = tr(\bar{Q}C_*)$, is equivalent to

$$tr(\bar{Q}C_*) + \langle C_* x, \bar{Q}C_* x\rangle = 0.$$

Now, fix $y \in G$. Then for any $x \in \mathbb{R}^J$, for all ϵ small enough, we have $y + \epsilon x \in G$. Substituting y and $y + \epsilon x$ into the above display and taking the difference, we have

$$\epsilon\langle C_* y, \bar{Q}C_* x\rangle + \epsilon\langle C_* x, \bar{Q}C_* y\rangle + \epsilon^2\langle C_* x, \bar{Q}C_* x\rangle = 0. \tag{30}$$

Dividing the above display by ϵ and taking the limit as $\epsilon \to 0$, we have

$$\langle C_* y, \bar{Q}C_* x\rangle + \langle C_* x, \bar{Q}C_* y\rangle = 0.$$

Substituting this back into (30), we have shown that for every $x \in \mathbb{R}^J$,

$$\langle C_* x, (\bar{N})^{-1}\bar{Q}C_* x\rangle = 0.$$

Since C_* is invertible, this shows that $\bar{Q}$ is skew symmetric.

Example 3.11. We note that solutions to the equations (21) and (22) are preserved under linear combinations. More precisely, suppose for $m \in \mathbb{N}$, each pair (b_i, H_i), $i = 1, \ldots, m$, satisfies the pair of equations (21) and (22). Then, for any $\lambda_i \in \mathbb{R}$, $i = 1, 2, \cdots, m$, the pair $(\sum_{i=1}^m \lambda_i b_i, \sum_{i=1}^m H_i)$ also satisfy the same pair of equations. Thus, taken together, the above examples identify a general class of drift vector fields whose adjoint has a solution of Gibbs form with potential being a quadratic form. For example, if $(\bar{N}, \bar{Q}), A$ satisfies the global skew-symmetric condition (24) and the Jacobian of the skew-transform of the drift $b(\cdot)$ is a constant symmetric matrix, that is, $[A - \bar{N}^{-1}\bar{Q}]^{-1}b(x) = C_* x + \mu_*$ for some symmetric matrix C_* and some constant vector μ_*. Then $b(x) = Cx + \mu$, where $C = [A - \bar{N}^{-1}\bar{Q}]C_*$ and $\mu = [A - \bar{N}^{-1}\bar{Q}]\mu_*$. By combining Examples 3.9 and 3.10, the adjoint PDE with drift $b(x) = Cx + \mu$ has a strictly positive solution $p(x) = e^{H(x)}$, where $H(x) = x^T C_* x + \langle \mu_*, x\rangle$.

4. Examples. In this section, we provide several examples of reflected diffusions in piecewise $\mathcal{C}^1$ domains with continuous reflection for which the submartingale problem is well posed and Assumption 2 is satisfied, so that Theorems 1 and 2 provide a characterization of their stationary distributions. The examples serve to illustrate the range of applicability of the results of the paper. The first and fourth examples consider families of semimartingale reflected diffusions, whereas the remaining examples describe reflected diffusions that could fail to be semimartingales. The last example involves a cusp-like domain that was specifically identified in [47] as a two-dimensional example not covered by the methods therein. To the best of our knowledge, prior to this work, there existed no characterization of the stationary distribution of the processes described in Examples 4.3, 4.5, 4.6 and 4.7.

EXAMPLE 4.1 (Reflected diffusions in smooth domains). We start with the simple case of reflected diffusions in smooth domains addressed in [47]. Let G be a bounded open set in $\mathbb{R}^J$ such that $G = \{x \in \mathbb{R}^J : \ \phi(x) > 0\}$, where $\phi \in \mathcal{C}_b^2(\mathbb{R}^J)$ and $|\nabla\phi| \geq 1$ on ∂G. Then $\nabla\phi(x)$ is an inward normal vector at $x \in \partial G$. Let $\gamma(\cdot)$ be a bounded Lipschitz continuous vector field that satisfies $\langle \nabla\phi(x), \gamma(x)\rangle > 0$ on ∂G. By [43] (see Theorems 3.1 and 5.4 therein) the associated submartingale problem with $\mathcal{L}$ as in (1) and $\mathcal{V} = \emptyset$ is well posed. Now, $(G, d(\cdot))$ is a $\mathcal{C}^1$ domain with continuous reflection and, since $\mathcal{U} = \partial G$, Assumption 2 is trivially satisfied with $\mathcal{V} = \emptyset$.

EXAMPLE 4.2 (RBM in a 2-dimensional wedge). Consider a wedge $G \subset \mathbb{R}^2$ given in polar coordinates by

$$G = \{(r, \theta) : \ 0 \leq \theta \leq \zeta, \ r \geq 0\},$$

where $\zeta \in (0, \pi)$ is the angle of the wedge. Then G admits the representation

$G = G_1 \cap G_2$, where G_1 and G_2 are the two half planes

$$\begin{aligned} G_1 &= \{(r,\theta):\ 0 \leq \theta \leq \pi,\ r \geq 0\}, \\ G_2 &= \{(r,\theta):\ \zeta - \pi \leq \theta \leq \zeta,\ r \geq 0\}, \end{aligned}$$

whose unit inward normals we denote by n^1 and n^2, respectively. Let the directions of reflection on ∂G_1 and ∂G_2 be specified as constant vectors γ^1 and γ^2, normalized such that for $j = 1, 2$, $\langle \gamma^j, n^j \rangle = 1$. For $j = 1, 2$, define the angle of reflection θ_j to be the angle between n^j and γ^j, such that θ_j is positive if and only if γ^j points towards the origin. Note that $-\pi/2 < \theta_j < \pi/2$. Define $\alpha = (\theta_1 + \theta_2)/\zeta$. It was proved in Theorem 3.10 of [46] that the submartingale problem with $\mathcal{L} = \frac{1}{2}\Delta$ and $\mathcal{V} = \{0\}$ is well posed if and only if $\alpha < 2$. Since $\partial G \setminus \mathcal{U} \subseteq \mathcal{V}$ and $\mathcal{V}$ contains only one element, Assumption 2 holds. Note that when $\alpha \in [1, 2)$, the RBM is not a semimartingale.

Example 4.3 (RBM in a 2-dimensional polygon). Consider a two dimensional polygon $G \subset \mathbb{R}^2$ with vertices $a_1, \ldots, a_K$ (in counterclockwise order). For $k = 1, \ldots, K-1$, define *side* k as the open line segment between a_k and a_{k+1}. Similarly, side K is the line segment between a_K and a_1, excluding the endpoints. Let ξ_k denote the interior angle made by the two sides meeting at vertex a_k. Also given are angles $\theta_1, \ldots, \theta_K$ satisfying $|\theta_k| < \pi/2$ which will determine the directions of reflection on each side. For each $k = 1, \ldots, K$, θ_k is the angle between the inward normal n^k and the constant direction of reflection γ^k associated with side k and θ_k is positive if and only if γ^k points towards the vertex a_{k+1}. It was established in Theorem 3.7 of [22] that when $\theta_{k-1} < \theta_k + 2\xi_k$ for all $k = 1, \ldots, K$, the submartingale problem with $\mathcal{L} = \frac{1}{2}\Delta$ and $\mathcal{V} = \{a_1, \ldots, a_k\}$ is well posed. Note that at each vertex a_k, there exists a unit vector v_{a_k} such that $\langle v_{a_k}, \gamma^{k-1} \rangle \geq 0$ and $\langle v_{a_k}, \gamma^k \rangle \geq 0$. In addition, $\partial G \setminus \mathcal{U} \subseteq \mathcal{V}$. Thus, Assumption 2 holds. Note that if there exists k such that $(\theta_{k-1} - \theta_k)/\xi_k \in [1, 2)$, then the associated RBM is not a semimartingale. A subclass of these RBMs, which arise as diffusion approximations of closed networks, were also investigated in [40].

Example 4.4 (SRBM in polyhedral domains). We now describe a class of semimartingale RBMs (SRBMs for short) that arise as diffusion approximations of queueing networks [50]. In this case, $G = \mathbb{R}^J_+$ is the nonnegative orthant in $\mathbb{R}^J$, which admits the representation $G = \bigcap_{i=1}^J G_i$, where $G_i \doteq \{x \in \mathbb{R}^J : x_i \geq 0\}$, and the direction vector field γ^i on G_i is a constant vector field, pointing in a direction $d^i \in \mathbb{R}^J$. Moreover, the matrix D with column d^i is assumed to satisfy the completely-$\mathcal{S}$ condition, which implies that $\mathcal{U} = \partial G$. It was shown in [45] that the reflected Brownian motion associated with G and $d(\cdot)$ admits a weak solution that is unique in law. Therefore, by Theorem 2 of [28] it follows that the submartingale problem with $\mathcal{L} = \frac{1}{2}\Delta$ and $\mathcal{V} = \emptyset$ is well posed. By Remark 3.4, Assumption 2 is trivially satisfied with $\mathcal{V} = \emptyset$.

EXAMPLE 4.5 (Nonsemimartingale reflected diffusions in polyhedral domains). Here, we first consider a class of RBMs that were shown in [16, 17, 18, 36, 37] to arise as reflected diffusion approximations of multiclass queueing networks using the so-called generalized processor scheduling policy that is used in high-speed networks for efficient sharing of resources amongst traffic of different classes. The state space G associated with the GPS ESP has the representation

$$G = \bigcap_{i=1}^{J+1} \{x \in \mathbb{R}^J : \ \langle x, n^i \rangle > 0\},$$

where $n^i = e_i$ for $i = 1, \dots, J$ (here $\{e_i, i = 1, \dots, J\}$ is the standard orthonormal basis in $\mathbb{R}^J$) and $n_{J+1} = \sum_{i=1}^J e_i/\sqrt{J}$. The reflection vector field is piecewise constant on each face, governed by the vectors $\{\gamma^i, \ i = 1, ..., J+1\}$ that are defined as follows: $\gamma^{J+1} = \sum_{i=1}^J e_i/\sqrt{J}$ and $\{\gamma^i, i = 1, \dots, J\}$ are defined in terms of a "weight" vector $\bar{\alpha} \in \mathbb{R}_+^J$ that satisfies $\bar{\alpha}_i > 0$ for each $i = 1, ..., J$ and $\sum_{i=1}^J \bar{\alpha}_i = 1$: for $i, j = 1, \dots, J$,

$$\gamma_j^i = \begin{cases} -\dfrac{\bar{\alpha}_j}{1 - \bar{\alpha}_i} & \text{for } j \neq i, \\ \hfill 1 & \text{for } j = i. \end{cases}$$

The fact that the associated stochastic differential equation with reflection has a pathwise unique solution follows from Corollary 4.4 of [35]. Hence, well-posedness of the submartingale problem with $\mathcal{L}$ as in (1) and $\mathcal{V} = \{0\}$ follows from Theorem 2 of [28]. Moreover, Lemma 3.4 of [35] shows that $\partial G \setminus \mathcal{U} = \mathcal{V} = \{0\}$ and $\mathcal{V}$ only contains one element. Hence, Assumption 2 holds. It was shown in [26] that this process is not a semimartingale. The two-dimensional case corresponds to the case $\alpha = 1$ and $\zeta = \pi/2$ in Example 4.2.

EXAMPLE 4.6 (Nonsemimartingale RBMs in curved domains). We now consider a class of reflected diffusions in curved domains introduced by Burdzy and Toby in [7]. Suppose that L and R are twice continuously differentiable real functions defined on $\mathbb{R}$ and such that $L(0) = R(0) = 0$ and $L(y) < R(y)$ for all $y > 0$. The domain has the form $G = G_1 \cap G_2 \cap G_3$, where

$$G_1 = \{(x,y) : x > L(y)\}, \quad G_2 = \{(x,y) : \ x < R(y)\},$$

$$G_3 = \{(x,y) : \ y > 0, x \in \mathbb{R}\}.$$

For $j = 1,2,3$ and $z \in \partial G_j$, let $n^j(z)$ denote the unit inward normal vector to ∂G_j at z. Let $\gamma^1(\cdot) = (1,0)'$, $\gamma^2(\cdot) = (-1,0)'$ and $\gamma^3(\cdot) = (0,1)'$. In [7] the RBM in such a domain was characterized as the pathwise unique strong solution to the associated stochastic differential equations with reflection. Using techniques similar to Theorem 2 of [28], it can be shown that the associated submartingale problem with $\mathcal{L} = \frac{1}{2}\Delta$ and $\mathcal{V} = \{0\}$ is well posed, and Proposition 4.13 of [6] shows that the process is not a semimartingale. Since $\mathcal{U} = \partial G \setminus \mathcal{V}$, and $\mathcal{V}$ is a singleton, Assumption 2 is trivially satisfied.

