Directional Derivatives of Oblique Reflection Maps

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Given an oblique reflection map Γ and functions $\psi, \chi \in \mathcal{D}_{\lim}$ (the space of $\mathbb{R}^K$-valued functions that have finite left and right limits at every point), the directional derivative $\nabla_\chi \Gamma(\psi)$ of Γ along χ, evaluated at ψ, is defined to be the pointwise limit, as $\varepsilon \downarrow 0$, of the family of functions $\nabla^\varepsilon_\chi \Gamma(\psi) \doteq \varepsilon^{-1} [\Gamma(\psi + \varepsilon\chi) - \Gamma(\psi)]$. Directional derivatives are shown to exist and lie in $\mathcal{D}_{\lim}$ for oblique reflection maps associated with reflection matrices of the so-called Harrison-Reiman class. When ψ and χ are continuous, the convergence of $\nabla^\varepsilon_\chi \Gamma(\psi)$ to $\nabla_\chi \Gamma(\psi)$ is shown to be uniform on compact subsets of continuity points of the limit $\nabla_\chi \Gamma(\psi)$ and the derivative $\nabla_\chi \Gamma(\psi)$ is shown to have an autonomous characterization as the unique fixed point of an associated map. Directional derivatives arise as functional central limit approximations to time-inhomogeneous queueing networks. In this case ψ and χ correspond, respectively, to the functional strong law of large numbers and functional central limits of the so-called netput process. In this work it is also shown how the various types of discontinuities of the derivative $\nabla_\chi \Gamma(\psi)$ are related to the reflection matrix and properties of the function $\Gamma(\psi)$. In the queueing network context, this describes the influence of the topology of the network and the states (of underloading, overloading or criticality) of the various queues in the network on the discontinuities of the directional derivative. Directional derivatives have also been found useful for identifying optimal controls for fluid approximations of time-inhomogoeneous queueing networks and are also of relevance to the study of differentiability of stochastic flows of obliquely reflected Brownian motions.

Key words: reflection map; Skorokhod map; time-inhomogeneous queueing networks; non-stationarity; time-varying rates; diffusion limits; heavy traffic; directional derivatives; differentiability of stochastic flows of reflected diffusions

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1. Introduction

1.1 Background and Motivation Most real-world queueing systems are time-inhomogeneous in the sense that they evolve according to transition laws that themselves vary with time. However, the majority of queueing research has been devoted to time-homogeneous models, in which the transition laws are assumed to be independent of time. While such models may provide reasonable approximations for slowly varying systems, they completely fail to capture many important phenomena such as surges in demand, sudden node failures and periodicity. The explicit analysis of even time-homogeneous networks is usually intractable. Instead, one usually resorts to appropriate asymptotic approximations that capture the essential features of network behavior that are of interest. A commonly used asymptotic scaling is one in which arrival and service rates are scaled proportionately, but the number of servers at each queue is kept constant. Over the past two decades, much progress has been made on this kind of approximation for time-homogeneous networks with fairly general arrival, service and routing processes that satisfy a so-called heavy-traffic condition. In particular, under an additional initial assumption on the queues that guarantees that the first-order asymptotic limit (or fluid limit) is trivially zero, it has been shown that the second-order asymptotic limits associated with various classes of of time-homogeneous queueing networks are reflected Brownian motions (RBMs) or reflected Lévy processes (see, for example, [4, 14, 23, 24, 25, 32] and references therein). In contrast, the analysis of time-inhomogeneous networks remains challenging even in a Markovian setting. In particular, there has been relatively little work done on second-order approximations to time-inhomogeneous queueing networks with a fixed number of servers. Such networks arise frequently as models of transportation, telecommunication and computer systems; see [9, 16, 20].

The single queue with time-varying arrival and service rates has been studied by various authors under different assumptions [12, 17, 18, 19, 26, 27]. The detailed asymptotic analysis carried out in Mandelbaum and Massey [17] is pathwise and uses strong approximations. It shows that the so-called fluid limit or first-order approximation of a time-dependent Markovian queue alternates between phases of overloading, critical loading and underloading and that the second-order correction to the fluid limit can have discontinuous paths and exhibits different characteristics in each of the three different phases of loading. This second-order correction admits an interpretation as the directional derivative of the one-dimensional reflection map Γ. It is natural to expect that such an interpretation would continue to hold in the network setting, in the sense that the corresponding second-order corrections in the asymptotic approximations to a class of time-inhomogeneous networks would take the form of directional derivatives of associated multi-dimensional reflection maps (see Section 2.1 for a formal discussion of this connection). The main objectives of this work are to introduce and characterize properties of directional derivatives of the class of so-called Harrison-Reiman multi-dimensional reflection maps (which are associated with single-class open queueing networks), and to illustrate the practical insights that can be obtained from such an analysis.

The representation obtained in Mandelbaum and Massey [17] for the directional derivative of the one-dimensional reflection map Γ relied heavily on the following explicit form for Γ obtained by Skorokhod:

$$\Gamma(\psi)(t) = \psi(t) + \theta(t) \tag{1}$$

for càdlàg functions ψ, where the constraining term θ that keeps $\Gamma(\psi)$ non-negative is given by

$$\theta(t) = \max\left(\sup_{s\in[0,t]} [-\psi(s)], 0\right). \tag{2}$$

In contrast, in the multi-dimensional setting there is no explicit expression for the oblique reflection map, making characterization of its directional derivatives considerably more involved. In fact, derivatives of reflection maps associated with even feedforward tandem networks cannot always be expressed simply as a composition of directional derivatives of one-dimensional reflection maps (see Section 3.3.2 for further discussion of this issue). The network setting also introduces additional complications due to the dependence on network topology and leads to interesting new questions about when and how effects propagate through the network. Consequently, new techniques need to be developed to analyze derivatives of multi-dimensional reflection maps.

Another motivation for studying directional derivatives arises from the fact that, as shown in Cudina and Ramanan [5], they are useful for the identification of optimal controls for fluid approximations to time-inhomogeneous networks. Directional derivatives of multi-dimensional reflection maps are also potentially useful for the study of differentiability of stochastic flows of multi-dimensional reflected diffusions in non-smooth domains (see, for example, Deuschel and Zambotti [6] for the case of normal reflection in $\mathbb{R}_+^K$).

1.2 Outline of the Paper The outline of the rest of the paper is as follows. The basic notation used throughout the paper is first collected in Section 1.3. In Section 1.4 the definitions of the multi-dimensional oblique reflection map and its directional derivative are introduced. The main results of the paper, Theorems 1.1 and 1.2, are presented in Section 1.5. Theorem 1.1 establishes the existence of directional derivatives $\nabla_\chi\Gamma(\psi)$ of Harrison-Reiman reflection maps and, under additional conditions on ψ and χ, also provides an autonomous characterization of the derivative. Theorem 1.2 derives necessary conditions for the existence of discontinuities in the directional derivative when ψ and χ are continuous. In the queueing network context, ψ and χ correspond, respectively, to the functional strong law of large numbers and functional central limits of the so-called netput process. Indeed, Section 2 contains a brief discussion of the connection between approximations to time-inhomogeneous queueing networks and directional derivatives of multi-dimensional reflection maps. The examples presented in the section show that the directional derivative can be explicitly calculated in many cases and also illustrate some interesting features that arise in the multi-dimensional or network context. In the study of the optimality of fluid limits of time-inhomogeneous networks (see, e.g., Cudina and Ramanan [5]), ψ typically represents the fluid limit of the netput process that gives rise to the optimal path for a given control problem, while χ corresponds to an arbitrary allowable perturbation of the path. On the other hand, in the context of differentiability of stochastic flows, typically ψ is a sample path of a diffusion process and χ is a vector representing the difference in two initial conditions. The rest of the paper is essentially devoted to proving the two main results. General properties of Harrison-Reiman maps are summarized in Section 3.1 and the existence of the directional derivative is established in Section 3.2, with the proof of Theorem 1.1 presented in Section 3.4. Important ingredients of this proof are the notion of a generalized one-dimensional derivative (which is introduced in Section 3.3) and the representation of the one-dimensional derivative obtained in Theorem 3.2 (whose proof is given in Section 5.2). The proof also relies on an auxiliary result, which is established in Section 5.1. In Section 4 the discontinuities of the directional derivative are analyzed when ψ and χ are continuous, culminating in the proof of Theorem 1.2 in Section 4.3.