Example 4.7 (RBMs in Cusp-like domains). Consider a two-dimensional domain G with representation

$$G = \{(x,y) : \ x \geq 0, \ -x^\beta < y < x^\beta\}, \qquad \beta > 1.$$

The domain G has a cusp at the origin and $G = G_1 \cap G_2$, where

$$\begin{aligned} G_1 &= \{(x,y) : \ y < x^\beta \text{ when } x \geq 0 \text{ and } y < 0 \text{ when } x < 0\}, \\ G_2 &= \{(x,y) : \ y > -x^\beta \text{ when } x \geq 0 \text{ and } y > 0 \text{ when } x < 0\}. \end{aligned}$$

For each $j = 1,2$, and $z \in \partial G_j$, let $n^j(z)$ be the inward unit normal vector to ∂G_j and let $\gamma^j(z)$ make a constant angle $\theta_j \in (-\pi/2, \pi/2)$ with $n^j(z)$. We take $\theta_j > 0$ if and only if the first component of $\gamma^j(z)$ is negative, that is, $\gamma^j(z)$ points towards the origin for z in a small neighborhood of the origin. Since $\theta_j \neq \pm\pi/2$, we can without loss of generality assume the normalization $\langle \gamma^j(z), n^j(z) \rangle = 1$ holds. It was proved in [12] that the submartingale problem with $\mathcal{V} = \{0\}$ is well posed when $\theta_1 + \theta_2 \leq 0$. It is easy to check that $\partial G \setminus \mathcal{U} \subseteq \mathcal{V}$ and $\mathcal{V}$ contains only one element, and thus Assumption 2 holds.

5. Sufficiency of the Inequality Condition. Throughout this section, assume $(G, d(\cdot))$, $b(\cdot)$, $\sigma(\cdot)$ and a finite set $\mathcal{V} \subset \partial G$ are associated with a well-posed submartingale problem. Let π be a probability measure on $(\bar{G}, \mathcal{B}(\bar{G}))$ such that $\pi(\partial G) = 0$. In this section we show that if π also satisfies (2) for every $f \in \mathcal{H}$, then π is a stationary distribution for the well-posed submartingale problem. The proof consists of three main steps. First, in Section 5.1 (see Proposition 5.1) we show that the inequality (2) is equivalent to a certain generalized basic adjoint relation (BAR). Next, in Section 5.2, we use the generalized BAR to deduce the existence of a stationary process X that has marginals equal to π and satisfies some additional properties. We complete the proof in Section 5.3 by showing that the law of X is equal to $\mathbb{Q}_\pi$, the solution to the well-posed submartingale problem with initial distribution π.

5.1. *A Generalized Basic Adjoint Relation.* In what follows, let

$$\mathcal{K}_1 \doteq \{(x,u) \in \mathbb{R}^{2J} : \ x \in \partial G \setminus \mathcal{V}, \ u \in d(x), \ |u| = 1\}. \tag{31}$$

PROPOSITION 5.1. *Let $(G, d(\cdot)), b(\cdot), \sigma(\cdot)$ and a finite set $\mathcal{V}$ be associated with a well-posed submartingale problem and suppose that the associated set $\mathcal{H}$ defined in (3) satisfies Assumption 1. Given any probability measure π on $\bar{G}$ that satisfies $\pi(\partial G) = 0$, π satisfies the inequality (2) if and only if there exists a σ-finite (nonnegative) Borel measure μ on $\mathcal{K}_1$ such that*

$$\int_{\bar{G}} \mathcal{L} f(x) \pi(dx) + \int_{\mathcal{K}_1} \langle u, \nabla f(x) \rangle \mu(dx, du) = 0 \textit{ for each } f \in \mathcal{H}. \tag{32}$$

PROOF. The fact that (32) implies (2) is immediate because μ is a nonnegative measure, and $f \in \mathcal{H}$ implies $\langle u, \nabla f(x) \rangle \geq 0$ for $(x,u) \in \mathcal{K}_1$.

We now prove the converse. Suppose π satisfies (2) and let $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$, where $\mathcal{K}_1$ is defined in (31) and

$$\mathcal{K}_2 \doteq \{(x,u) \in \mathbb{R}^{2J} : \ x \in G, \ |u| = 1\}.$$

For each $f \in \mathcal{H}$, let $h_f : \mathcal{K} \mapsto \mathbb{R}$ be the function given by $h_f(x,u) = \langle u, \nabla f(x) \rangle$ for each $(x,u) \in \mathcal{K}$. Clearly, $h_f \in \mathcal{C}^1_c(\mathcal{K})$ for each $f \in \mathcal{H}$. Let $\mathcal{T}_0$ be the linear subspace of $\mathcal{C}_c(\mathcal{K})$ given by

$$\mathcal{T}_0 \doteq \left\{ g \in \mathcal{C}^1_c(\mathcal{K}) : g = \sum_{i=1}^n a_i h_{f_i}, \ n \in \mathbb{N}, f_i \in \mathcal{H}, \ a_i \in \mathbb{R}, i = 1, \dots, n \right\},$$

and for each $g \in \mathcal{T}_0$ that has a representation of the form $g = \sum_{i=1}^n a_i h_{f_i}$, define

$$\Lambda(g) \doteq - \int_{\bar{G}} \mathcal{L} \left(\sum_{i=1}^n a_i f_i \right)(x) \pi(dx). \tag{33}$$

We now show that the value of $\Lambda(g)$ does not depend on the chosen representation for g. Suppose we are given two representations of $g \in \mathcal{T}_0$ with

$g = \sum_{i=1}^n a_i h_{f_i} = \sum_{j=1}^m b_j h_{\tilde{f}_j}$. Then, by the definition of h_f,

$$\left\langle u, \nabla \left(\sum_{i=1}^n a_i f_i \right)(x) \right\rangle = \left\langle u, \nabla \left(\sum_{j=1}^m b_j \tilde{f}_j \right)(x) \right\rangle, \quad x \in G,\ |u| = 1.$$

This implies that $\nabla(\sum_{i=1}^n a_i f_i)(x) = \nabla(\sum_{j=1}^m b_j \tilde{f}_j)(x)$ for any $x \in G$, and hence $\mathcal{L}(\sum_{i=1}^n a_i f_i)(x) = \mathcal{L}(\sum_{j=1}^m b_j \tilde{f}_j)(x)$ for any $x \in G$. Since $\pi(\partial G) = 0$, the right-hand sides of (33) for the two representations coincide. Thus, Λ is well defined.

We show below that Λ is in fact a positive linear functional on $\mathcal{T}_0$ with respect to a suitable partial order. Linearity of Λ trivially follows from the definition. Let

$$\mathcal{P} = \{g \in \mathcal{C}_c(\mathcal{K}) :\ 0 \le g(x,u) \le h_f(x,u), (x,u) \in \mathcal{K}_1 \text{ for some } f \in \mathcal{H}\}.$$

Since the mapping from f to h_f is linear by the definition of h_f, it is easy to verify that (1) $g, \tilde{g} \in \mathcal{P}$ implies $g + \tilde{g} \in \mathcal{P}$; and (2) $g \in \mathcal{P}$ and $a > 0$ implies $ag \in \mathcal{P}$. Thus, $\mathcal{P}$ is a positive cone in $\mathcal{C}_c(\mathcal{K})$. On $\mathcal{C}_c(\mathcal{K})$, we consider the partial order $\le$ defined by $h \le g$ if $g - h \in \mathcal{P}$. To show that Λ is positive on $\mathcal{T}_0$, let $g \in \mathcal{T}_0 \cap \mathcal{P}$. Since $g \in \mathcal{T}_0$, it admits a representation of the form $g = \sum_{i=1}^n a_i h_{f_i}$ for some $a_i \in \mathbb{R}$, $f_i \in \mathcal{H}$. Since each $f_i \in \mathcal{H}$, clearly $\sum_{i=1}^n a_i f_i \in \mathcal{C}_c^2(\bar{G}) \oplus \mathbb{R}$. Moreover, $g \in \mathcal{P}$ implies $g \ge 0$, which in turn implies $\langle u, \sum_{i=1}^n a_i \nabla f_i(x) \rangle \ge 0$ for each $x \in \partial G \setminus \mathcal{V}$ and $u \in d(x)$ with $|u| = 1$. As a consequence, $\sum_{i=1}^n a_i f_i \in \mathcal{H}$. By (2) and (33), this implies that $\Lambda(g) \ge 0$ which shows that Λ is positive.

We now verify an additional condition that will allow us to apply a version of the Hahn-Banach theorem.

Claim 1. For each $h \in \mathcal{C}_c(\mathcal{K})$, there exists $g \in \mathcal{T}_0$ such that $g - h \in \mathcal{P}$.

Proof of Claim 1. Fix $h \in \mathcal{C}_c(\mathcal{K})$. Then there exists a compact set $K \subset \mathcal{K}_1$ and a constant $0 < C < \infty$ such that $|h(x,u)| \le C\mathbb{I}_K(x,u)$ for each $(x,u) \in \mathcal{K}_1$. We can assume without loss of generality that there exist $r, s > 0$ such that

$$\{x \in \mathbb{R}^J : (x,u) \in K\} \subseteq \{x \in \partial G : |x| \le r, d(x, \mathcal{V}) \ge s\}. \tag{34}$$

Let $f = Cf_{r,s}$, where $f_{r,s} \in \mathcal{H} \cap \mathcal{C}_c^2(\bar{G})$ is the function from property 2 of Assumption 1. Then clearly $|h(x,u)| \le C\mathbb{I}_K(x,u) \le h_f(x,u) = \langle u, \nabla f(x) \rangle$ for each $(x,u) \in \mathcal{K}_1$. Choose $g = h_f$. Then $g \in \mathcal{T}_0$, $0 \le g - h \le 2h_f = h_{2f}$ on $\mathcal{K}_1$, and $2f \in \mathcal{H}$. Thus, $g - h \in \mathcal{P}$. This establishes the claim.

By the claim and an application of the positive cone version of the Hahn-Banach theorem (cf. Theorem 2.1 of [3]), Λ can be extended to a positive linear functional on $\mathcal{C}_c(\mathcal{K})$, which we denote again by Λ. An application of the Riesz representation theorem then shows that there exists a unique regular Borel measure μ on $\mathcal{K}$ such that

$$\Lambda(g) = \int_{\mathcal{K}} g(x,u)\mu(dx,du) \text{ for each } g \in \mathcal{C}_c(\mathcal{K}). \tag{35}$$

Now, for each $g \in \mathcal{C}_c(\mathcal{K}_2)$, both g and $-g$ are identically zero on $\mathcal{K}_1$ and hence lie in $\mathcal{P}$. Therefore, $\Lambda(g) = 0$ for every $g \in \mathcal{C}_c(\mathcal{K}_2)$, which in turn implies $\mu(\mathcal{K}_2) = 0$. Now, for $f \in \mathcal{H}$, substituting $g = h_f \in \mathcal{T}_0$ into both the definition (33) and the representation (35) of Λ, and using the fact that $\mu(\mathcal{K}_2) = 0$, we obtain (32).

To see that μ is a sigma-finite measure, fix any constant, say $C = 1$, and a compact subset $K \subset \mathcal{K}_1$. Let $r, s > 0$ be such that (34) holds, and let $f = f_{r,s} \in \mathcal{H}$ be the function from Assumption 1(2). Then $h_f \in \mathcal{T}_0$, and substituting f in (32), we obtain

$$\mu(K) < \int_{\mathcal{K}_1} h_f(x,u)\mu(dx,du) = -\int_{\bar{G}} \mathcal{L}f(x)du < \infty,$$

where finiteness of the last integral holds because $\mathcal{L}f$ is continuous and has compact support in $\bar{G}$. □

REMARK 5.2. When $\mathcal{V} \neq \emptyset$ the condition in (32), which we will refer to as the generalized BAR, is somewhat more subtle than the usual BAR that has been established for semimartingale RBMs in the orthant [23]. In the latter setting, the measure μ in the BAR is a finite measure that is absolutely continuous with respect to the local time measure dL associated with the RBM Z on the boundary ∂G, and takes the form

$$\mu(A) = \mathbb{E}_\pi \left[\int_0^1 \mathbb{I}_{\{Z(u) \in A\}} dL(u) \right], \qquad A \in \mathcal{B}(\mathbb{R}^d).$$

However, when $\mathcal{V} \neq \emptyset$, the local time is typically not of bounded variation [26, 35], and so the local time does not define a finite measure on the boundary. However, since increments of the local time when the process is away from the set $\mathcal{V}$ can be shown to be of bounded variation [35], one can still associate a σ-finite measure μ which is, roughly speaking, associated with the local time of excursions of the reflected diffusion away from the set $\mathcal{V}$. Since functions in $\mathcal{H}$ are constant in a neighborhood of $\mathcal{V}$, the condition (32) is satisfied. This emphasizes the subtlety in the correct choice of test functions for characterization of the stationary distribution.

5.2. *Existence of a Stationary Process.* We now establish a corollary of the generalized BAR, the proof of which relies on the following approximation lemma. For $f \in \mathcal{H}$, the limit $\lim_{|x|\to\infty} f(x)$ clearly exists, and in what follows, we denote it by $f(\infty)$.

LEMMA 5.3. *The set $\mathcal{H}$ has a countable subset $\mathcal{H}_0$ with the property that for each $f \in \mathcal{H}$ and each $N \in \mathbb{N}$ such that $B_N(0)$ contains both an open neighborhood of $\mathcal{V}$ and an open neighborhood of $\mathrm{supp}(f - f(\infty))$, there exists a sequence $\{g_k : \ k \in \mathbb{N}\} \subset \mathcal{H}_0$ such that*

(36)
$$\lim_{k\to\infty} \sup_{x\in\bar{G}\cap B_N(0)} \max_{i,j=1}^{J} |f(x)-g_k(x)| \vee \left|\frac{\partial f(x)}{\partial x_i} - \frac{\partial g_k(x)}{\partial x_i}\right| \vee \left|\frac{\partial^2 f(x)}{\partial x_i \partial x_j} - \frac{\partial^2 g_k(x)}{\partial x_i \partial x_j}\right| = 0.$$

Moreover, the above property also holds with the set $\mathcal{D} \doteq \{f \in \mathcal{H} : \ f \geq 0\}$ in place of $\mathcal{H}$ and a countable subset $\mathcal{D}_0$ in place of $\mathcal{H}_0$.

The proof of Lemma 5.3 relies on the denseness of polynomials in $\mathcal{C}^2(\mathbb{R}^J)$ and standard mollification arguments, and hence is deferred to Appendix A.

COROLLARY 3. *Let the conditions of Proposition 5.1 be satisfied. Then there exists a stationary process X whose law $\tilde{\mathbb{Q}}_\pi$ on $(\mathcal{C}\,[0,\infty), \mathcal{M})$, satisfies the following properties:*

1. *The law of $\omega(0)$ under $\tilde{\mathbb{Q}}_\pi$ is π;*
2. *For every $f \in \mathcal{H}$, the process*
$$f(\omega(t)) - \int_0^t \mathcal{L} f(\omega(u))\, du, \quad t \geq 0,$$
is a $\tilde{\mathbb{Q}}_\pi$-submartingale on $(\mathcal{C}\,[0,\infty), \mathcal{M}, \{\mathcal{M}_t\})$;
3. $$\mathbb{E}^{\tilde{\mathbb{Q}}_\pi}\left[\int_0^\infty \mathbb{I}_{\mathcal{V}}(\omega(s))\, ds\right] = 0.$$

PROOF. Let μ be the non-negative σ-finite measure specified in Proposition 5.1, and extend μ from $\mathcal{K}$ to a σ-finite measure on $\bar{G} \times S_1(0)$ by defining $\mu(\bar{G} \times S_1(0) \setminus \mathcal{K}) = 0$. In the rest of the proof, we use μ to denote this extension. Since μ is σ-finite, it follows that there exists a continuous function ϕ defined on $\bar{G} \times S_1(0)$ that satisfies $\phi(x,u) \in (0,1)$ for each $x \in \bar{G} \setminus \mathcal{V}$ and $u \in S_1(0)$ and $\int_{\bar{G}\times S_1(0)} \phi(x,u)\mu(dx,du) < \infty$.

We will establish the corollary by verifying the assumptions of Theorem 1.7 of [30]. We first show that the five clauses of Condition 1.2 of [30] are satisfied with $\mathcal{D} = \{f \in \mathcal{H} : f \geq 0\}$, $E = \bar{G}$, $U = S_1(0)$, and operators A and B from $\mathcal{D} \subset \mathcal{C}_b(\bar{G})$ to $\mathcal{C}(\bar{G} \times S_1(0))$, that are defined as follows:

$$Af(x,u) = \mathcal{L}f(x) \text{ and } Bf(x,u) = \frac{1}{\phi(x,u)}\langle u, \nabla f(x)\rangle.$$

Clearly, $1 \in \mathcal{D}$ and $A1 = B1 = 0$. Thus, (i) of Condition 1.2 of [30] holds. Let $\psi_A = \psi_B \equiv 1$. Note that for each function $f \in \mathcal{D}$, ∇f is zero outside a compact set $K_f \subset \bar{G} \setminus \mathcal{V}$ and f, ∇f and (due to the continuity of the drift and dispersion coefficients) $\mathcal{L}f$, are all uniformly bounded on $\bar{G}$, say by a constant $a_f < \infty$. Hence, $|Af(x,u)| \leq a_f = a_f \psi_A(x,u)$ and $|Bf(x,u)| \leq b_f = b_f \psi_B(x,u)$ for every $(x,u) \in \bar{G} \times S_1(0)$, where

$$b_f \doteq \sup_{x\in\bar{G},u\in S_1(0)} \frac{|\nabla f(x)|}{|\phi(x,u)|} \leq \left(\frac{a_f}{\inf_{x\in K_f, u\in S_1(0)} |\phi(x,u)|} \right),$$

which is finite because $0 < \phi < 1$ by construction, and the infimization can be replaced by a minimization since ϕ is continuous and $K_f \times S_1(0)$ is compact. It follows from the second part of Lemma 5.3 that the set $\{(f, Af, Bf) : f \in \mathcal{D}\}$ is separable in the sense that there exists a countable subset $\mathcal{D}_0 \subset \mathcal{D}$ such that the set $\{(f, Af, Bf) : f \in \mathcal{D}\}$ is contained in the bounded, pointwise closure of the linear span of $\{(f, Af, Bf) : f \in \mathcal{D}_0\}$. This verifies (iii) of Condition 1.2 of [30]. From the definitions of A and B, it clear that (iv) of Condition 1.2 of [30] is also satisfied (see, for instance, Example 1.4 of [30]). Lastly, it is clear that $\mathcal{D}$ is closed under multiplication. Also, by 1 of Assumption 1, $\mathcal{H}$ separates points, and since for any $f \in \mathcal{H}$ and $c \geq 0$, $f - \min_{z\in\mathbb{R}^J} f(z) + c \in \mathcal{D}$, $\mathcal{D}$ also separates points. Thus, the last property of Condition 1.2 of [30] follows.

We next define two measures on $\bar{G} \times S_1(0)$. Let η_0 be the unique rotationally invariant probability measure on $S_1(0)$ and let μ_0 be the probability measure on $\bar{G} \times S_1(0)$ given by

$$\mu_0(dx, du) = \eta_0(du)\pi(dx).$$

Then for each $f \in \mathcal{D}$, we have

$$\int_{\bar{G}\times S_1(0)} Af(x,u)\mu_0(dx,du) = \int_{\bar{G}\times S_1(0)} \mathcal{L}f(x)\mu_0(dx,du) = \int_{\bar{G}} \mathcal{L}f(x)\pi(dx).$$

Also, let μ_1 be the finite measure on $\bar{G} \times S_1(0)$ given by

$$\mu_1(dx, du) = \phi(x,u)\mu(dx,du).$$

It is clear that

$$\int_{\bar{G}\times S_1(0)} \psi_A(x,u)\mu_0(dx,du) + \int_{\bar{G}\times S_1(0)} \psi_B(x,u)\mu_1(dx,du) < \infty$$

and by (32), for each $f \in \mathcal{D} \subset \mathcal{H}$,

$$\begin{aligned}\int_{\bar{G}\times S_1(0)} & Af(x,u)\mu_0(dx,du) + \int_{\bar{G}\times S_1(0)} Bf(x,u)\mu_1(dx,du) \\ &= \int_{\bar{G}} \mathcal{L}f(x)\pi(dx) + \int_{\bar{G}\times S_1(0)} \langle u, \nabla f(x)\rangle \mu(dx,du) = 0.\end{aligned}$$

Let $\mathcal{U} = \bar{G} \times S_1(0)$. Obviously, $\mu_i(\mathcal{U}) = \mu_i(\bar{G} \times S_1(0))$ for $i = 0, 1$. We have verified all the assumptions of Theorem 1.7 of [30], and so it follows from that theorem that there exists a stationary process X such that $X(0)$ has distribution π and $\{f(X(t)) - \int_0^t \int_{B_1(0)} Af(X(s),u)\eta_0(du)ds, t \geq 0\} = \{f(X(t)) - \int_0^t \mathcal{L}f(X(s))ds, t \geq 0\}$ is a submartingale for each $f \in \mathcal{D}$. Since $f - \min_{x\in\mathbb{R}^J} f(x) \in \mathcal{D}$ for each $f \in \mathcal{H}$, it follows that $\{f(X(t)) - \int_0^t \mathcal{L}f(X(s))\, ds, t \geq 0\}$ is a submartingale for each $f \in \mathcal{H}$. To conclude the proof, we note that by the stationarity of X, the assumption $\pi(\partial G) = 0$ and the fact that $X(0)$ has distribution π,

$$\mathbb{E}\left[\int_0^\infty \mathbb{I}_{\mathcal{V}}(X(s))ds\right] = \int_0^\infty \mathbb{E}\left[\mathbb{I}_{\mathcal{V}}(X(s))\right] ds = \int_0^\infty \pi(\mathcal{V})ds = 0.$$

The above discussion shows that $\tilde{Q}_\pi$, the law of X, satisfies the three properties stated in the corollary. □

5.3. *Proof of Theorem 1.* The necessity of the condition (2) follows from the discussion prior to the statement of Theorem 1. So, it only remains to prove sufficiency. If (2) and the assumptions of the theorem hold, then by Corollary 3 there exists a stationary process X whose law, $\tilde{\mathbb{Q}}_\pi$ satisfies the three properties stated therein. To complete the proof of Theorem 1 it only remains to show that $\tilde{\mathbb{Q}}_\pi$ is equal to $\mathbb{Q}_\pi$, the solution to the well-posed submartingale theorem.

For each $\omega \in \mathcal{C}\,[0,\infty)$, let $\tilde{\mathbb{Q}}_\omega$ be a regular conditional probability distribution of $\tilde{\mathbb{Q}}_\pi$ given $\mathcal{M}_0$. Then, for each $\omega \in \mathcal{C}\,[0,\infty)$,

$$\tilde{\mathbb{Q}}_\omega(\omega'(0) = \omega(0)) = 1. \tag{37}$$

Moreover, disintegrating $\tilde{\mathbb{Q}}_\pi$ and using property 1 of $\tilde{\mathbb{Q}}_\pi$, we obtain

$$\tilde{\mathbb{Q}}_\pi(\cdot) = \int_{\mathcal{C}[0,\infty)} \tilde{\mathbb{Q}}_\omega(\cdot)\tilde{\mathbb{Q}}_\pi(d\omega) = \int_{\mathcal{C}[0,\infty)} \tilde{\mathbb{Q}}_\omega(\cdot)\tilde{\mathbb{P}}^\pi(d\omega), \tag{38}$$

where $\tilde{\mathbb{P}}^{\pi}$ is the probability measure on $(\mathcal{C}[0,\infty), \mathcal{M}_0)$ obtained as the restriction of $\tilde{\mathbb{Q}}_{\pi}$ to $\mathcal{M}_0$ defined as follows: for $A_0 \in \mathcal{B}(\mathbb{R}^J)$,

$$\tilde{\mathbb{P}}^{\pi}(A) \doteq \pi(A_0 \cap \bar{G}), \qquad \text{if } A = \{\omega \in \mathcal{C}[0,\infty) : \omega(0) \in A_0\}. \tag{39}$$

It then follows from property 3 of $\tilde{\mathbb{Q}}_{\pi}$ that

$$0 = \mathbb{E}^{\tilde{\mathbb{Q}}_{\pi}}\left[\int_0^{\infty} \mathbb{I}_{\mathcal{V}}(\omega(s))\,ds\right] = \int_{\mathcal{C}[0,\infty)} \mathbb{E}^{\tilde{\mathbb{Q}}_{\omega}}\left[\int_0^{\infty} \mathbb{I}_{\mathcal{V}}(\omega'(s))\,ds\right]\tilde{\mathbb{P}}^{\pi}(d\omega). \tag{40}$$