1.3 Basic Notation In this section, for convenience, we compile all the common notation used throughout the paper. For $a, b \in \mathbb{R}$, let $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. Given a vector $x \in \mathbb{R}^K$, x^i or $[x]^i$ is used to denote the ith component of the vector. For $a \in \mathbb{R}^K$, the norm $|a|$ is defined by

$$|a| \doteq \max_{i=1,\dots,K} |a^i| \tag{3}$$

where, for $a^i \in \mathbb{R}$, $|a^i|$ denotes the absolute value of a_i. Given a $K \times K$ matrix R, R^T denotes its transpose, $\sigma(R)$ its spectral radius and R_{ij} represents the entry in the ith row and jth column of R. The matrix I represents the $K \times K$ identity matrix, and $\{e_i,\ i = 1, \dots, K\}$ is the standard orthonormal basis in $\mathbb{R}^K$. Inequalities of vectors and matrices should be interpreted componentwise. Vectors are always expressed as column vectors. The K-dimensional orthant is denoted by $\mathbb{R}^K_+$:

$$\mathbb{R}^K_+ \doteq \left\{x \in \mathbb{R}^K : x^i \geq 0 \text{ for every } i = 1, \dots, K\right\}. \tag{4}$$

The notation $\uparrow$ (respectively, $\downarrow$) is used to denote monotone nondecreasing (respectively, nonincreasing) convergence of a family of real numbers to a limit. We adopt the convention that the infimum and supremum of an empty set are ∞ and $-\infty$, respectively. The notation 0 is used to denote both the number zero as well as the identically zero function – the use should be clear from the context.

Given a function f on $[0, \infty)$ that takes values in $\mathbb{R}^K$, f^i denotes the ith coordinate function. For any $\mathbb{R}^K$-valued function f and $T < \infty$, $||f||_T$ denotes the supremum norm: $||f||_T \doteq \sup_{s\in[0,T]} |f(s)|$, where $|\cdot|$ is the norm defined above in (3). In addition, the notation $\overline{f}$ is used to denote the supremum function:

$$\overline{f}(t) \doteq \sup_{s\in[0,t]} f(s). \tag{5}$$

The analysis in this paper involves the use of many different functions spaces, which are summarized below:

$\mathcal{D}_{\lim}$	the space of all functions on $[0, \infty)$ taking values in $\mathbb{R}^K$ that have finite left and right limits for every $t \in [0, \infty)$;
$\mathcal{D}^+_{\lim}$	the subspace of functions $f \in \mathcal{D}_{\lim}$ with $f(0) \in \mathbb{R}^K_+$;
$\mathcal{D}_r$	the subspace of right continuous functions in $\mathcal{D}_{\lim}$;
$\mathcal{D}_{\ell,r}$	the subspace of functions that are either right continuous or left continuous at every $t \in [0, \infty)$;
$\mathcal{D}_{usc}$	the subspace of functions in $\mathcal{D}_{\lim}$ such that each coordinate function f^i is upper semicontinuous (i.e., $f(t) \geq f(t-) \vee f(t+)$ for every $t \in [0, \infty)$);
$\mathcal{D}_{c,\lim}$	the subspace of piecewise constant functions in $\mathcal{D}_{\lim}$ with a finite number of jumps;
$\mathcal{D}_c$	the subspace of piecewise constant functions in $\mathcal{D}_r$ with a finite number of jumps;
$\mathcal{I}^+$	the subspace of functions in $\mathcal{D}^+_{\lim}$ such that each coordinate function is non-decreasing;
$\mathcal{I}^+_0$	the subspace of functions $f \in \mathcal{I}^+$ such that $f(0) = 0$;
$\mathcal{C}$	the subspace of continuous functions in $\mathcal{D}_{\lim}$;
$\mathcal{BV}$	the subspace of functions in $\mathcal{D}_{\lim}$ that have bounded variation on every bounded interval of $[0, \infty)$.

When the functions take values in $\mathbb{R}$ instead of $\mathbb{R}^K$, then we will emphasize this by writing $\mathcal{D}_{\lim}(\mathbb{R})$, $\mathcal{C}(\mathbb{R})$, etc. For $f \in \mathcal{BV}$, $|f|_t$ denotes the total variation norm on $[0, t]$, with respect to the norm $|\cdot|$ on $\mathbb{R}^K$ defined in (3). A function $f \in \mathcal{D}_{\lim}$ is said to have a *separated discontinuity* at a point $t \in [0, \infty)$ if for some $i = 1, \dots, K$, $f^i(t)$ does not lie in the interval created by $f^i(t-)$ and $f^i(t+)$: that is, if

$$f^i(t) \notin [f^i(t-) \wedge f^i(t+), f^i(t-) \vee f^i(t+)].$$

For $f \in \mathcal{D}_{\lim}$, let $Disc(f)$ (respectively $LDisc(f)$, $RDisc(f)$ and $SDisc(f)$) denote the set of points of discontinuity (respectively left discontinuity, right discontinuity and separated discontinuity) of f. Clearly, $Disc(f) = LDisc(f) \cup RDisc(f)$, and for $f \in \mathcal{D}_{usc}$, it is easy to see that $SDisc(f) = LDisc(f) \cap RDisc(f)$.

The left and right regularizations of any function $g \in \mathcal{D}_{\lim}$, denoted by g_l and g_r respectively, are defined by

$$g_l(s) \doteq g(s-) \qquad \text{and} \qquad g_r(s) \doteq g(s+), \qquad s \in [0,\infty). \tag{6}$$

It is easy to see that $g_l(s-) = g_l(s) = g(s-)$ and $g_l(s+) = g(s+)$, and likewise $g_r(s+) = g_r(s) = g(s+)$ and $g_r(s-) = g(s-)$. Thus $g_l \in \mathcal{D}_l$, $g_r \in \mathcal{D}_r$ and

$$g \in \mathcal{D}_l \Rightarrow g_l = g, \qquad \text{and} \qquad g \in \mathcal{D}_r \Rightarrow g_r = g. \tag{7}$$

Lastly, given a real-valued function f, a point $t \in [0,\infty)$ is said to be a point of strict left increase if there exists $\delta > 0$ such that $f(s) < f(t)$ for every $s \in [(t-\delta)^+, t)$, and of strict right increase if there exists $\delta > 0$ such that $f(t) < f(s)$ for every $s \in (t, t+\delta]$. Moreover, f is said to be flat to the left of t, which is represented by the notation $\Delta f(t-) = 0$, if there exists $\delta \in (0,t)$ such that $f(s) = f(t)$ for all $s \in (t-\delta, t]$ and, analogously, f is said to be flat to the right of t, which is denoted by $\Delta f(t+) = 0$, if there exists $\delta > 0$ such that $f(s) = f(t)$ for all $s \in [t, t+\delta)$. We will also use the shorthand notations $\Delta f(t-) \neq 0$ and $\Delta f(t+) \neq 0$, respectively, to denote the fact that f is not flat to the left and right of t.

1.4 Definition of the Oblique Reflection Map and its Directional Derivative In this section we state the precise definitions of the oblique reflection map and its directional derivatives. Let $R \in \mathbb{R}^{K\times K}$ be a matrix whose ith column is the vector r_i, which represents the constraint direction on the face $F_i = \{x \in \mathbb{R}^K_+ : x^i = 0\}$ of the boundary of the non-negative orthant $\mathbb{R}^K_+$. Roughly speaking, given a trajectory $\psi \in \mathcal{D}_{\lim}$, the oblique reflection problem (ORP) associated with the constraint matrix R defines a constrained version ϕ of ψ that is restricted to live in $\mathbb{R}^K_+$ by a constraining term that pushes along the direction r_i only when ϕ lies on the face F_i. We will assume that for every $i = 1,\ldots,K$, $R_{ii} = r^i_i > 0$. This ensures that from any point in the relative interior of the face F_i, the vector r_i points into the orthant $\mathbb{R}^K_+$. This condition is without loss of generality since it is clearly a necessary condition for the existence of a constrained version ϕ of ψ that takes values in $\mathbb{R}^K_+$. The rigorous definition of the ORP is as follows. Recall the definitions of $\mathcal{D}^+_{\lim}$ and $\mathcal{I}^+_0$ given in Section 1.3.

Definition 1.1 (Oblique Reflection Problem) *Given $R \in \mathbb{R}^{K\times K}$ with $R_{ii} > 0$ for $i = 1,\ldots,K$ and $\psi \in \mathcal{D}^+_{\lim}$, $(\phi,\theta) \in \mathcal{D}^+_{\lim} \times \mathcal{I}^+_0$ solve the oblique reflection problem associated with the constraint matrix R for ψ if $\phi(0) = \psi(0)$, and if for all $t \in [0,\infty)$,*

(*i*) $\phi(t) \in \mathbb{R}^K_+$;

(*ii*) $\phi(t) = \psi(t) + R\theta(t)$, *where for every $i = 1,\ldots,K$*

$$\int_{(0,t]} 1_{(0,\infty)}(\phi^i(s))d\theta^i(s) = 0. \tag{8}$$

The condition $\phi(0) = \psi(0)$ is imposed for simplicity; it can be relaxed by allowing a jump in θ at 0. Note that the condition (8) simply states that the constraining term θ^i can increase at time t only if $\phi^i(t) = 0$. From the definition above it is clear that one can without loss of generality assume that $R_{ii} = 1$ for $i = 1,\ldots,K$. Indeed, we shall assume this normalization throughout the rest of the paper. When a unique solution to the ORP exists for every $\psi \in \mathcal{D}^+_{\lim}$, we say the ORP is well-defined and refer to the mapping $\Gamma : \psi \to \phi$ as the *reflection map* (RM). We also use $\Theta : \psi \to \theta$ to denote the mapping that takes ψ to the corresponding constraining term θ.

In this work we focus mainly on oblique reflection problems (ORPs) associated with reflection matrices R that satisfy the so-called Harrison-Reiman (H-R) condition stated below as Definition 1.2, which was first introduced by Harrison and Reiman [11]. As shown in Theorem 3.1, ORPs in this class are well-defined, and in fact have Lipschitz continuous RMs (with respect to the uniform topology on path space on both the domain and range).

Definition 1.2 *(H-R condition) A constraint matrix $R \in \mathbb{R}^{K\times K}$ is said to satisfy the H-R condition if $P \doteq I - R \geq 0$ and the spectral radius of the matrix P is less than one.*

Remark 1.1 If R satisfies the H-R condition and $P \doteq I - R$, then there exists a diagonal matrix A with strictly positive diagonal elements such that each row sum of the matrix $\tilde{P} \doteq A^{-1}PA$ is strictly less than 1 (see Lemma 3 of Veinnot [31]).

Remark 1.2 The ORP was introduced by Harrison and Reiman [11] to characterize functional central limits of single-class open queueing networks (see Figure 3). Single-class open queueing networks with K queues in which, on average, a fraction q_{ij} of the departures from queue i are sent to queue j, and a fraction $1-\sum_{j=1}^K q_{ij}$ of the departures from queue i exit the network give rise to ORPs with an $\mathbb{R}^{K\times K}$ constraint matrix R given by

$$R_{ij} \doteq \begin{cases} -q_{ji}, & j \neq i, \\ \quad 1 & \textit{otherwise.} \end{cases}$$

We now precisely state what we mean by a directional derivative of the multi-dimensional reflection map.

Definition 1.3 (Directional Derivative) *Consider an ORP whose reflection map Γ is well defined on $\mathcal{D}_{\lim}$. Given paths $\psi \in \mathcal{D}^+_{\lim}, \chi \in \mathcal{D}_{\lim}$, define*

$$\nabla^\varepsilon_\chi \Gamma(\psi) \doteq \frac{1}{\varepsilon}\left[\Gamma(\psi+\varepsilon\chi) - \Gamma(\psi)\right], \qquad \varepsilon > 0. \tag{9}$$

The derivative of Γ along χ evaluated at ψ is the pointwise limit of the sequence $\{\nabla^\varepsilon_\chi\Gamma(\psi)\}$, as $\varepsilon \downarrow 0$.

1.5 Main Results of the Paper

1.5.1 Existence of the derivative Mandelbaum and Massey [17] showed that when Γ is the one-dimensional RM, ψ, χ are continuous, and $\Gamma(\psi)(0) = \psi(0) = 0$, then the directional derivative has the explicit form

$$\nabla_\chi\Gamma(\psi)(t) = \chi(t) + \sup_{s\in\Phi(t)}[-\chi(s)] \vee 0, \tag{10}$$

where

$$\Phi(t) \doteq \{s \in [0,t] : \phi(s) = 0 \text{ and } \theta(s) = \theta(t)\},$$

with $\phi = \Gamma(\psi)$ and θ defined as in (1) and (2), respectively. In reality, this was shown by Mandelbaum and Massey [17] under the additional restrictions that $\nabla_\chi\Gamma(\psi)$ has only a finite number of discontinuities in any compact interval and $\phi(0) = 0$. However, as shown in Theorem 3.2 (see also Whitt [33, Theorem 9.3.1]), these conditions can be relaxed. When ψ is the fluid limit of the netput process and χ is the functional central limit of the (scaled and centered) netput process associated with a time-varying queue, $\nabla_\chi\Gamma(\psi)$ characterizes the second-order approximation to the time-varying queue. In this case, $\phi = \Gamma(\psi)$ has an interpretation as the fluid limit of the queue and θ as the corresponding cumulative potential outflow lost (due to idleness of the server) during the period $[0,t]$. Thus, in this context, $\Phi(t)$ represents the set of all times s in the interval $[0,t]$ when the fluid queue was zero, but the server was fully utilized in the interval $[s,t]$. Observe that when $\phi(0) = 0$, due to the representation for the one-dimensional RM given in (1) and (2), we have $\Gamma(\psi)(t) = \psi(t) + \theta(t) = \psi(t) + \overline{-\psi}(t) \geq 0$ for every $t \in [0,\infty)$. Thus, $\Phi(t)$ can be rewritten as $\Phi(t) = \Phi_{-\psi}(t)$, where for $f \in \mathcal{D}_{\lim}$, we set

$$\Phi_f(t) \doteq \{s \in [0,t] : f(s) = \bar{f}(t)\}. \tag{11}$$

When $\psi, \chi \in \mathcal{D}_{\lim}$ are not necessarily continuous, the directional derivative of the one-dimensional RM can be shown to still exist (see Theorem 3.2) but, in addition to sets of the form Φ_f, its explicit representation also involves sets of the form

$$\Phi^L_f(t) \doteq \{s \in [0,t] : f(s-) = \bar{f}(t)\}, \tag{12}$$

$$\tilde{\Phi}^R_f(t) \doteq \{s \in [0,t) : f(s+) = \bar{f}(t)\}. \tag{13}$$

Now, consider the multi-dimensional setting when Γ is the RM associated with an ORP that has an H-R constraint matrix $R \in \mathbb{R}^{K\times K}$ and (ϕ,θ) solve the ORP for a given $\psi \in \mathcal{D}_{\lim}$. When the matrix R is associated with an open queueing network of K queues, for $i = 1,\ldots,K$, θ^i represents the cumulative potential outflow lost from the ith queue during $[0,t]$, and the set

$$\Phi^i(t) \doteq \left\{s \in [0,t] : \phi^i(s) = 0 \text{ and } \theta^i(s) = \theta^i(t)\right\} \tag{14}$$

represents the times $s \in [0,t]$ at which the ith fluid queue is zero but the ith server is fully utilized during $[s,t]$. As stated below in Theorem 1.1, when ψ and χ are continuous, the directional derivative in the multi-dimensional case can be expressed in terms of these sets. The proof of Theorem 1.1 is given in Section 3.4. Recall, from Section 1.3, that $\Delta f(t+) \neq 0$ denotes the condition that the function f is not flat to the right of t, and also recall the convention that $\inf\emptyset = \infty$.

THEOREM 1.1 (EXISTENCE AND CHARACTERIZATION OF THE DERIVATIVE) *Let $R \in \mathbb{R}^{K\times K}$ be a reflection matrix that satisfies the H-R condition stated in Definition 1.2, let $P \doteq I - R$ and let Γ be the associated RM. Then the following properties hold.*

(*i*) *Given $\psi, \chi \in \mathcal{D}_{\lim}$, the directional derivative $\nabla_\chi \Gamma(\psi)$ exists and lies in $\mathcal{D}_{\lim}$. In addition, for every $\psi \in \mathcal{D}_{\lim}$, the derivative $\nabla_\chi\Gamma(\psi)$ is Lipschitz in χ (with respect to the uniform topology on both the domain and range). Furthermore, the following scaling property is satisfied: for every $\alpha, \beta > 0$,*

$$\nabla_{\alpha\chi}\Gamma(\beta\psi) = \alpha\nabla_\chi\Gamma(\psi). \tag{15}$$

(*ii*) *When $\psi, \chi \in \mathcal{C}$, the convergence of $\nabla^\varepsilon_\chi\Gamma(\psi)$ to $\nabla_\chi\Gamma(\psi)$ is uniform on compact subsets of continuity points of $\nabla_\chi\Gamma(\psi)$. Moreover, if (ϕ,θ) solve the ORP for ψ, then*

$$\nabla_\chi\Gamma(\psi) = \chi + R\gamma_{(1)}(\psi,\chi), \tag{16}$$

where $\gamma_{(1)} \doteq \gamma_{(1)}(\psi,\chi)$ lies in $\mathcal{D}_{usc}$ and is the unique solution to the system of equations

$$\gamma^i(t) = \begin{cases} 0 & \text{if } t \in (0, t^i_l), \\ \sup_{s\in\Phi^i(t)} \left[-\chi^i(s) + [P\gamma]^i(s)\right] \vee 0 & \text{if } t \in [t^i_l, t^i_u], \\ \sup_{s\in\Phi^i(t)} \left[-\chi^i(s) + [P\gamma]^i(s)\right] & \text{if } t \in (t^i_u, \infty), \end{cases} \tag{17}$$

for $i = 1,\ldots,K$, with Φ^i defined as in (14) and

$$t^i_l \doteq \inf\{t \geq 0 : \phi^i(t) = 0\}, \tag{18}$$
$$t^i_u \doteq \inf\left\{t \geq 0 : \theta^i(t) > 0\right\}. \tag{19}$$

REMARK 1.3 Note that the complementarity condition (8) ensures that $t^i_u \geq t^i_\ell$. Also, the case when ψ, χ are continuous and $\phi(0) = 0$, $\theta(t) = 0$ for every $t \in (0,\infty)$, corresponds to the case when all fluid queues are initially empty and are subsequently always in heavy traffic or, equivalently, are always critically loaded. In this case, $t^i_\ell = 0$, $t^i_u = \infty$, $\Phi^i(t) = [0,t]$, $t \in [0,\infty)$, for $i = 1,\ldots,K$, and hence $\gamma_{(1)}$ is the unique solution to the system of equations

$$\gamma^i(t) = \sup_{s\in[0,t]} \left[-\chi^i(s) + [P\gamma]^i(s)\right] \vee 0, \qquad i = 1,\ldots,K.$$

It then follows from Theorem 3.1 (see also equations (13)-(15) and Theorem 1 of Harrison and Reiman [11]) that the derivative is simply the reflected or constrained version of χ:

$$\nabla_\chi\Gamma(\psi) = \chi + R\gamma_{(1)} = \Gamma(\chi),$$

which is consistent with the well-known reflected Brownian motion characterization of heavy traffic limits of stationary open single-class queueing networks (see, [11, 25]).