For each $N \in \mathbb{N}$, consider the stopping time

$$\chi^N(\omega) = \inf\{t \geq 0 : \ \omega(t) \notin B_N(0)\}, \qquad \omega \in \mathcal{C}[0,\infty), \tag{41}$$

where we adopt the convention that the infimum over an empty set is infinity. Let $\mathcal{H}_0$ be the countable subset of $\mathcal{H}$ described in Lemma 5.3. By property 2 of $\tilde{\mathbb{Q}}_{\pi}$ and the optional stopping theorem, $\{f(\omega(t \wedge \chi^N(\omega))) - \int_0^{t\wedge\chi^N(\omega)} \mathcal{L}f(\omega(u))\,du,\ t \geq 0\}$ is a $\tilde{\mathbb{Q}}_{\pi}$-submartingale for each $f \in \mathcal{H}_0$. By (38) and the fact that $\mathcal{H}_0$ is countable, there exists $F_0^N \in \mathcal{M}_0$ with $\tilde{\mathbb{P}}^{\pi}(F_0^N) = 0$ such that for every $\omega \notin F_0^N$ and each $f \in \mathcal{H}_0$, $\{f(\omega'(t \wedge \chi^N(\omega'))) - \int_0^{t\wedge\chi^N(\omega')} \mathcal{L}f(\omega'(u))\,du,\ t \geq 0\}$ is a $\tilde{\mathbb{Q}}_{\omega}$-submartingale. Since functions in $\mathcal{H}$ are bounded and, by Lemma 5.3, can be approximated by functions in $\mathcal{H}_0$, it follows that for every $\omega \notin F_0^N$ and each $f \in \mathcal{H}$, $\{f(\omega'(t \wedge \chi^N(\omega'))) - \int_0^{t\wedge\chi^N(\omega')} \mathcal{L}f(\omega'(u))\,du,\ t \geq 0\}$ is a $\tilde{\mathbb{Q}}_{\omega}$-submartingale. Let $F_0 \doteq \cup_N F_0^N \cup \{\omega : \chi^N(\omega) \not\to \infty\}$. Then $\tilde{\mathbb{P}}^{\pi}(F_0) = 0$ and for each $\omega \notin F_0$ and $f \in \mathcal{H}$, by passing to the limit as $N \to \infty$, we conclude that $\{f(\omega'(t)) - \int_0^t \mathcal{L}f(\omega'(u))\,du,\ t \geq 0\}$ is a $\tilde{\mathbb{Q}}_{\omega}$-submartingale. In addition, by (40), without loss of generality by enlarging F_0 to another $\tilde{\mathbb{P}}^{\pi}$-null set, we may assume that for each $\omega \notin F_0$,

$$\mathbb{E}^{\tilde{\mathbb{Q}}_{\omega}}\left[\int_0^{\infty} \mathbb{I}_{\mathcal{V}}(\omega'(s))\,ds\right] = 0.$$

Thus, for each $\omega \notin F_0$, we see that $\tilde{\mathbb{Q}}_{\omega}$ satisfies all three properties of Definition 2.1 with $z = \omega(0)$. By the well-posedness of the submartingale problem, this implies that

$$\tilde{\mathbb{Q}}_{\omega} = \mathbb{Q}_{\omega(0)}$$

and then by (38) and (39),

$$\tilde{\mathbb{Q}}_{\pi}(\cdot) = \int_{\mathcal{C}[0,\infty)} \mathbb{Q}_{\omega(0)}(\cdot)\tilde{\mathbb{P}}^{\pi}(d\omega) = \int_{\bar{G}} \mathbb{Q}_z(\cdot)\pi(dz) = \mathbb{Q}_{\pi}(\cdot).$$

This shows that $\tilde{\mathbb{Q}}_{\pi} = \mathbb{Q}_{\pi}$ and completes the proof of Theorem 1.

6. A Boundary Property. The main result of this section shows that for a large class of domains and reflection fields $(G, d(\cdot))$ and subsets $\mathcal{V} \subset \partial G$ associated with a well-posed submartingale problem, the solution to the submartingale problem spends zero Lebesgue time on the boundary of the domain. In this section for simplicity we assume that the uniform ellipticity condition (12) holds. The boundary property is first stated precisely in Section 6.1 (see Proposition 6.1), and its proof, which is given in Section 6.3, is preceded by some supporting results that are established in Section 6.2.

6.1. *Statement of the Boundary Property.* We state the boundary property and show that it is equivalent to the property stated in (47) below.

PROPOSITION 6.1. *Suppose that $(G, d(\cdot))$ is a piecewise $\mathcal{C}^2$ domain with continuous reflection, $\mathcal{V} \subset \partial G$ satisfies $\partial G \setminus \mathcal{U} \subseteq \mathcal{V}$ and the submartingale problem associated with $(G, d(\cdot))$, $\mathcal{V}$, $b(\cdot)$ and $\sigma(\cdot)$ is well posed. If $\{\mathbb{Q}_x, x \in \bar{G}\}$ is the solution to the associated submartingale problem, then for each $x \in \bar{G}$,*

$$\mathbb{E}^{\mathbb{Q}_x}\left[\int_0^\infty \mathbb{I}_{\partial G}(\omega(s))\, ds\right] = 0. \tag{42}$$

Due to property 3 of the submartingale problem and the assumption that $\partial G \setminus \mathcal{U} \subseteq \mathcal{V}$, to show (42) it suffices to show that for each $x \in \bar{G}$,

$$\mathbb{E}^{\mathbb{Q}_x}\left[\int_0^\infty \mathbb{I}_{\mathcal{U}}(\omega(s))\, ds\right] = 0. \tag{43}$$

Without loss of generality, we may assume in this section that $\partial G \setminus \mathcal{V} = \mathcal{U}$. Recall the definition of $\mathcal{I}$ given in (8) and for each $\delta > 0$, let

$$\mathcal{U}_\delta \doteq \left\{ x \in \mathcal{U} : \begin{array}{c} \mathcal{I}(y) \subseteq \mathcal{I}(x) \text{ for all } y \in B_\delta(x) \cap \partial G \text{ and } \exists\, n \in n(x) \\ \text{such that } n = \sum_{i \in \mathcal{I}(x)} \theta_i n^i(x), \text{ where } \theta_i \geq 0, i \in \mathcal{I}(x), \\ \sum_{i \in \mathcal{I}(x)} \theta_i = 1, \text{ and } \langle n, d \rangle \geq \delta |d| \text{ for all } d \in d(x) \end{array} \right\}, \tag{44}$$

and for each $\mathcal{J} \subseteq \mathcal{I}$, $\mathcal{J} \neq \emptyset$, let

$$\mathcal{U}_\delta^{\mathcal{J}} \doteq \{x \in \mathcal{U}_\delta : \mathcal{I}(x) = \mathcal{J}\}. \tag{45}$$

It is immediate from the definition that any two elements in $\{\mathcal{U}_\delta^{\mathcal{J}}, \mathcal{J} \subseteq \mathcal{I}, \mathcal{J} \neq \emptyset\}$ are disjoint, and

$$\mathcal{U}_\delta = \bigcup_{\mathcal{J} \subseteq \mathcal{I}, \mathcal{J} \neq \emptyset} \mathcal{U}_\delta^{\mathcal{J}}, \qquad \mathcal{U} = \bigcup_{\delta > 0} \mathcal{U}_\delta. \tag{46}$$

In light of (46), to prove (43) and hence Proposition 6.1, it is clearly sufficient to show that for every $x \in \bar{G}$, $\delta > 0$ and $\mathcal{J} \subseteq \mathcal{I}$, $\mathcal{J} \neq \emptyset$, such that $\mathcal{U}_\delta^{\mathcal{J}} \neq \emptyset$,

$$\mathbb{E}^{\mathbb{Q}_x}\left[\int_0^\infty \mathbb{I}_{\mathcal{U}_\delta^{\mathcal{J}}}(\omega(s))\,ds\right] = 0. \tag{47}$$

Indeed, taking first the sum in (47) over $\mathcal{J} \subseteq \mathcal{I}$, $\mathcal{J} \neq \emptyset$, next the limit as $\delta \to 0$ in (47) and then applying Fatou's lemma, we obtain (43).

6.2. *Supporting Results.* We now state some preliminary results that will be used in the proof of Proposition 6.1. Throughout, we assume $(G, d(\cdot))$ is a piecewise $\mathcal{C}^2$ domain with continuous reflection. We start with an elementary observation, whose proof we include for completeness.

LEMMA 6.2. *For each $\delta > 0$ and $\mathcal{J} \subseteq \mathcal{I}$, $\mathcal{J} \neq \emptyset$, $\mathcal{U}_\delta^{\mathcal{J}}$ is closed.*

PROOF. Fix $\delta > 0$ and $\mathcal{J} \subseteq \mathcal{I}$, $\mathcal{J} \neq \emptyset$, such that $\mathcal{U}_\delta^{\mathcal{J}} \neq \emptyset$, and let a point $x \in \mathbb{R}^J$ and the sequence $\{x_k\}_{k\in\mathbb{N}} \subseteq \mathcal{U}_\delta^{\mathcal{J}}$ be such that $x_k \to x$ as $k \to \infty$. Clearly, $x \in \partial G$ because $\{x_k\}_{k\in\mathbb{N}} \subseteq \partial G$ and ∂G is closed. Let $N_1 < \infty$ be such that for all $k \geq N_1$, $x \in B_\delta(x_k) \cap \partial G$. Then for $k \geq N_1$, it follows from (44) and (45) that $\mathcal{I}(x) \subseteq \mathcal{I}(x_k) = \mathcal{J}$. When combined with the upper-semicontinuity of the set-function $\mathcal{I}(\cdot)$, this implies that $\mathcal{I}(x) = \mathcal{J}$. Moreover, given any $y \in B_\delta(x) \cap \partial G$, since $B_\delta(x)$ is open and $x_k \to x$ as $k \to \infty$, there exists $N_2 > N_1$ such that $y \in B_\delta(x_k) \cap \partial G$ for all $k \geq N_2$. Since $x_k \in U_\delta^{\mathcal{J}}$, this implies that $\mathcal{I}(y) \subseteq \mathcal{I}(x_k) = \mathcal{J} = \mathcal{I}(x)$. Finally, $x_k \in \mathcal{U}_\delta^{\mathcal{J}}$ also implies that there exists $n_k \in n(x_k)$ such that $\langle n_k, d\rangle \geq \delta|d|$ for all $d \in d(x_k)$. Since $n^i(\cdot)$ and $\gamma^i(\cdot)$, $i \in \mathcal{J}$, are continuous, and $x_k \to x$ as $k \to \infty$, and $\mathcal{I}(x) = \mathcal{I}(x_k) = \mathcal{J}$, it follows from the definitions of $n(x)$ in (9) and $d(x)$ in (10) and the continuity of the vector fields $\gamma^i, i \in \mathcal{I}(x)$, that there exists $n \in n(x)$ such that $\langle n, d\rangle \geq \delta|d|$ for every $d \in d(x)$. Thus, we have shown that $x \in \mathcal{U}_\delta^{\mathcal{J}}$, and hence, that $\mathcal{U}_\delta^{\mathcal{J}}$ is closed. $\square$

In the next lemma, we construct a family of test functions that lie in the set $\mathcal{H}$. The proof of the lemma is purely analytic and hence, is relegated to Appendix B. Some properties of the test functions are stated in terms of another class of functions, which we now define. Recall that ϕ^i, $i \in \mathcal{I}$, are the functions that characterize the domains G_i, as defined in Definition 3.3. For $x \in \mathcal{U}$, let $\theta_i(x) > 0$, $i \in \mathcal{I}(x)$, be constants such that for each $j \in \mathcal{I}(x)$,

$$\left\langle \sum_{i\in\mathcal{I}(x)} \theta_i(x)\frac{\nabla\phi^i(x)}{|\nabla\phi^i(x)|}, \gamma^j(x)\right\rangle > 0. \tag{48}$$

Such constants exist by the definition (6) of $\mathcal{U}$. Then, for $x \in \mathcal{U}$, define

$$g^x(y) \doteq \sum_{i \in \mathcal{I}(x)} \frac{\theta_i(x)}{|\nabla \phi^i(x)|} \phi^i(y), \; y \in \mathbb{R}^J. \tag{49}$$

LEMMA 6.3. *There exists a function $\kappa : (0,1) \mapsto (0,1/2)$ with $\kappa(\varepsilon) < \varepsilon/2$ for every $\varepsilon \in (0,1)$ such that for each $x \in \mathcal{U}$, there exist constants $0 < r'_x < r_x < \operatorname{dist}(x, \mathcal{V})$, $0 < c_x < \infty$, $\beta_x > 0$, and a family of functions $\{q_{\varepsilon,x} \in \mathcal{H} : \varepsilon \in (0,1)\}$ that has the following properties:*

1. $\operatorname{supp}[q_{\varepsilon,x}] \cap \bar{G} \subset \bar{G} \cap B_{r_x}(x)$;
2. $-\varepsilon^2 - \varepsilon^{3/2} \leq q_{\varepsilon,x} \leq 0$;
3. $|\nabla q_{\varepsilon,x}| \leq c_x \varepsilon$;
4. *for every $y \in \overline{G} \cap B_{r'_x}(x)$,*

$$\sum_{i,j=1}^{J} a_{ij}(y) \frac{\partial^2 q_{\varepsilon,x}}{\partial x_i \partial x_j}(y) \geq \begin{cases} 2\alpha\beta_x - c_x\varepsilon & \text{if } 0 \leq g^x(y) \leq \varepsilon/2, \\ -c_x\varepsilon & \text{if } \varepsilon/2 < g^x(y) < \varepsilon - \kappa(\varepsilon), \end{cases}$$

and

$$\left| \sum_{i,j=1}^{J} a_{ij}(y) \frac{\partial^2 q_{\varepsilon,x}}{\partial x_i \partial x_j}(y) \right| \leq c_x \sqrt{\varepsilon} \qquad \text{if } g^x(y) \geq \varepsilon - \kappa(\varepsilon).$$

For each $\delta > 0$ and $x \in \mathcal{U}_\delta$, let r'_x be the constant from Lemma 6.3. The neighbourhoods $\{B_{r'_x}(x) : \; x \in \mathcal{U}_\delta\}$ form an open cover of the closed set $\mathcal{U}_\delta$. The next lemma shows that we can choose a countable open sub-cover that has certain properties. For each nonempty subset $\mathcal{J}$ of $\mathcal{I}$, recall the definition of $\mathcal{U}_\delta^{\mathcal{J}}$ given in (45).