1.5.2 Discontinuities of the derivative $\nabla_\chi\Gamma(\psi)$ for continuous ψ, χ Theorem 1.1 shows that even when $\psi, \chi \in \mathcal{C}$, convergence of $\nabla^\varepsilon_\chi\Gamma(\psi)$ to $\nabla_\chi\Gamma(\psi)$ is pointwise and is uniform only on compact subsets of continuity points of the derivative $\nabla_\chi\Gamma(\psi)$. In order to establish functional central limit theorems for non-stationary queueing networks, it would be useful to establish convergence with respect to stronger topologies than the pointwise topology. This requires an understanding of the structure of the discontinuities of $\nabla_\chi\Gamma(\psi)$. The next main result of the paper, Theorem 1.2, describes the various types of discontinuities exhibited by the derivative. It turns out that discontinuities in $\nabla_\chi\Gamma(\psi)$ can occur only at points at which there is a change in certain regimes associated with the solution (ϕ,θ) to the ORP with input ψ. These regimes, which are introduced in Definition 1.4 below, are described in terms of the following set-valued functions. For $t \in [0,\infty)$, define

$$\begin{aligned} \mathcal{O}(t) &\doteq \{i \in \{1,\ldots,K\} : \phi^i(t) > 0\}, \\ \mathcal{U}(t) &\doteq \{i \in \{1,\ldots,K\} : \phi^i(t) = 0, \Delta\theta^i(t+) \neq 0, \Delta\theta^i(t-) \neq 0\}, \\ \mathcal{C}(t) &\doteq \{1,\ldots,K\} \setminus [\mathcal{O}(t) \cup \mathcal{U}(t)] \\ \mathcal{EO}(t) &\doteq \{i \in \mathcal{C}(t) : \exists\delta > 0 \text{ such that } \phi^i(s) > 0 \quad \forall s \in (t-\delta, t)\} \\ \mathcal{SU}(t) &\doteq \{i \in \mathcal{C}(t) : \Delta\theta^i(t-) = 0, \Delta\theta^i(t+) \neq 0\}. \end{aligned} \tag{20}$$

When ψ is the fluid limit of the so-called netput process associated with a queueing network that is modelled by the ORP, then $\mathcal{O}(t)$ represents the set of queues that are overloaded at time t, $\mathcal{U}(t)$ is the set of queues that are underloaded (and therefore idling) at time t and $\mathcal{C}(t)$ is the set of queues that are critical at time t (a critical queue is one that is empty, but whose server is working at full capacity). Moreover, $\mathcal{SU}(t)$ represents the set of queues that are at the start of underloading and $\mathcal{EO}(t)$ the set of queues that are at the end of overloading at time t. The terminology used in the following definition relies on this interpretation of the various regimes of (ϕ, θ).

Definition 1.4 (Regimes of (ϕ, θ)) *Given an ORP associated with an H-R reflection matrix, let (ϕ, θ) be the solution to the ORP for a given input trajectory $\psi \in \mathcal{D}_{\lim}$. Then $i \in \mathcal{I}$ is said to be overloaded (respectively critical, underloaded, at the start of underloading, at the end of overloading) at time t if and only if $i \in \mathcal{O}(t)$ (respectively, $i \in \mathcal{C}(t)$, $i \in \mathcal{U}(t)$, $i \in \mathcal{SU}(t)$, $i \in \mathcal{EO}(t)$).*

Strong approximations for the uniformly accelerated $M_t/M_t/1$ queue with integrable average instantaneous arrival and service rates $\lambda(\cdot)$ and $\mu(\cdot)$ were obtained in Mandelbaum and Massey [17]. The second-order term in the expansion for the queue length process obtained in Mandelbaum and Massey [17] admits an interpretation as the directional derivative $\nabla_\chi \Gamma(\psi)$ of the one-dimensional RM Γ, where $\psi \in \mathcal{C}$ is equal to the fluid netput process given by

$$\psi(t) = \int_0^t \lambda(s)ds - \int_0^t \mu(s)ds, \quad t \in [0, \infty), \tag{21}$$

In Mandelbaum and Massey [17], the queue $\phi = \Gamma(\psi)$ is said to be overloaded, critical or underloaded depending on whether the traffic intensity function

$$\rho^*(t) \doteq \sup_{s\in[0,t]} \frac{\int_s^t \lambda(r)dr}{\int_s^t \mu(r)dr}, \quad t \in [0, \infty),$$

is greater than, equal to or less than 1, respectively. By comparing Proposition 7.2 of Mandelbaum and Massey [17] with Lemma 4.2 of this paper, it can be shown that the regimes introduced above in Definition 1.4 coincide with the definition given in Mandelbaum and Massey [17] for the one-dimensional case when ψ has the particular form (21). However, Definition 1.4 allows for more general $\psi \in \mathcal{C}$ that are not necessarily even of bounded variation and is also addresses the multi-dimensional setting.

Given a one-dimensional RM Γ, functions ψ of the form (21) and $\chi \in \mathcal{C}$, it was shown in Mandelbaum and Massey [17] that under the additional assumption that the derivative has only a finite number of discontinuities in a bounded interval, the one-dimensional derivative $\nabla_\chi \Gamma(\psi)$ is either right or left continuous at every point. In the multi-dimensional setting, the situation is considerably more complex with components of $\nabla_\chi \Gamma(\psi)$ even admitting points with separated discontinuities (see case (S3) of Theorem 1.2). The following concept of critical and sub-critical chains captures the relevant aspects of the reflection matrix R (or, equivalently, of the topology of the associated network) that influence the nature of discontinuities of the derivative $\nabla_\chi \Gamma(\psi)$.

Definition 1.5 (Critical and Sub-critical Chains) *Given an H-R constraint matrix $R \in \mathbb{R}^{K\times K}$, $P \doteq I - R$, associated RM Γ and $\psi \in \mathcal{C}$, let $\phi \doteq \Gamma(\psi)$. Then a sequence $j_0, j_1, j_2, \ldots, j_m$, with $j_k \in \{1, \ldots, K\}$ for $k = 0, 1, \ldots, m$, that satisfies $P_{j_{k-1}j_k} > 0$ for $k = 1, \ldots, m$ is said to be a chain. The chain is said to be a cycle if there exist distinct $k_1, k_2 \in \{0, \ldots, m\}$ such that $j_{k_1} = j_{k_2}$, the chain is said to precede i if $j_0 = i$ and is said to be empty at t if $\phi^{j_k}(t) = 0$ for every $k = 1, \ldots, m$. For $i = 1, \ldots, K$ and $t \in [0, \infty)$, we consider the following two types of chains.*

(*i*) *An empty chain preceding i is said to be critical at time t if it is either cyclic or j_m is at the end of overloading at t.*

(*ii*) *An empty chain preceding i is said to be sub-critical at time t if it is either cyclic or j_m is at the start of underloading at t.*

We now state the second main result of the paper, which specifies necessary conditions for the existence of left and right discontinuities of $\nabla_\chi \Gamma(\psi)$ when ψ, χ are continuous. The proof of Theorem 1.2 is given in Section 4.3. In what follows $\tilde{t}_u^k$, $k = 1, \ldots, K$ are times that are defined in (71).

Theorem 1.2 (Necessary Conditions for Discontinuities in $\nabla_\chi \Gamma(\psi)$) *Given an H-R constraint matrix R with associated reflection map Γ and functions $\psi, \chi \in \mathcal{C}$, the directional derivative $\nabla\Gamma \doteq \nabla_\chi \Gamma(\psi)$ satisfies the following properties:*

(L) If $\nabla\Gamma^i$ has a left discontinuity at $t \in (0,\infty)$, either $t \in \{\tilde{t}_u^k, k = 1,\ldots,K\}$ or one of the following conditions must hold at t:
(a) i is at the end of overloading in which case

$$\nabla\Gamma^i(t-) < \nabla\Gamma^i(t) = 0; \tag{22}$$

(b) i is not underloaded and a critical chain precedes i; if, in addition, i is overloaded then

$$\nabla\Gamma^i(t) < \nabla\Gamma^i(t-); \tag{23}$$

(R) If $\nabla\Gamma^i$ has a right discontinuity at $t \in [0,\infty)$, then one of the following conditions must hold at t:
(a) i is at the start of underloading in which case

$$\nabla\Gamma^i(t) > \nabla\Gamma^i(t+) = 0; \tag{24}$$

(b) i is not underloaded and a sub-critical chain precedes i; if i is also overloaded then

$$\nabla\Gamma^i(t) < \nabla\Gamma^i(t+); \tag{25}$$

(LR) If $\nabla\Gamma^i$ has both a right and left discontinuity at $t \in [0,\infty)$, then one of the following conditions must hold at t:
(a) i is at the end of overloading, and a sub-critical chain precedes i, in which case

$$\nabla\Gamma^i(t-) < \nabla\Gamma^i(t) = 0 < \nabla\Gamma^i(t+);$$

(b) i is at the start of underloading and a critical chain precedes i, in which case

$$\nabla\Gamma^i(t-) > \nabla\Gamma^i(t) > \nabla\Gamma^i(t+) = 0;$$

(c) i is not underloaded and there exist both critical and sub-critical chains preceding i; if, in addition, i is overloaded then the discontinuity is a separated discontinuity of the form

$$\nabla\Gamma^i(t) < \min\left[\nabla\Gamma^i(t-), \nabla\Gamma^i(t+)\right]. \tag{26}$$

Finally, if i is underloaded at $t \in [0,\infty)$ then $\nabla\Gamma^i(t-) = \nabla\Gamma^i(t) = \nabla\Gamma^i(t+) = 0$, which implies t is a point of continuity for $\nabla\Gamma$.

2. Connection with Queueing Networks In Section 2.1 we provide a heuristic description of how directional derivatives of multi-dimensional reflection maps arise in the characterization of second-order (or functional central limit) approximations to non-stationary queueing networks. In Section 2.2 we present two examples to illustrate how the topology of a queueing network associated with a reflection map Γ and the various states of the fluid (ϕ,θ) associated with a continuous netput process ψ can influence the nature of discontinuities of the associated directional derivative $\nabla_\chi\Gamma(\psi)$.

2.1 Directional Derivatives and Functional Central Limits Second-order or diffusion approximations of many classes of queueing networks can be obtained by the following general procedure. Consider a family of queueing networks defined in terms of their primitives (i.e., the random processes defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$ that describe arrivals, services and routing, as well as the scheduling rules). For each queueing network in the family, one constructs from the primitives a certain netput process, $\tilde{X}^\varepsilon$, where roughly speaking, the ith component of $\tilde{X}^\varepsilon$ represents the cumulative net arrivals minus the potential services at the ith queue (see, for example, [4, 23, 24, 25, 32] for precise definitions of netput processes associated with various queueing networks). The evolution of the corresponding queue length process, $\tilde{Z}^\varepsilon$, coincides with the evolution of the netput process $\tilde{X}^\varepsilon$ whenever all queues are non-empty, but in general the queue length process is a more complicated functional of the netput process: $\tilde{Z}^\varepsilon = \Gamma(\tilde{Z}^\varepsilon(0) + \tilde{X}^\varepsilon)$, where the functional Γ is the multi-dimensional oblique reflection mapping associated with the queueing network. In many cases, the family of netput processes $\{\tilde{X}^\varepsilon\}$ can be assumed to satisfy a functional strong law of large numbers (FSLLN) and functional central limit theorem (FCLT). For example, for $\varepsilon > 0$, consider the so-called uniformly accelerated version $\bar{X}^\varepsilon$ of $\tilde{X}^\varepsilon$, where $\bar{X}^\varepsilon/\varepsilon^2$ is defined to be the Markovian process whose instantaneous transition rates are equal to the instantaneous transition rates of $\tilde{X}^\varepsilon$ scaled by $1/\varepsilon^2$. (The references [5, 13, 18, 19, 20] contain further discussion on the uniform acceleration scaling applied to queueing networks.) Note that in the time-homogeneous setting, $\bar{X}^\varepsilon$ can equivalently be defined as

$$\bar{X}^\varepsilon(t) \doteq \varepsilon^2 \tilde{X}^\varepsilon(t/\varepsilon^2), \qquad t \in [0,\infty).$$

The FSLLN for the family of netput processes then takes the form

$$\bar{X}^\varepsilon \to \bar{X} \qquad \text{as } \varepsilon \to 0,$$

where the limit is in the sense of $\mathbb{P}$-a.s. convergence with respect to an appropriate topology on path space (e.g., uniform convergence on compact sets). Similarly, the FCLT for the netput process takes the form

$$\hat{X}^\varepsilon \Rightarrow \hat{X} \qquad \text{as } \varepsilon \to 0, \tag{27}$$

where the limit is in the sense of weak convergence and

$$\hat{X}^\varepsilon \doteq \frac{1}{\varepsilon}\left[\bar{X}^\varepsilon - \bar{X}\right] \tag{28}$$

is a rescaled centered version of the netput process that captures the fluctuations around its FSSLN limit.

In order to obtain a corresponding FSLLN and FCLT for the queue length process, in analogy with $\bar{X}^\varepsilon$, $\bar{Z}^\varepsilon$ is first defined to be the corresponding uniformly accelerated version of $\tilde{Z}^\varepsilon$. Specifically, the homogeneity of the reflection map Γ with respect to space and time can be used to show that $\overline{Z}^\varepsilon$ can be represented as

$$\bar{Z}^\varepsilon = \Gamma(\bar{Z}^\varepsilon(0) + \bar{X}^\varepsilon). \tag{29}$$

Then, under the assumption that $\bar{Z}^\varepsilon(0) \to \bar{Z}(0)$ as $\varepsilon \to 0$, the FSLLN for the queue length process is obtained by establishing the $\mathbb{P}$-a.s. convergence

$$\bar{Z}^\varepsilon = \Gamma(\bar{Z}^\varepsilon(0) + \bar{X}^\varepsilon) \to \bar{Z} \doteq \Gamma(\bar{Z}(0) + \bar{X}) \qquad \text{as } \varepsilon \to 0, \tag{30}$$

where $\bar{X}$ is the FSLLN limit of the netput process. The process $\bar{Z}$ provides a first-order approximation to the queueing network and is often referred to as the fluid limit of the queueing network. To capture the fluctuations of the queue lengths around the fluid limit, one then considers the centered sequence $\{\hat{Z}^\varepsilon\}$ of queue lengths defined by

$$\hat{Z}^\varepsilon \doteq \frac{1}{\varepsilon}\left[\bar{Z}^\varepsilon - \bar{Z}\right] \qquad \text{for } \varepsilon > 0. \tag{31}$$

The above display, together with (28), (29) and (30), then yields the relation

$$\hat{Z}^\varepsilon = \frac{1}{\varepsilon}\left[\Gamma(\bar{Z}^\varepsilon(0) + \bar{X}^\varepsilon) - \Gamma(\bar{Z}(0) + \bar{X})\right] = \frac{1}{\varepsilon}\left[\Gamma\left(\bar{Z}^\varepsilon(0) + \bar{X} + \varepsilon\hat{X}^\varepsilon\right) - \Gamma(\bar{Z}(0) + \bar{X})\right].$$

In many cases, using continuity properties of Γ and the FCLT (27), it is then possible to show that (with respect to a suitable topology on path space) the limit $\hat{Z} \doteq \lim_{\varepsilon\to 0} \hat{Z}^\varepsilon$ exists and satisfies

$$\hat{Z} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[\Gamma\left(\bar{Z}(0) + \bar{X} + \varepsilon\hat{X}\right) - \Gamma(\bar{Z}(0) + \bar{X})\right] = \nabla_{\hat{X}}\Gamma(\bar{Z}(0) + \bar{X}), \tag{32}$$

where $\nabla_{\hat{X}}\Gamma(\bar{X})$ is the directional derivative of the reflection map Γ (see Definition 1.3) in the direction $\hat{X}$, evaluated at $\bar{Z}(0) + \bar{X}$.

In summary, under appropriate conditions, the fluid limit or first-order approximation, $\bar{Z}$, and the functional central limit or second-order correction $\hat{Z}$ to the fluid limit of the queue length process have the representations

$$\bar{Z} = \Gamma(\bar{Z}(0) + \bar{X}) \qquad \text{and} \qquad \hat{Z} = \nabla_{\hat{X}}\Gamma(\bar{Z}(0) + \bar{X}), \tag{33}$$

where $\bar{X}$ and $\hat{X}$ are the functional strong law and functional central limits, respectively, of the netput process. As explained in Remark 1.3, for time-homogeneous networks, under so-called heavy traffic conditions, the representations for fluid and functional central limits for the queueing network take the simpler, more familiar form $\bar{Z} \equiv 0$ and $\hat{Z} = \Gamma(\hat{X})$. On the other hand, in order to analyze time-inhomogeneous networks or transient behaviour in time-homogeneous networks (i.e., when $\bar{Z}(0) \neq 0$), the fluid limit is in general not trivial, and so the second-order approximation is no longer equal to the image of $\hat{X}$ under the reflection map, but instead involves a certain directional derivative of the oblique reflection map.