LEMMA 6.4. *For each $\delta > 0$ and $\mathcal{J} \subseteq \mathcal{I}$, $\mathcal{J} \neq \emptyset$, there exists a countable set of points $\mathcal{S}_\delta^{\mathcal{J}} \subset \mathcal{U}_\delta^{\mathcal{J}}$ such that*

$$\mathcal{U}_\delta^{\mathcal{J}} \subseteq \cup_{x \in \mathcal{S}_\delta^{\mathcal{J}}} B_{r'_x}(x),$$

and there exists a measurable mapping $\kappa_\delta^{\mathcal{J}}$ from $\mathcal{U}_\delta^{\mathcal{J}}$ onto $\mathcal{S}_\delta^{\mathcal{J}}$ such that $x \in B_{r'_{\kappa_\delta^{\mathcal{J}}(x)}}(\kappa_\delta^{\mathcal{J}}(x))$ and $\mathcal{I}(x) = \mathcal{I}(\kappa_\delta^{\mathcal{J}}(x))$ for each $x \in \mathcal{U}_\delta^{\mathcal{J}}$.

PROOF. Fix $\delta > 0, x \in \mathcal{U}$. Let $r'_x > 0$ be the constant from Lemma 6.3 and pick $\mathcal{J} \subseteq \mathcal{I}, \mathcal{I} \neq \emptyset$ such that $\mathcal{U}_\delta^{\mathcal{J}} \neq \emptyset$. Then $\mathcal{U}_\delta^{\mathcal{J}}$ is a closed set by Lemma 6.2, and so $\mathcal{U}_\delta^{\mathcal{J}} \cap B_n(0)$ is compact for each $n \in \mathbb{N}$. Since $\{B_{r'_x}(x), \; x \in \mathcal{U}_\delta^{\mathcal{J}} \cap B_n(0)\}$ is a covering of the compact set $\mathcal{U}_\delta^{\mathcal{J}} \cap B_n(0)$, there exists a finite subset $\mathcal{S}_{n,\delta}^{\mathcal{J}}$

of $\mathcal{U}_\delta^{\mathcal{J}} \cap B_n(0)$ such that $\{B_{r'_x}(x),\ x \in \mathcal{S}_{n,\delta}^{\mathcal{J}}\}$ covers $\mathcal{U}_\delta^{\mathcal{J}} \cap B_n(0)$. It is clear that the countable set $\mathcal{S}_\delta^{\mathcal{J}} = \cup_{n\in\mathbb{N}} \mathcal{S}_{n,\delta}^{\mathcal{J}}$ satisfies the stated property. We can further choose the set $\mathcal{S}_\delta^{\mathcal{J}}$ to be minimal in the sense that for each strict subset C of $\mathcal{S}_\delta^{\mathcal{J}}$, $\cup_{x\in C} B_{r'_x}(x)$ does not cover $\mathcal{U}_\delta^{\mathcal{J}}$. Denote $\mathcal{S}_\delta^{\mathcal{J}} = \{x_k,\ k \in \mathbb{N}\}$. Let $D_k = B_{r'_{x_k}}(x_k) \setminus (\cup_{i=0}^{k-1} B_{r'_{x_i}}(x_i)) \cap \mathcal{U}_\delta^{\mathcal{J}}$ for each $k \in \mathbb{N}$. Then $\{D_k, k \in \mathbb{N}\}$ is a partition of $\mathcal{U}_\delta^{\mathcal{J}}$, and so for each $x \in \mathcal{U}_\delta^{\mathcal{J}}$ there is a unique index $k(x)$ such that $x \in D_{k(x)}$. Define $\kappa^{\mathcal{J}}(x) = x_{k(x)}$. Then $\kappa^{\mathcal{J}}$ is a measurable mapping from $\mathcal{U}_\delta^{\mathcal{J}}$ onto $\mathcal{S}_\delta^{\mathcal{J}}$ that satisfies the stated property. □

6.3. *Proof of Proposition 6.1.* We first introduce a sequence of stopping times. Fix $\delta > 0$ and $\mathcal{J} \subseteq \mathcal{I}, \mathcal{J} \neq \emptyset$, such that $\mathcal{U}_\delta^{\mathcal{J}} \neq \emptyset$. Let $\mathcal{S}_\delta^{\mathcal{J}}$, $\{B_{r'_x}(x) : x \in \mathcal{S}_\delta^{\mathcal{J}}\}$ and the measurable mapping $\kappa_\delta^{\mathcal{J}}$ be as in Lemma 6.4. Now, set $\sigma_0 \doteq 0$ and for $n \in \mathbb{N}$, recursively define

$$\tau_n \doteq \inf\{t \geq \sigma_{n-1} : \omega(t) \in \mathcal{U}_\delta^{\mathcal{J}}\}, \tag{50}$$

$$\sigma_n \doteq \inf\left\{t \geq \tau_n : \omega(t) \notin B_{r'_{\kappa_\delta^{\mathcal{J}}(\omega(\tau_n))}}(\kappa_\delta^{\mathcal{J}}(\omega(\tau_n)))\right\}. \tag{51}$$

Since $\mathcal{U}_\delta^{\mathcal{J}}$ is a closed set by Lemma 6.2 and $B_{r'_{\kappa_\delta^{\mathcal{J}}(\omega(\tau_n))}}(\kappa_\delta^{\mathcal{J}}(\omega(\tau_n)))$ is an $\mathcal{F}_{\tau_n}$-measurable open ball, $\{\tau_n,\ n \in \mathbb{N}\}$ and $\{\sigma_n,\ n \in \mathbb{N} \cup \{0\}\}$ are two nested sequences of stopping times.

Now, fix $x \in \bar{G}$ and let $\mathbb{Q}_x$ be the solution to the well-posed submartingale problem associated with $(G, d(\cdot)), \mathcal{V}$, $b(\cdot)$ and $\sigma(\cdot)$. From the discussion in Section 6.1, it suffices to establish (47), for which we will use a proof by induction. Note that for $n = 1$, $\sigma_{n-1} = 0$ and so we trivially have

$$\mathbb{E}^{\mathbb{Q}_x}\left[\int_0^{\sigma_{n-1}} \mathbb{I}_{\mathcal{U}_\delta^{\mathcal{J}}}(\omega(s))\, ds\right] = 0. \tag{52}$$

Now, suppose that (52) holds for some $n \in \mathbb{N}$. We will show that then (52) also holds with n replaced by $n+1$. Since under $\mathbb{Q}_x$, $\omega(t) \notin \mathcal{U}_\delta^{\mathcal{J}}$ for $t \in [\sigma_{n-1}, \tau_n)$, it is clear that

$$\mathbb{E}^{\mathbb{Q}_x}\left[\int_{\sigma_{n-1}}^{\tau_n} \mathbb{I}_{\mathcal{U}_\delta^{\mathcal{J}}}(\omega(s))\, ds\right] = 0. \tag{53}$$

Next, for each $y \in \mathcal{U}$, let the constant $c_y \in (0, \infty)$ and the family of test functions $q_{\varepsilon,y}$, $\varepsilon \in (0,1)$, be as specified in Lemma 6.3. Since $q_{\varepsilon,y} \in \mathcal{H}$, $q_{\varepsilon,y}$ is constant in a neighborhood of $\mathcal{V}$ and $\langle d, \nabla q_{\varepsilon,y}(z)\rangle \geq 0$ for all $d \in d(z)$ and

$z \in \partial G$, by property 3 of Definition 2.1 and the optional stopping theorem, for each $y \in \mathcal{U}$ and $\varepsilon \in (0,1)$,

$$q_{\varepsilon,y}(\omega(t \wedge \sigma_n)) - \int_0^{t\wedge\sigma_n} \mathcal{L}q_{\varepsilon,y}(\omega(u))\,du$$

is a $\mathbb{Q}_x$-submartingale. In fact, the above submartingale is integrable because $q_{\varepsilon,y} \in \mathcal{H}$ implies $q_{\varepsilon,y}$ is uniformly bounded. Now, let $\{\varepsilon_k,\ k \in \mathbb{N}\}$ be a sequence of real numbers in $(0,1)$ such that $\varepsilon_k \to 0$ as $k \to \infty$. The same argument that is used in the proof of Theorem 1.2.10 of [44] can be applied to show that for the regular conditional probability distribution $\{\mathbb{Q}_\omega, \omega \in \mathcal{C}[0,\infty)\}$ of $\mathbb{Q}_x$ given $\mathcal{M}_{\tau_n}$, there exists a $\mathbb{Q}_x$-null set $F \in \mathcal{M}_{\tau_n}$ such that for each $\omega \notin F$, $y \in \mathcal{S}_\delta^{\mathcal{J}}$ and ε_k, $k \in \mathbb{N}$,

(54)
$$\left\{q_{\varepsilon_k,y}(\omega'(t\wedge\sigma_n(\omega'))) - q_{\varepsilon_k,y}(\omega'(t\wedge\tau_n(\omega'))) - \int_{t\wedge\tau_n(\omega')}^{t\wedge\sigma_n(\omega')} \mathcal{L}q_{\varepsilon_k,y}(\omega'(u))\,du,\ t\geq 0\right\}$$

is a $\mathbb{Q}_\omega$-submartingale and

(55)
$$\mathbb{Q}_w\left(\omega' \in \mathcal{C}[0,\infty) : \tau_n(\omega') = \tau_n(\omega) \text{ and } \omega'(t) = \omega(t),\ 0 \leq t \leq \tau_n(\omega)\right) = 1.$$

Due to (52) and (53), it follows that

(56)
$$\mathbb{E}^{\mathbb{Q}_x}\left[\int_0^{\sigma_n} \mathbb{I}_{\mathcal{U}_\delta^{\mathcal{J}}}(\omega(s))\,ds\right] = \mathbb{E}^{\mathbb{Q}_x}\left[\mathbb{I}_{\{\tau_n(\omega)<\infty\}}\mathbb{E}^{\mathbb{Q}_\omega}\left[\int_{\tau_n(\omega)}^{\sigma_n(\omega')} \mathbb{I}_{\mathcal{U}_\delta^{\mathcal{J}}}(\omega'(s))\,ds\right]\right].$$

Consider $\omega \notin F$ such that $\tau_n(\omega) < \infty$. Note that $\omega(\tau_n(\omega)) \in \mathcal{U}_\delta^{\mathcal{J}}$. Let $\bar{x} \in \mathcal{S}_\delta^{\mathcal{J}}$ be such that $\bar{x} \doteq \kappa_\delta^{\mathcal{J}}(\omega(\tau_n(\omega)))$ and recall that $\mathcal{I}(\omega(\tau_n(\omega))) = \mathcal{I}(\bar{x}) = \mathcal{J}$. Fix $t > \tau_n(\omega)$ and note from (55) that for $\mathbb{Q}_\omega$ almost surely every ω', $t > \tau_n(\omega') = \tau_n(\omega)$. Since, under $\mathbb{Q}_\omega$, $\omega'(s) \in \bar{G} \cap B_{r'_{\bar{x}}}(\bar{x})$ for every $s \in [\tau_n(\omega), \sigma_n(\omega'))$, it follows from the submartingale property of (54) and property (2) of $q_{\varepsilon_k,\bar{x}}$ in Lemma 6.3 that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}_\omega}\left[\int_{\tau_n(\omega)}^{t\wedge\sigma_n(\omega')} \mathcal{L}q_{\varepsilon_k,\bar{x}}(\omega'(u))\,du\right] \\
&\leq \mathbb{E}^{\mathbb{Q}_\omega}\left[q_{\varepsilon_k,\bar{x}}(\omega'(t\wedge\sigma_n(\omega'))) - q_{\varepsilon_k,\bar{x}}(\omega'(\tau_n(\omega)))\right] \\
&\leq 2\varepsilon_k^2 + 2\varepsilon_k^{3/2}.
\end{aligned}$$

On the other hand, note that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}_\omega}\left[\int_{\tau_n(\omega)}^{t\wedge\sigma_n(\omega')} \mathcal{L}q_{\varepsilon_k,\bar{x}}(\omega'(u))\,du\right] \\
&= \mathbb{E}^{\mathbb{Q}_\omega}\left[\int_{\tau_n(\omega)}^{t\wedge\sigma_n(\omega')} \frac{1}{2}\sum_{i,j=1}^{J} a_{ij}(\omega'(u))\frac{\partial^2 q_{\varepsilon_k,\bar{x}}(\omega'(u))}{\partial x_i\partial x_j}\,du\right] \\
&\quad+\mathbb{E}^{\mathbb{Q}_\omega}\left[\int_{\tau_n(\omega)}^{t\wedge\sigma_n(\omega')} \sum_{j=1}^{J} b_j(\omega'(u))\frac{\partial q_{\varepsilon_k,\bar{x}}(\omega'(u))}{\partial x_j}\,du\right].
\end{aligned}$$

Combining the last two displays with property (3) of $q_{\varepsilon_k,\bar{x}}$ in Lemma 6.3, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}_\omega}\left[\int_{\tau_n(\omega)}^{t\wedge\sigma_n(\omega')} \sum_{i,j=1}^{J} a_{ij}(\omega'(u))\frac{\partial^2 q_{\varepsilon_k,\bar{x}}(\omega'(u))}{\partial x_i\partial x_j}\,du\right] \\
&\quad\le 4\varepsilon_k^2 + 4\varepsilon_k^{3/2} + 2c_{\bar{x}}t \sup_{z\in\bar{G}\cap B_{r'_{\bar{x}}}} |b(z)|\varepsilon_k.
\end{aligned}$$

Together with property (4) of $q_{\varepsilon_k,\bar{x}}$ in Lemma 6.3, this implies that

$$\begin{aligned}
&(2\alpha\beta_{\bar{x}} - c_{\bar{x}}\varepsilon_k)\mathbb{E}^{\mathbb{Q}_\omega}\left[\int_{\tau_n(\omega)}^{t\wedge\sigma_n(\omega')} \mathbb{I}_{\{0\le g^{\bar{x}}(\omega'(u))\le\varepsilon_k/2\}}\,du\right] \\
&\quad\le c_{\bar{x}}t\sqrt{\varepsilon_k} + c_{\bar{x}}t\varepsilon_k + 4\varepsilon_k^2 + 4\varepsilon_k^{3/2} + 2c_{\bar{x}}t \sup_{z\in\bar{G}\cap B_{r'_{\bar{x}}}} |b(z)|\varepsilon_k.
\end{aligned}$$

Letting first $k\to\infty$ and then $t\to\infty$, we obtain

$$\mathbb{E}^{\mathbb{Q}_\omega}\left[\int_{\tau_n(\omega)}^{\sigma_n(\omega')} \mathbb{I}_{\{g^{\bar{x}}(\omega'(u))=0\}}\,du\right] = 0.$$

From the definition of σ_n and $g^{\bar{x}}$ given in (51) and (49), respectively, it follows that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}_\omega}\left[\int_{\tau_n(\omega)}^{t\wedge\sigma_n(\omega')} \mathbb{I}_{\cap_{j\in\mathcal{J}}\partial G_j}(\omega'(u))\,du\right] \\
&\quad= \mathbb{E}^{\mathbb{Q}_\omega}\left[\int_{\tau_n(\omega)}^{t\wedge\sigma_n(\omega')} \mathbb{I}_{\cap_{j\in\mathcal{I}(\bar{x})}\partial G_j}(\omega'(u))\,du\right] \\
&\quad= \mathbb{E}^{\mathbb{Q}_\omega}\left[\int_{\tau_n(\omega)}^{t\wedge\sigma_n(\omega')} \mathbb{I}_{\{g^{\bar{x}}(\omega'(u))=0\}}\,du\right].
\end{aligned}$$

Thus, it follows that

$$\mathbb{E}^{\mathbb{Q}_\omega}\left[\int_{\tau_n(\omega)}^{\sigma_n(\omega')} \mathbb{I}_{\mathcal{U}_\delta^{\mathcal{J}}}(\omega'(u))\,du\right] \le \mathbb{E}^{\mathbb{Q}_\omega}\left[\int_{\tau_n(\omega)}^{\sigma_n(\omega')} \mathbb{I}_{\cap_{j\in\mathcal{J}}\partial G_j}(\omega'(u))\,du\right] = 0.$$

When combined with (56), this shows that (52) holds with n replaced by $n+1$. Since $\sigma_n(\omega) \to \infty$ as $n \to \infty$ for $\mathbb{Q}_x$ almost every ω, the proposition follows by induction.

7. Proof of Theorem 2. The proof of Theorem 2 relies on the construction of certain local test functions, whose existence we first establish.

PROPOSITION 7.1. *For each $x \in \bar{G}$, there exist a constant $r_x > 0$, increasing continuous functions $\alpha_x : (0, r_x) \mapsto (0,\infty)$ and $\kappa_x : (0, r_x) \mapsto (0,\infty)$ such that $\kappa_x < \alpha_x$, $\lim_{r\to 0}\alpha_x(r) = 0$, and a collection of nonnegative functions $\{g_{x,r} \in \mathcal{C}_c^2(\bar{G}), r \in (0, r_x)\}$ that satisfy the following properties:*

1. *$supp[g_{x,r}] \cap \bar{G} \subset B_{\alpha_x(r)}(x) \cap \bar{G}$;*
2. *$-g_{x,r} \in \mathcal{H}$;*
3. *$0 \le g_{x,r}(y) \le 1$ for $y \in \mathbb{R}^J$ and $g_{x,r}(y) = 1$ for each $y \in B_{\kappa_x(r)}(x) \cap \overline{G}$.*

Moreover, if $x \in \partial G \setminus \mathcal{V}$, we can choose $\alpha_x(r) = r$ for $r \in (0, r_x)$.

PROOF. We split the proof into two cases, depending on whether x lies in the interior or the boundary of G.

Case 1: $x \in G$. Let ξ be a bounded $\mathcal{C}^\infty$ function on $\mathbb{R}$ such that $\xi(z) = 1$ when $z \le 1/2$, $\xi(z) = 0$ when $z > 1$, and ξ is strictly decreasing in the interval $(1/2, 1)$. Note that then $\|\xi'\|_\infty < \infty$ and $\|\xi''\|_\infty < \infty$. For each $x \in G$ and $0 < r < (\mathrm{dist}(x, \partial G))^2$, define $g_{x,r}(y) \doteq \xi(|y-x|^2/r)$ for $y \in \mathbb{R}^J$. We now verify that $g_{x,r}$ satisfies properties (1)–(3) of the proposition, with $r_x = \mathrm{dist}(x, \partial G))^2$, $\alpha_x(r) = \sqrt{r}$ and $\kappa_x(r) = \sqrt{r}/2$. The first property holds because $|x-y|^2/r > 1$ when $y \notin B_{\sqrt{r}}(x)$ and $\xi(z) = 0$ when $z > 1$. The third property is satisfied because $0 \le \xi(z) \le 1$ for $z \in \mathbb{R}$ and $y \in B_{\sqrt{r}/2}(x)$ implies $(y-x)^2/r \le 1/4$, and $\xi(z) = 1$ for $z \le 1/4$. It is clear that $g_{x,r} \in \mathcal{C}_c^2(G)$ and $\mathrm{supp}[g_{x,r}] \cap G$. Hence, $-g_{x,r} \in \mathcal{H}$. This completes the proof of Case 1.

Case 2: $x \in \partial G$. Fix $x \in \partial G$. If $x \in \mathcal{V}$, recall v_x and $\rho_x > 0$ from Assumption 2. If $x \notin \mathcal{V}$, recall (48) and let $v_x \doteq \sum_{i\in\mathcal{I}(x)} \theta_i(x)\frac{\nabla\phi^i(x)}{|\nabla\phi^i(x)|}$. Without loss of generality, we may assume that $\sum_{i\in\mathcal{I}(x)}\theta_i(x) \in (0,1]$. Thus, we have $|v_x| \le 1$ in both cases. In addition, since $\mathcal{V}$ is a finite set, if $x \in \mathcal{V}$, we may assume, without loss of generality, that $\mathcal{V} \cap B_{\rho_x}(x) = \{x\}$. In this case, we will show that Proposition 7.1 is satisfied for some $r_x < \bar{r}_x$, $\alpha_x(r) = r$ and $\kappa_x(r) = r/8$, where

$$\bar{r}_x \doteq \begin{cases} \operatorname{dist}\big(x, \mathcal{V} \cup \cup_{i \notin \mathcal{I}(x)} (\partial G \cap G_i\big) & \text{if } x \notin \mathcal{V}, \\ \rho_x & \text{if } x \in \mathcal{V}. \end{cases}$$

It follows from Assumption 2, (48) and the continuity of $\gamma^j(\cdot)$, $j \in \mathcal{I}$, that there exists $r_x < \bar{r}_x \wedge \frac{8}{14}$ such that $\langle v_x, d\rangle \geq 0$ for each $d \in d(y) \cap S_1(0)$ and $y \in B_{r_x}(x)$. For each $r \in (0, r_x)$, consider the following function f_r defined on $\mathbb{R}^J$ by:

$$f_r(y) = \begin{cases} -\langle v_x, y - x\rangle + 7r/8 & \text{if } r/4 < |y - x| \leq 7r/8, \\ 1 & \text{if } |y - x| \leq r/4, \\ 0 & \text{if } |y - x| > 7r/8. \end{cases}$$

Since $|v_x| \leq 1$ and $r < r_x < 8/14$, when $r/4 < |y - x| \leq 7r/8$, we have

$$-\langle v_x, y - x\rangle + 7r/8 \leq |y - x| + 7r/8 \leq 7r/8 + 7r/8 = 14r/8 < 1$$

and

$$-\langle v_x, y - x\rangle + 7r/8 \geq -|y - x| + 7r/8 \geq 0.$$

Clearly, $0 \leq f_r \leq 1$. Let $\{\phi_m \in \mathcal{C}_c^\infty(\mathbb{R}^J), m \in \mathbb{N}\}$ be a sequence of nonnegative functions, where each ϕ_m satisfies $\int_{\mathbb{R}^J} \phi_m(x)\,dx = 1$ and has compact support in $B_{c_m}(0)$ and $\{c_m, m \in \mathbb{N}\}$ is a sequence such that $c_m \to 0$ as $m \to \infty$. For each $r \in (0, r_x)$, choose $m_r \in \mathbb{N}$ such that $c_{m_r} < r/8$. Define

$$g_{x,r} = f_r * \phi_{m_r},$$

where $*$ denotes the convolution operation. Then it is clear that $g_{x,r} \in \mathcal{C}_c^\infty(\mathbb{R}^J)$, $0 \leq g_{x,r} \leq 1$, $g_{x,r}(y) = 1$ for each $y \in B_{r/8}(x)$, and $\operatorname{supp}[g_{x,r}] \subset B_r(x)$. This shows that properties (1) and (3) hold. To show that $-g_{x,r} \in \mathcal{H}$, note that f_r is locally integrable and so has a distributional derivative ∇f_r, which is given explicitly by

$$\nabla f_r(y) = \begin{cases} -v_x & \text{if } r/4 < |y - x| < 7r/8, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\nabla g_{x,r} = \nabla f_r * \phi_{m_r}$ and $\nabla g_{x,r}(y) = v_x$ for each y such that $3r/8 < |y - x| < 3r/4$. It is clear that $\langle d, \nabla f_r(y)\rangle \leq 0$ for each $y \in \partial G$ and $d \in d(y)$. This implies that $-g_{x,r} \in \mathcal{H}$. $\square$

REMARK 7.2. From the proof note that for $x \in G$, the function $g_{x,r}$ constructed above is translation invariant in G in the sense that for $z \in G - x$, $g_{x,r}(y) = g_{x+z,r}(y + z)$ for each $y \in B_{\sqrt{r}}(x)$ and $r < \min(r_x, r_{x+z})$.

We now turn to the proof of Theorem 2.

PROOF OF THEOREM 2. Suppose that $(G, d(\cdot))$ is piecewise $\mathcal{C}^1$ with continuous reflection and Assumption 2 holds. We first show that $\mathcal{H}$ satisfies property 1 of Assumption 1. Let $x, y \in \bar{G}$ with $x \neq y$. Let r_x and $\alpha_x(\cdot)$ be as in Proposition 7.1, choose $r < r_x$ sufficiently small such that $\alpha_x(r) < |x - y|$ and let $g_{x,r}$ be the function in Proposition 7.1. Then, $g_{x,r}$ takes the value 0 at y by property (1) and it takes the value 1 at x by property (3). Thus, the function $-g_{x,r} \in \mathcal{H}$ separates x and y.