This philosophy is likely to be applicable in other settings where the process of interest is not necessarily given by a reflection map but another Lipschitz continuous map. In that setting as well, the directional derivative of the corresponding map is likely to be useful for establishing functional central limits as well as for identifying optimal controls of time-inhomogeneous fluid limits (an example of the latter can be found in Cudina and Ramanan [5]).

2.2 Illustrative Examples We provide two examples to illustrate how directional derivatives associated with two time-inhomogeneous networks can be computed. The first is a two-station tandem queueing network, which is presented in Section 2.2.1, and the second is a three-station "join" network, which is given in Section 2.2.2. In both examples, Q is the routing matrix of the network, $R = I - Q^T$ the associated reflection matrix and $P \doteq I - R$. Moreover, λ^i denotes the mean exogenous arrival rate to station i and μ^i is the mean potential service rate at station i. The netput process, ψ, which represents the cumulative net arrivals minus the cumulative potential services that the queues would have seen had they been non-empty throughout, is then defined by the equations $\psi^i(t) = \int_0^t \left(\lambda^i(s) + [P\mu(s)]^i - \mu^i(s)\right) ds$ for $t \in [0,T]$ and $i = 1, \ldots, K$.

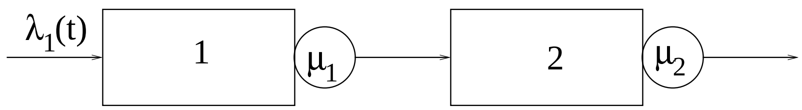

Figure 1: A two-station tandem network.

2.2.1 A tandem queueing model Consider the two-station tandem queueing system illustrated in Figure 1, which has routing matrix Q and reflection matrix $R = I - Q^T$ given by

$$Q \doteq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad R \doteq \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}. \tag{34}$$

Since $Q^T \geq 0$ and has spectral radius zero, R satisfies the Harrison-Reiman condition stated in Definition 1.2. Let Γ denote the associated reflection map (see Figure 2 for the geometry of the associated ORP).

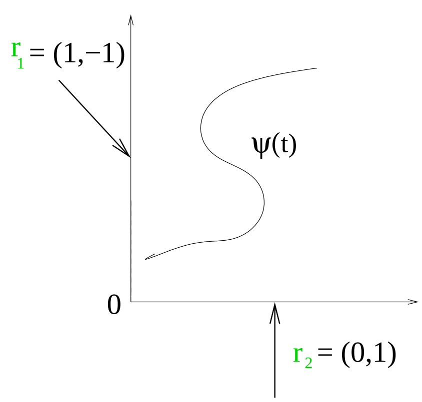

Figure 2: The oblique reflection problem (ORP) associated with a tandem queueing network.

We consider a model in which there are no exogeneous arrivals to station 2, arrivals to station 1 occur at a time-dependent mean rate of $\lambda^1(\cdot)$ given by

$$\lambda^1(t) \doteq \begin{cases} 3 & \text{for } t \in [0,1), \\ 1 & \text{for } t \in [1,3], \end{cases} \tag{35}$$

and the mean potential service rates at station 1 and station 2 are constant and given by $\mu^1 = 2$ and $\mu^2 = 1$, respectively. If ψ is the netput process and (ϕ, θ) solve the ORP for ψ then, as shown in Figure 3, it is easy to see that $\psi^2(t) = t$ and $\theta^2(t) = 0$ for $t \in [0,3]$ and

$$\psi^1(t) = \begin{cases} t & \text{for } t \in [0,1), \\ 1-(t-1) & \text{for } t \in [1,3], \end{cases} \qquad \phi^1(t) = \begin{cases} t & \text{for } t \in [0,1), \\ 1-(t-1) & \text{for } t \in [1,2), \\ 0 & \text{for } t \in [2,3], \end{cases}$$

$$\theta^1(t) = \begin{cases} 0 & \text{for } t \in [0,2), \\ (t-2) & \text{for } t \in [2,3], \end{cases} \qquad \phi^2(t) = \begin{cases} t & \text{for } t \in [0,2), \\ 2 & \text{for } t \in [2,3]. \end{cases}$$

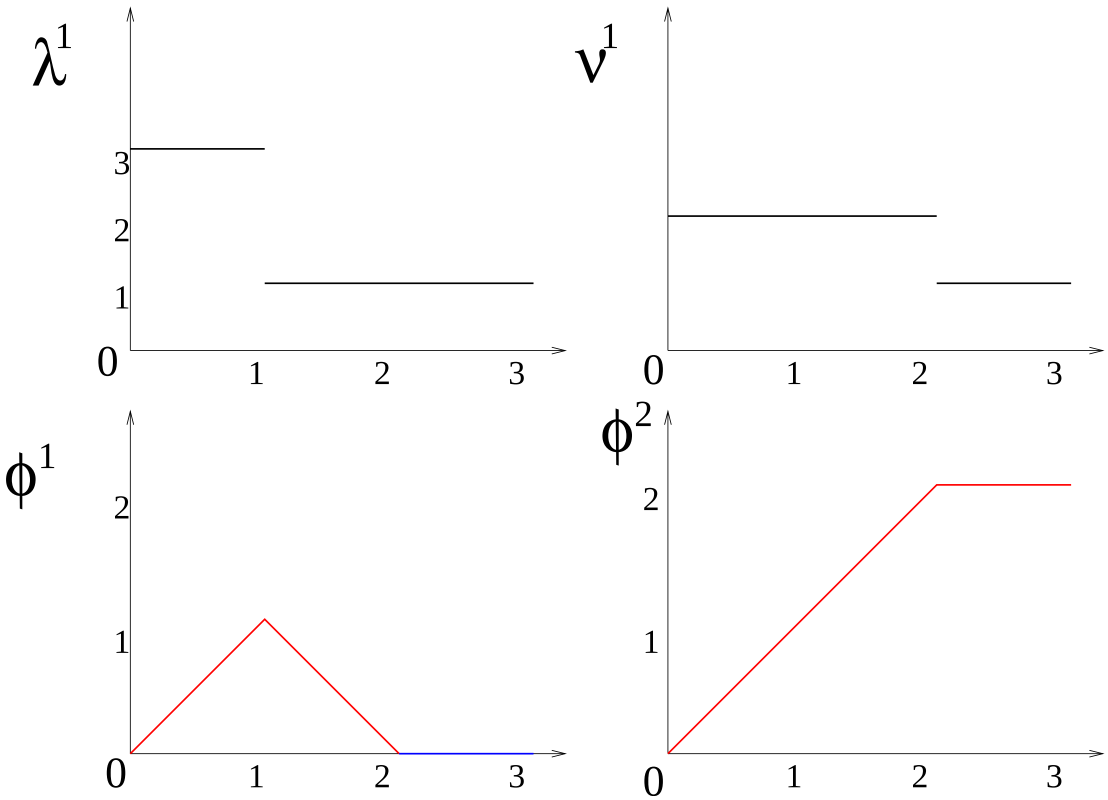

Figure 3: The time-varying exogenous arrival rate λ^1 to and departure rate ν^1 from the first queue, along with the contents ϕ^1 and ϕ^2 of the first and second queue in the tandem network.

The above relations also show that $t_l^1 = t_l^2 = 0$, $t_u^1 = 2$, $t_u^2 = \infty$ and, by the representation (14) for Φ^i, we see that $\Phi^2(t) = \{0\}$ for $t \in [0, \infty)$ and

$$\Phi^1(t) \doteq \begin{cases} \{0\} & \text{for } t \in [0,2), \\ \{0,2\} & \text{for } t = 2, \\ \{t\} & \text{for } t \in (2,3]. \end{cases}$$

Now, fix $\chi \in \mathcal{C}$. Since $\psi \in \mathcal{C}$, by Theorem 1.1 $\nabla\Gamma = \nabla\Gamma_\chi(\psi) = \chi + R\gamma$, where $\gamma = \gamma_{(1)}$ is characterized by (17), with the sets Φ^i, $i = 1, 2$, as given above. Thus, $\nabla\Gamma^1(t) = \chi^1(t) + \gamma^1(t)$, where

$$\gamma^1(t) = \begin{cases} [-\chi^1(0)] \vee 0 & \text{for } t \in [0,2), \\ \left[-\chi^1(2)\right] \vee [-\chi^1(0)] \vee 0 & \text{for } t = 2, \\ -\chi^1(t) & \text{for } t \in (2,3]; \end{cases}$$

while $\nabla\Gamma^2(t) = \chi^2(t) + \gamma^2(t) - \gamma^1(t) = \chi^2(t) + \gamma^2(0) - \gamma^1(t)$, where

$$\gamma^2(0) = \left[-\chi^2(0) + \gamma^1(0)\right] \vee 0 = \left[-\chi^2(0) + [-\chi^1(0)] \vee 0\right] \vee 0.$$

We now refer to the various types of discontinuities mentioned in Theorem 1.2. From the above expressions, it is clear that at $t = 2$, $[-\chi^1(2)] > [-\chi^1(0)] \vee 0$ is a necessary and sufficient condition for $\nabla\Gamma^1$ to have a left discontinuity (of type (La)) as well as for $\nabla\Gamma^2$ to have a left discontinuity (of type (Lb)), while the reverse inequality, $[-\chi^1(2)] < [-\chi^1(0)] \vee 0$, is necessary and sufficient for $\nabla\Gamma^1$ to have a right discontinuity (of type (Ra)) as well as for $\nabla\Gamma^2$ to have a right discontinuity (of type (Rb)). Observe that the necessary conditions mentioned in Theorem 1.2 are indeed satisfied since at $t = 2$, queue 1 is at the end of overloading and at the start of underloading, while queue 2 is overloaded and has critical and sub-critical chains preceding it.

2.2.2 A merge or join network This example servers to illustrate how a separated discontinuity could arise in the directional derivative. Consider a scenario in which two upstream queues feed into a common buffer (see Figure 4). The upstream queues experience a surge in the arrival rate for an initial period, and the arrival rate subsequently subsides to a lower rate. However, just as the surge ends, the server at one of the upstream queues, queue 2, undergoes a partial failure, resulting in the queue maintaining criticality. It is shown below that in such a scenario there can be a discontinuity in the derivative of the downstream queue at the time congestion ends in the upstream queues.

There are no exogeneous arrival rates to queue 3 and the mean exogenous arrival rate λ^i to queue i for $i = 1, 2$ is given by

$$\lambda^1(t) \quad = \quad \begin{cases} 1 & \text{for } t \in [0,1), \\ 1/2 & \text{for } t \in [1,2], \end{cases} \qquad \text{and} \qquad \lambda^2(t) \quad = \quad \begin{cases} 3/2 & \text{for } t \in [0,1/2), \\ 1/2 & \text{for } t \in [1/2,1), \\ 1/3 & \text{for } t \in [1,2]. \end{cases}$$

Moreover, we assume that queues 1 and 3 have constant service rates $\mu^1(t) = \mu^3(t) = 1$ for $t \in [0, 2]$, while queue 2 has service rate

$$\mu^2(t) = \begin{cases} 1 & \text{for } t \in [0,1), \\ 1/3 & \text{for } t \in [1,2]. \end{cases} \tag{36}$$

Since the departures from queues 1 and 2 feed into queue 3 (see Figure 4), the routing matrix Q and reflection matrix $R = I - Q^T$ are given by

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad R = I - Q^T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is trivial to verify that Q^T is an H-R matrix. Let Γ denote the associated reflection map, ψ the netput process and let $\phi = \Gamma(\psi)$. Then it follows from the definitions that

$$\psi^1(t) \quad = \quad \begin{cases} 0 & \text{for } t \in [0,1), \\ -\frac{1}{2}(t-1) & \text{for } t \in [1,2], \end{cases} \qquad \theta^1(t) \quad = \quad \begin{cases} 0 & \text{for } t \in [0,1), \\ \frac{1}{2}(t-1) & \text{for } t \in [1,2], \end{cases}$$

and ϕ^1, θ^2 and θ^3 are identically zero on $[0, 2]$. Moreover, $\phi^2 = \psi^2$ and $\phi^3 = \psi^3$ are given by

$$\phi^2(t) \quad = \quad \begin{cases} \frac{1}{2}t & \text{for } t \in [0,1/2), \\ \frac{1}{4} - \frac{1}{2}(t-1/2) & \text{for } t \in [1/2,1), \\ 0 & \text{for } t \in [1,2], \end{cases} \qquad \phi^3(t) \quad = \quad \begin{cases} t & \text{for } t \in [0,1), \\ 1 - \frac{1}{6}(t-1) & \text{for } t \in [1,2]. \end{cases}$$

Figure 5 provides an illustration of the fluid limit ϕ of the three queues.

The above calculations also readily show that $\Phi^3(t) = \{0\}$ for $t \in [0, 2]$,

$$\Phi^1(t) \quad = \quad \begin{cases} [0,t] & \text{for } t \in [0,1], \\ \{t\} & \text{for } t \in (1,2], \end{cases} \qquad \text{and} \qquad \Phi^2(t) \quad = \quad \begin{cases} \{0\} & \text{for } t \in [0,1), \\ \{0\} \cup [1,t] & \text{for } t \in [1,2]. \end{cases}$$

Note that at $t = 1$, queue 3 is overloaded and, since queue 2 is at the end of overloading and queue 1 is at the start of underloading at $t = 1$, the chain $2, 3$ is a critical chain and $2, 1$ is a subcritical chain preceding 3. Thus the necessary condition for a separated discontinuity stated in (LRc) of Theorem 1.2 is satisfied. Below, explicit calculations are provided to show that the separated discontinuity can indeed occur in this example.

By Theorem 1.1(ii) for $\chi \in \mathcal{C}$, the explicit form of $\nabla\Gamma = \nabla_\chi \Gamma(\psi)$ is given by

$$\nabla\Gamma^1(t) = \chi^1(t) + \sup_{s \in \Phi^1(t)} \left[-\chi^1(s)\right], \qquad \nabla\Gamma^2(t) = \chi^2(t) + \sup_{s \in \Phi^2(t)} \left[-\chi^2(s)\right]$$

and $\nabla\Gamma^3(t) = \chi^3(t) - \gamma^1(t) - \gamma^2(t) + \gamma^3(t)$, with $\gamma^3(t) = \sup_{s \in \Phi^3(t)} \left[-\chi^3(s) + [P\gamma]^3(s)\right]$. Thus, we have

$$\begin{aligned} \nabla\Gamma^3(t) \quad = \quad & \chi^3(t) - \sup_{s \in \Phi^2(t)} \left[-\chi^2(s)\right] - \sup_{s \in \Phi^1(t)} \left[-\chi^1(s)\right] \\ & + \sup_{s \in \Phi^3(t)} \left[-\chi^3(s) + \sup_{r \in \Phi^2(s)} \left[-\chi^2(r)\right] + \sup_{r \in \Phi^1(s)} \left[-\chi^1(r)\right]\right]. \end{aligned}$$

From the above expressions it is straightforward to deduce that

$$\nabla\Gamma^3(1) - \nabla\Gamma^3(1-) = \left[\chi^2(1) - \chi^2(0)\right] \wedge 0 \qquad \text{and} \qquad \nabla\Gamma^3(1) - \nabla\Gamma^3(1+) = -\chi^1(1) - \sup_{s \in [0,1]} [-\chi^1(s)].$$

Therefore, if $\chi^2(1) < \chi^2(0)$ and $\sup_{s\in[0,1]}\left[-\chi^1(s)\right] > -\chi^1(1)$, then

$$\nabla\Gamma^3(1) - \nabla\Gamma^3(1-) = \chi^2(1) - \chi^2(0) < 0 \quad \text{and} \quad \nabla\Gamma^3(1) - \nabla\Gamma^3(1+) = -\chi^1 - \sup_{s\in[0,1]}\left[-\chi^1(s)\right] < 0,$$

which implies $\nabla\Gamma^3$ is neither right nor left continuous at $t = 1$. In fact, it has a separated discontinuity at that point since $\nabla\Gamma^3(1) < \nabla\Gamma^3(1-) \wedge \nabla\Gamma^3(1+)$, as anticipated by condition (LRc) of Theorem 1.2. In the context of functional central limit theorems, χ will be a Brownian motion and so you expect the conditions on χ to be satisfied with positive probability.

It is worthwhile to note that a separated discontinuity can arise only in the multi-dimensional setting, and not in the one-dimensional setting. This has important ramifications for the mode of convergence of $\nabla^{\varepsilon}_{\chi}(\psi)$ to $\nabla_{\chi}\Gamma(\psi)$ when ψ, χ are continuous. Specifically, as remarked earlier, it was shown in Mandelbaum and Massey [17] that for the one-dimensional map, $\nabla^{\varepsilon}_{\chi}(\psi)$ converges to $\nabla_{\chi}\Gamma(\psi)$ in the M_1 topology (see Whitt [32] for a definition of this topology). When ψ, χ are continuous, $\nabla^{\varepsilon}_{\chi}(\psi)$ is also continuous for every $\varepsilon > 0$. Since $\mathcal{D}_{l,r}$, the space of functions that are either left or right continuous at every point, is complete under the M_1 topology (cf., Whitt [32]), and continuous functions clearly lie in $\mathcal{D}_{l,r}$, while functions with separated discontinuities do not lie in $\mathcal{D}_{l,r}$, this example demonstrates that one cannot in general expect M_1 convergence in the multi-dimensional setting.