We now establish property 2 of Assumption 1. Since $d(\cdot) \cap S_1(0)$ in (10) is continuous for piecewise $\mathcal{C}^1$ domains with continuous reflection, by Remark 3.2 it suffices to show that for every $x \in \partial G \setminus \mathcal{V}$ there exists $f \in \mathcal{H}$ such that $\langle d, \nabla f(x)\rangle > 0$ for every $d \in d(x) \cap S_1(0)$. By (6), there exists $n \in n(x)$ such that $\langle d, n\rangle > 0$ for every $d \in d(x) \setminus \{0\}$. Choose $g_{x,r}$ as in Proposition 7.1 for some $r \in (0, r_x)$ and define

$$f(y) = g_{x,r}(y)(C_1 + \langle n, y\rangle), \qquad y \in \bar{G},$$

where C_1 is selected so that $C_1 + \langle n, y\rangle < 0$ on $supp[g_{x,r}]$. Then for each $y \in \partial G$ and $d \in d(y)$,

$$\langle d, \nabla f(y)\rangle = (C_1 + \langle n, y\rangle)\langle d, \nabla g_{x,r}(y)\rangle + g_{x,r}(y)\langle d, n\rangle \geq 0.$$

Thus, $f \in \mathcal{H}$. Moreover, by property (3) of $g_{x,r}$ in Proposition 7.1, we have that $g_{x,r}(x) = 1$ and $\nabla g_{x,r}(x) = 0$. This implies that

$$\inf_{d \in d(x) \cap S_1(0)} \langle d, \nabla f(x)\rangle = \inf_{d \in d(x) \cap S_1(0)} \langle d, n\rangle > 0.$$

This establishes property 2 of Assumption 1. The second part of the theorem follows directly from the boundary property stated in Proposition 6.1 and the stationarity of π. $\square$

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APPENDIX A: AN APPROXIMATION LEMMA

In this section, we prove the approximation result stated in Lemma 5.3. Let $\mathcal{Q}$ be the set of rational numbers in $\mathbb{R}$ and for $s \in \mathbb{R}$, let $\mathcal{Q}_s$ be the subset of rational numbers less than s. Since $\mathcal{V} = \{v_1, \cdots, v_K\}$ is at most finite, let $\mathcal{Q}_{\mathcal{V}} = \{r \in \mathcal{Q}^J : \mathcal{V} \subset B_r(0)\}$. For each $s \in \mathcal{Q}$, let $B(\mathcal{V}, s)$ denote the set

$$B(\mathcal{V}, s) = \{x \in \mathbb{R}^J : \operatorname{dist}(x, \mathcal{V}) \leq s\}.$$

Then there exists $s_0 > 0$ such that for every $r \in \mathcal{Q}_{s_0}$, $B(\mathcal{V}, s)$ is a disjoint union of $\{B_s(v_l) : l = 1, \cdots, K\}$. For each $r \in \mathcal{Q}_{\mathcal{V}}$ and rational number $s < s_0$, it follows from property 2 of Assumption 1 that there exists a function $h_{r,s} \in \mathcal{H} \cap \mathcal{C}_c^2(\bar{G})$ such that

(57)

$$\langle d, \nabla h_{r,s}(x)\rangle > 1 \text{ for all } d \in d(x) \cap S_1(0) \text{ and } x \in [\partial G \cap B_r(0)] \setminus B(\mathcal{V}, s).$$

Recall that $\{\phi_m \in \mathcal{C}_c^\infty(\mathbb{R}^J), m \in \mathbb{N}\}$ is a sequence of non-negative functions, where each ϕ_m satisfies $\int_{\mathbb{R}^J} \phi_m(x)\, dx = 1$ and has compact support in $B_{c_m}(0)$ and $\{c_m, m \in \mathbb{N}\}$ is a sequence such that $c_m \to 0$ as $m \to \infty$. Also, let $\mathbb{L}$ be the countable set of all polynomials with rational coefficients. Now, given any $m \in \mathbb{N}$ and $w = (r, s, \{a_\ell, \ell = 1, \ldots, K\}) \in \mathcal{Q}_{\mathcal{V}} \times \mathcal{Q}_{s_0} \times \mathcal{Q}^K$, we define the mappings $\mathcal{S}_w : \mathbb{L} \mapsto \mathbb{R}^{\mathbb{R}^J}$ and $\mathcal{S}_{w,m} : \mathbb{L} \mapsto \mathcal{C}^\infty(\mathbb{R}^J)$ as follows: given any polynomial $q \in \mathbb{L}$,

$$(\mathcal{S}_w q)(x) = \begin{cases} q(x) & \text{if } x \in (\bar{G} \cap B_r(0)) \setminus B(\mathcal{V}, s), \\ 0 & \text{if } x \in (\bar{G} \cap B_r(0))^c, \\ a_l & \text{if } x \in B_s(v_l), l = 1, \ldots, K, \end{cases}$$

and

$$(\mathcal{S}_{w,m} q) = \mathcal{S}_w * \phi_m.$$

Then clearly, $(\mathcal{S}_w q)$ is a function on $\mathbb{R}^J$ and $(\mathcal{S}_{w,m} q) \in \mathcal{C}^\infty(\mathbb{R}^J)$. Fixing $w = (r, s, \{a_\ell, \ell = 1, \ldots, K\})$ as above, without loss of generality, by taking m large enough so that $c_m < \min\{\operatorname{dist}(B_s(v_\ell), B_s(v_j)), j, \ell \in \{1, \ldots, K\}, j \neq \ell\}/2$ we can guarantee that for each $\ell = 1, \ldots, K$, $\mathcal{S}_{w,m} q(x) = a_\ell$ in an open neighborhood of v_ℓ and $(\nabla \mathcal{S}_{w,m} q)(x) = 0$ for every $x \notin \bar{G} \cap B_{r+c_m}(0)$. Now define $\mathcal{H}_0$ by

$$\mathcal{H}_0 \doteq \left\{ \begin{array}{l} \mathcal{S}_{w,m} q + \left[\sup_{x \in \partial G \cap B_{r+c_m}(0)} \sup_{d \in d(x) \cap S_1(0)} \langle d, \nabla \mathcal{S}_{w,m} q\rangle^-\right] h_{r,s} + b : \\ q \in \mathbb{L}, w = (r, s, \{a_\ell\}) \in \mathcal{Q}_{\mathcal{V}} \times \mathcal{Q}_{s_0} \times \mathcal{Q}^K, m \in \mathbb{N}, b \in \mathbb{Q} \end{array} \right\}.$$

It is easy to see that $\mathcal{H}_0$ is a countable subset of $\mathcal{H}$.

We now show that $\mathcal{H}_0$ has the required property. Fix an $f \in \mathcal{H}$. Without loss of generality, we may assume that $f \in \mathcal{C}_c^2(\bar{G})$. Fix $B_N(0)$ containing an open neighborhood of $\mathcal{V}$ and an open neighborhood of $\operatorname{supp}(f)$. For any function $h \in \mathcal{C}^2(\mathbb{R}^J)$, we define the norm

$$||h||_{\mathcal{C}^2(\bar{G} \cap B_N(0))} = \sup_{x \in \bar{G} \cap B_N(0)} \max\{|D^\beta f(x)| : |\beta| \leq 2\},$$

where $D^\beta f(x)$ is the partial derivative corresponding to the multi-index β. Given $\epsilon > 0$, by Theorem 1 of [2], there exists a sequence of polynomials $\{q^{(k)} :\ k \in \mathbb{N}\}$ in $\mathbb{L}$ and $k_0 > 0$ such that for any $k \geq k_0$,

$$||f - q^{(k)}||_{\mathcal{C}^2(\bar{G}\cap B_N(0))} \leq \frac{\varepsilon}{4}. \tag{58}$$

For each $\ell = 1, \ldots, K$, choose $a_\ell \in \mathcal{Q}$ such that $|f(v_\ell) - a_\ell| \leq \epsilon/4$, choose $s \in \mathcal{Q}_{s_0}$, $r \in \mathbb{Q}$, $r < N$ such that $B_r(0)$ contains both an open neighborhood of $\mathcal{V}$ and an open neighborhood of $\mathrm{supp}(f)$, and set $w = (r, s, \{a_\ell\})$. For $k \in \mathbb{N}$, define $\tilde{q}_w^{(k)} = \mathcal{S}_w q^{(k)}$ and $\tilde{q}_{w,m}^{(k)} = \mathcal{S}_{w,m} q^{(k)}$. Then for each $x \in \mathbb{R}^J$, for $w = (r, s, \{a_\ell\})$,

$$\tilde{q}_w^{(k)}(x) - f(x) = \begin{cases} q^{(k)}(x) - f(x) & \text{if } x \in (\bar{G} \cap B_r(0)) \setminus B(\mathcal{V}, s), \\ 0 & \text{if } x \in (\bar{G} \cap B_r(0))^c, \\ a_l - f(x) & \text{if } x \in B_s(v_l), l = 1, \cdots K. \end{cases}$$

Thus, for each $k \geq k_0$,

$$\sup_{x\in\mathbb{R}^J} |\tilde{q}_w^{(k)}(x) - f(x)| \leq \epsilon/4.$$

It follows that for each $x \in \bar{G} \cap B_N(0)$ and $k \geq k_0$,

$$\begin{aligned} |\tilde{q}_{w,m}^{(k)}(x) - q^k(x)| &\leq |(\tilde{q}_w^{(k)}(x) - f) * \phi_m(x)| + |f * \phi_m(x) - f(x)| + |f(x) - q^{(k)}(x)| \\ &\leq \varepsilon/2 + |(f * \phi_m)(x) - f(x)|. \end{aligned}$$

Since the convolution operation commutes with differentiation, analogous arguments can be used to show that the above holds with $\tilde{q}_{w,m}^{(k)}$, $q^{(k)}$, $\tilde{q}_w^{(k)}$ and f replaced by $D^\beta \tilde{q}_{w,m}^{(k)}$, $D^\beta q^{(k)}$, $D^\beta \tilde{q}_w^{(k)}$ and $D^\beta f$, respectively, for any multi-index $\beta = (\beta_1, \ldots, \beta_J)$. So, in particular,

$$||\tilde{q}_{w,m}^{(k)} - q^{(k)}||_{\mathcal{C}^2(\bar{G}\cap B_N(0))} \leq \sup_{x\in\mathcal{C}^2(\bar{G}\cap B_N(0))} ||f(x) - f * \phi(x)||_{\mathcal{C}^2(\bar{G}\cap B_N(0))} + \frac{\varepsilon}{2}.$$

On the other hand, since f is bounded and uniformly continuous, $f * \phi_m \to f$ uniformly as $m \to \infty$. Thus, for each $k \geq k_0$, we can choose m_k large enough such that $r + c_{m_k} < N$ and

$$||\tilde{q}_{w,m_k}^{(k)} - q^{(k)}||_{\mathcal{C}^2(\bar{G}\cap B_N(0))} \leq \frac{3\varepsilon}{4}.$$

Combining this with (58), we have that for each $k \geq k_0$,

$$||\tilde{q}_{w,m_k}^{(k)} - f||_{\mathcal{C}^2(\bar{G}\cap B_N(0))} \leq \varepsilon. \tag{59}$$

In addition, from (59) for each $x \in \bar{G} \cap B_N(0)$ and $d \in d(x) \cap S_1(0)$, $\langle d, \nabla f(x)\rangle \geq 0$, we have $\left\langle d, \nabla \tilde{q}^{(k)}_{w,m_k}(x)\right\rangle^- \leq \epsilon$. For each $k \geq k_0$, let

$$g_k \doteq \tilde{q}^{(k)}_{w,m_k} + \sup_{y\in\partial G\cap B_{r+cm_k}(0)} \sup_{d\in d(y),|d|=1} \left\langle d, \nabla \tilde{q}^{(k)}_{w,m_k}(y)\right\rangle^- h_{r,s}. \tag{60}$$

Then $g_k \in \mathcal{H}_0$ for each $k \geq k_0$ and $\{g_k : \ k \geq k_0\}$ satisfies (36).

For the second part of the lemma, let $\mathcal{D}_0 \doteq \{f + \sup_{y\in\mathbb{R}^J} f^-(y) : \ f \in \mathcal{H}_0\}$. Then $\mathcal{D}_0$ is a countable subset of $\mathcal{D}$. For each $f \in \mathcal{D}$, by the first part of the lemma, there exists a sequence $\{g_k : \ k \in \mathbb{N}\} \subset \mathcal{H}_0$ such that (36) holds, where g_k is given by (60). For each $k \in \mathbb{N}$, let $p_k = g_k + \sup_{y\in\mathbb{R}^J} g_k^-(y)$. Then, the sequence $\{p_k : \ k \in \mathbb{N}\} \subset \mathcal{D}_0$. Since $f \geq 0$, $\sup_{y\in\bar{G}\cap B_N(0)} g_k^-(y) \to 0$ as $k \to \infty$. For each $x \notin \bar{G} \cap B_N(0)$ and $w = (r, s, \{a_\ell\})$, we have $\tilde{q}^k_{w,m_k}(x) = 0$ and hence

$$g_k^-(x) = \sup_{y\in\partial G\cap B_{r+cm_k}(0)} \sup_{d\in d(y),|d|=1} \left\langle d, \nabla \tilde{q}^k_{w,m_k}(y)\right\rangle^- h_{r,s}(x)^-.$$

So it follows that

$$\begin{aligned}
&\lim_{k\to\infty} \sup_{x\notin\bar{G}\cap B_N(0)} g_k^-(x) \\
&\quad = \lim_{k\to\infty} \sup_{y\in\partial G\cap B_{r+cm_k}(0)} \sup_{d\in d(y),|d|=1} \left\langle d, \nabla \tilde{q}^k_{w,m_k}(y)\right\rangle^- \sup_{x\notin\bar{G}\cap B_N(0)} h_{r,s}(x)^- \\
&\quad = 0.
\end{aligned}$$

Thus, $\{p_k : \ k \geq k_0\}$ satisfies (36) with $\mathcal{D}$ in place of $\mathcal{H}$ and $\mathcal{D}_0$ in place of $\mathcal{H}_0$. This completes the proof of the second part of the lemma.

APPENDIX B: CONSTRUCTION OF TEST FUNCTIONS

In this section, we prove the existence of test functions with properties as stated in Lemma 6.3. Fix $x \in \mathcal{U} \subseteq \partial G \setminus \mathcal{V}$ and $\varepsilon \in (0, 1)$. The test function $q_{\varepsilon,x}$ will be defined in terms of the function d_ε on $(-\infty, \infty)$ given by

$$d_\varepsilon(s) = \begin{cases} s^2 & \text{if } 0 \leq s \leq \varepsilon, \\ \varepsilon^2 + \varepsilon^{3/2} - \sqrt{\varepsilon}(\varepsilon + \sqrt{\varepsilon} - s)^2 & \text{if } \varepsilon < s \leq \varepsilon + \sqrt{\varepsilon}, \\ \varepsilon^2 + \varepsilon^{3/2} & \text{if } \varepsilon + \sqrt{\varepsilon} < s, \end{cases}$$

and

$$d_\varepsilon(s) = d_\varepsilon(-s) \text{ if } s < 0.$$

We first summarize the properties of d_ε that we will require. It is easy to verify that

$$\begin{aligned}
&0 \le d_\varepsilon(s) \le \varepsilon^2 + \varepsilon^{3/2}, && \text{for } s \in (-\infty, \infty),\\
&0 \le d'_\varepsilon(s) \le 2\varepsilon, && \text{for } s \in [0, \infty),\\
&d'_\varepsilon(s) = 0, && \text{for } s > \varepsilon + \sqrt{\varepsilon}.
\end{aligned} \tag{61}$$

Also, note that $d_\varepsilon \in \mathcal{C}^1(\mathbb{R})$ and d'_ε is piecewise differentiable with the second derivative

$$d''_\varepsilon(s) = \begin{cases} 2 & \text{if } 0 \le s < \varepsilon,\\ -2\sqrt{\varepsilon} & \text{if } \varepsilon < s < \varepsilon + \sqrt{\varepsilon},\\ 0 & \text{if } \varepsilon + \sqrt{\varepsilon} < s. \end{cases}$$

We now use a standard mollification argument to construct a $\mathcal{C}^2(\mathbb{R})$ function with similar properties. Let $\{\phi_n \in \mathcal{C}_c^\infty(\mathbb{R}), n \in \mathbb{N}\}$ be a sequence of non-negative functions with $\int_{\mathbb{R}} \phi_n(x)\, dx = 1$ and compact supports that shrink to $\{0\}$. Define

$$d_\varepsilon^n \doteq \phi_n * d_\varepsilon,$$

where $*$ denotes the convolution operation. Then $d_\varepsilon^n - \varepsilon^2 - \varepsilon^{3/2} \in \mathcal{C}_c^\infty(\mathbb{R})$ and for n sufficiently large, there exist $\kappa_n(\varepsilon) > 0$ with $\lim_{n\to\infty} \kappa_n(\varepsilon) = 0$ such that

$$\begin{aligned}
&0 \le d_\varepsilon^n(s) \le \varepsilon^2 + \varepsilon^{3/2} && \text{if } s \in (-\infty, \infty),\\
&d_\varepsilon^n(s) = \varepsilon^2 + \varepsilon^{3/2} && \text{if } s \ge 2(\varepsilon + \sqrt{\varepsilon}),\\
&0 \le (d_\varepsilon^n)'(s) \le 2\varepsilon && \text{if } s \in (-\infty, \infty),\\
&(d_\varepsilon^n)''(s) = 2 && \text{if } 0 \le s \le \varepsilon/2,\\
&(d_\varepsilon^n)''(s) \ge 0 && \text{if } 0 \le s \le \varepsilon - \kappa_n(\varepsilon),\\
&|(d_\varepsilon^n)''(s)| \le 2\sqrt{\varepsilon} && \text{if } s \ge \varepsilon - \kappa_n(\varepsilon),\\
&(d_\varepsilon^n)''(s) = 0 && \text{if } s \ge 2(\varepsilon + \sqrt{\varepsilon}).
\end{aligned} \tag{62}$$

Now, for the chosen $x \in \partial G \setminus \mathcal{V}$, let r_x and $r'_x = \kappa_x(r_x)$ be the two constants in Proposition 7.1, let g_{x,r_x} be the associated function and recall from Proposition 7.1 that we can assume the function α_x satisfies $\alpha_x(r) = r$. Also, let g^x be the function defined in (49), with the associated $\theta_i(x), i \in \mathcal{I}(x)$. Choose n sufficiently large so that (62) holds and for each $y \in \mathbb{R}^J$, let

$$p_{\varepsilon,x}(y) \doteq d_\varepsilon^n(g^x(y)) \text{ and } q_{\varepsilon,x}(y) \doteq (p_{\varepsilon,x}(y) - \varepsilon^2 - \varepsilon^{3/2}) g_{x,r_x}(y).$$

It follows from the properties of g_{x,r_x} and (62) that $q_{\varepsilon,x} \in \mathcal{C}_c^2(\bar{G})$, $\bar{G} \cap \operatorname{supp}[q_{\varepsilon,x}] \subset \bar{G} \cap B_{r_x}(x)$, $-\varepsilon^2 - \varepsilon^{3/2} \le q_{\varepsilon,x} \le 0$, and

$$q_{\varepsilon,x}(y) = p_{\varepsilon,x}(y) - \varepsilon^2 - \varepsilon^{3/2}, \quad y \in \bar{G} \cap B_{r'_x}(x). \tag{63}$$

An elementary calculation shows that for $y \in \mathbb{R}^J$,

$$\nabla q_{\varepsilon,x}(y) = \nabla p_{\varepsilon,x}(y) g_{x,r_x}(y) + (p_{\varepsilon,x}(y) - \varepsilon^2 - \varepsilon^{3/2}) \nabla g_{x,r_x}(y), \tag{64}$$

where

$$\nabla p_{\varepsilon,x}(y) = (d_\varepsilon^n)'(g^x(y)) \sum_{i \in \mathcal{I}(x)} \frac{\theta_i(x)}{|\nabla \phi^i(x)|} \nabla \phi^i(y). \tag{65}$$

Since $(d_\varepsilon^n)' \geq 0$ by (62), $\theta_i(x) \geq 0$ and $\nabla \phi^i$ is proportional to n^i, it follows that $\langle d, \nabla p_{\varepsilon,x}(y) \rangle \geq 0$ for each $d \in d(y)$ and $y \in \mathcal{U}$. When combined with the facts that the function g_{x,r_x} is nonnegative, the function $p_{\varepsilon,x} - \varepsilon^2 - \varepsilon^{3/2}$ is nonpositive and $-g_{x,r_x}$ lies in $\mathcal{H}$, this implies that $\langle d, \nabla q_{\varepsilon,x}(y) \rangle \geq 0$ for each $d \in d(y)$ and $y \in \mathcal{U}$. Thus, $q_{\varepsilon,x} \in \mathcal{H}$.

Next, note that $\nabla \phi^i$ is bounded on the support of g_{x,r_x}, $0 \leq g_{x,r_x} \leq 1$ by property (3) of Proposition 7.1 and $d'_\varepsilon(s) \in [0, 2\varepsilon]$ by (61), it follows that there exists $\tilde{c}_x < \infty$ such that $|\nabla p_{\varepsilon,x}(y) g_{x,r_x}(y)| \leq \tilde{c}_x \varepsilon$ for all $y \in \mathbb{R}^J$. On the other hand, since $g_{x,r_x} \in \mathcal{C}_c^2(\bar{G})$, $\nabla g_{x,r_x}$ is bounded (say by M_x), and it follows that $||p_{\varepsilon,x} \nabla g_{x,r_x}||_\infty \leq M_x \varepsilon^2 \leq M_x \varepsilon$ if $\varepsilon < 1$. This implies that $||\nabla q_{\varepsilon,x}||_\infty \leq \tilde{a}_x \varepsilon$ for some $\tilde{a}_x < \infty$. Moreover, due to (63), for every $y \in \bar{G} \cap B_{r'_x}(x)$,

$$\begin{aligned} \sum_{i,j=1}^J a_{ij}(y) \frac{\partial^2 q_{\varepsilon,x}}{\partial x_i \partial x_j}(y) &= \sum_{i,j=1}^J a_{ij}(y) \frac{\partial^2 p_{\varepsilon,x}}{\partial x_i \partial x_j}(y) \\ &= (d_\varepsilon^n)''(g^x(y)) \langle \nabla g^x(y), a(y) \nabla g^x(y) \rangle \\ &\quad + (d_\varepsilon^n)'(g^x(y)) \sum_{i,j=1}^J a_{ij}(y) \frac{\partial^2 g^x}{\partial x_i \partial x_j}(y). \end{aligned}$$

By the fourth property in (62), the second property in (61), the uniform ellipticity of a and the bound on the second derivatives of g^x on $B_{r'_x}(x)$, by redefining c_x to be larger if necessary, we deduce that if $0 \leq g^x(y) \leq \varepsilon/2, y \in \bar{G} \cap B_{r'_x}(x)$, then

$$\begin{aligned} \sum_{i,j=1}^J a_{ij}(y) \frac{\partial^2 q_{\varepsilon,x}(y)}{\partial x_i \partial x_j} &\geq 2\alpha |\nabla g^x(y)| - 2\varepsilon \left| \sum_{i,j=1}^J a_{ij}(y) \frac{\partial^2 g^x(y)}{\partial x_i \partial x_j} \right| \\ &\geq 2\alpha \beta_x - c_x \varepsilon, \end{aligned} \tag{66}$$

where α is the positive constant in (12) and $\beta_x = \inf_{y \in \bar{G} \cap B_{r'_x}(x)} |\nabla g^x(y)| > 0$. Moreover, by the third and sixth properties in (62), it is clear that, by choosing c_x yet larger if necessary, for each $y \in \bar{G} \cap B_{r'_x}(x)$ with $g^x(y) \geq \varepsilon - \kappa_n(\varepsilon)$,

$$\begin{aligned}
&\left\| \sum_{i,j=1}^{J} a_{ij}(y) \frac{\partial^2 q_{\varepsilon,x}(y)}{\partial x_i \partial x_j} \right\| \\
&\leq 2\sqrt{\varepsilon} |\langle \nabla g^x(y), a(y) \nabla g^x(y) \rangle| + 2\varepsilon \left| \sum_{i,j=1}^{J} a_{ij}(y) \frac{\partial^2 g^x(y)}{\partial x_i \partial x_j} \right| \\
&\leq c_x \sqrt{\varepsilon}.
\end{aligned}$$

The fifth property in (62) shows that when $y \in \bar{G} \cap B_{r'_x}(x)$ and $\varepsilon/2 < g^x(y) < \varepsilon - \kappa_n(\varepsilon)$, then

$$\sum_{i,j=1}^{J} a_{ij}(y) \frac{\partial^2 q_{\varepsilon,x}(y)}{\partial x_i \partial x_j} \geq -2\varepsilon \left| \sum_{i,j=1}^{J} a_{ij}(y) \frac{\partial^2 g^x(y)}{\partial x_i \partial x_j} \right| \geq -c_x \varepsilon.$$

This completes the proof of the lemma.

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