3. Existence and Characterization of the Directional Derivative This section is devoted to the proof of Theorem 1.1. Relevant properties of H-R ORPs with inputs $\psi \in \mathcal{D}_{\lim}$ are first described in Section 3.1, and then existence of the associated directional derivative is established in Section 3.2. In Section 3.3 the notion of a generalized one-dimensional derivative is introduced and characterized, and the proof of Theorem 1.1 is presented in Section 3.4.

3.1 Properties of the Oblique Reflection Map The ORP associated with an H-R matrix $R \in \mathbb{R}^{K\times K}$ was introduced in Section 1.4. Here, we first establish a minor generalization of a well-known result of Harrison and Reiman [11] to show that RMs Γ associated with H-R reflection matrices are well defined on $\mathcal{D}_{\lim}$. Recall the notation $\overline{f}(t) = \sup_{s\in[0,t]} f(s)$.

Theorem 3.1 (Solutions to H-R ORPs) *Let $R \in \mathbb{R}^{K\times K}$ be an H-R constraint matrix and let $P \doteq I - R$. Given $\psi \in \mathcal{D}^{+}_{\lim}$, there exists a unique solution (ϕ, θ) to the ORP associated with R for ψ. Moreover, $\theta = \Theta(\psi)$ is the unique fixed point of the map $F(\psi, \cdot) : \mathcal{I}_0 \to \mathcal{I}_0$ given by*

$$F^i(\psi, \theta)(t) \doteq \left[\overline{-\psi^i + [P\theta]^i}(t)\right] \vee 0, \qquad i = 1, \ldots, K. \tag{37}$$

In other words, for $i = 1, \ldots, K$, θ^i satisfies

$$\theta^i(t) = \left[\overline{-\psi^i + [P\theta]^i}\ (t)\right] \vee 0\,. \tag{38}$$

Furthermore, the maps Γ and Θ are Lipschitz continuous with respect to the uniform topology on $\mathcal{D}_{\lim}$, i.e., there exists $L = L(R) < \infty$ such that for every $\psi_1, \psi_2 \in \mathcal{D}_{\lim}$ and $N < \infty$,

$$||\Gamma(\psi_1) - \Gamma(\psi_2)||_N \le L||\psi_1 - \psi_2||_N \quad \text{and} \quad ||\Theta(\psi_1) - \Theta(\psi_2)||_N \le L||\psi_1 - \psi_2||_N. \tag{39}$$

Lastly, if $\psi \in \mathcal{C}$ (respectively, $\mathcal{D}_c$), then $\phi, \theta \in \mathcal{C}$ (respectively, $\mathcal{D}_c$).

Proof. Since $\psi \in \mathcal{D}^{+}_{\lim}$, we have $-\psi^i(0) \le 0$ for every i, and hence $F(\psi, \theta)(0) = 0$. In addition, $F(\psi, \theta)$ is clearly increasing and so $F(\psi, \theta) \in \mathcal{I}_0$. Because $\mathcal{D}_{\lim}$ is complete with respect to the sup norm, the argument used in Harrison and Reiman [11] also shows that $F(\psi, \cdot)$ is a contraction mapping that maps $\mathcal{I}_0$ into $\mathcal{I}_0$, and thus has a unique fixed point. The proof of the fact that θ is a fixed point of $F(\psi, \cdot)$ if and only if $\theta = \Theta(\psi)$ also follows from a straightforward generalization (from $\mathcal{C}$ to $\mathcal{D}^{+}_{\lim}$) of the corresponding argument used in Harrison and Reiman [11], and is thus omitted. Lipschitz continuity of the maps Γ and Θ can be deduced from the explicit representation (37) for F^i and the fact that the matrix P is similar to a matrix whose row sums are strictly less than 1 (see Lemma 3.3 for similar arguments or Theorem 2.2 of Dupuis and Ramanan [8] for an alternative proof of Lipschitz continuity when $\psi \in \mathcal{D}_r$). The last assertion of the lemma (which considers $\psi \in \mathcal{C}$) holds due to the fact that $\mathcal{C}$ and $\mathcal{D}_r$ are closed subspaces of $\mathcal{D}_{\lim}$ (with respect to the topology of uniform convergence). The case when $\psi \in \mathcal{D}_c$ is easily verified directly (see, for example, the argument in Dupuis and Ishii [7]). $\square$

3.2 Existence of the Directional Derivative In order to show the existence of the derivative or, equivalently, to show the existence of a pointwise limit of the sequence $\nabla^{\varepsilon}_{\chi}(\psi) \in \mathcal{D}_{\lim}$ as $\varepsilon \downarrow 0$, it turns out to be more convenient to work with a closely related family of processes $\{\gamma_\varepsilon(\psi,\chi)\}_{\varepsilon>0}$. This family is introduced in Section 3.2.1 and is shown to have a pointwise limit $\gamma(\psi,\chi)$ in Section 3.2.2. In Section 3.2.3 the limit $\gamma(\psi,\chi)$ and the derivative $\nabla_\chi\Gamma(\psi)$ are shown to lie in $\mathcal{D}_{\lim}$ and satisfy certain continuity and scaling properties.

3.2.1 A related family $\{\gamma_\varepsilon\}_{\varepsilon>0}$ of functions Given an ORP with H-R constraint matrix R, let

$$\gamma_\varepsilon(\psi,\chi) \doteq \varepsilon^{-1}\left[\Theta(\psi+\varepsilon\chi) - \Theta(\psi)\right], \qquad \varepsilon > 0, \tag{40}$$

where Θ is the mapping introduced after Definition 1.1. Using the fact that $\Gamma(\psi) = \psi + R\Theta(\psi)$ for $\psi \in \mathcal{D}_{\lim}$, along with definition (9) of the sequence $\{\nabla^{\varepsilon}_{\chi}\Gamma(\psi)\}$, one obtains the relation

$$\nabla^{\varepsilon}_{\chi}\Gamma(\psi) = \chi + R\gamma_\varepsilon(\psi,\chi), \qquad \varepsilon > 0. \tag{41}$$

Thus, in order to establish existence of the derivative, it clearly suffices to show that $\gamma_\varepsilon(\psi,\chi)$ has a pointwise limit as $\varepsilon \downarrow 0$.

Now, fix $\psi,\chi \in \mathcal{D}_{\lim}$. For conciseness, let $\theta \doteq \Theta(\psi)$ and for $\varepsilon > 0$, let $\theta_\varepsilon \doteq \Theta(\psi+\varepsilon\chi)$ and $\gamma_\varepsilon \doteq \gamma_\varepsilon(\psi,\chi)$. From (38), it follows that

$$\theta^i = \left[\overline{-\psi^i + [P\theta]^i}\right] \vee 0 \quad \text{ and } \quad \theta^i_\varepsilon = \left[\overline{-\psi^i - \varepsilon\chi^i + [P\theta_\varepsilon]^i}\right] \vee 0.$$

For $i = 1,\ldots,K$, define

$$\xi^i \doteq \psi^i - [P\theta]^i, \tag{42}$$

and rewrite θ^i and θ^i_ε in terms of ξ^i as follows:

$$\theta^i = \left[\overline{-\xi^i}\right] \vee 0, \qquad \theta^i_\varepsilon = \left[\overline{-\xi^i - \varepsilon\chi^i + [P(\theta_\varepsilon - \theta)]^i}\right] \vee 0.$$

Together with (40), this shows that for $i = 1,\ldots,K$,

$$\gamma^i_\varepsilon = \gamma^i_\varepsilon(\psi,\chi) = \varepsilon^{-1}\left[\theta^i_\varepsilon - \theta^i\right] = \overline{-\varepsilon^{-1}\xi^i - \chi^i + [P\gamma_\varepsilon]^i} \vee 0 - \overline{-\varepsilon^{-1}\xi^i} \vee 0. \tag{43}$$

3.2.2 Pointwise convergence of $\{\gamma_\varepsilon\}_{\varepsilon>0}$ for H-R ORPs In this section some basic properties of the families $\{\gamma_\varepsilon\}_{\varepsilon>0}$ and $\{\nabla^{\varepsilon}_{\chi}\Gamma(\psi)\}_{\varepsilon>0}$ are established. The existence of a pointwise limit is shown to be a consequence of the uniform boundedness of the sequence $\{\gamma_\varepsilon(t)\}_{\varepsilon>0}$ proved in Lemma 3.1 and the monotonicity property established in Lemma 3.3.

Lemma 3.1 (Uniform Boundedness) *Let $\nabla^{\varepsilon}_{\chi}\Gamma(\psi)$ and $\gamma_\varepsilon(\psi,\chi)$ be defined as in (9) and (40), respectively, and let $L < \infty$ be the constant that satisfies (39). Then for any $\xi,\chi,\chi_1,\chi_2 \in \mathcal{D}_{\lim}$ and $T < \infty$, the following inequalities hold:*

$$\sup_{\varepsilon>0} ||\nabla^{\varepsilon}_{\chi_1}\Gamma(\psi) - \nabla^{\varepsilon}_{\chi_2}\Gamma(\psi)||_T \le L||\chi_1 - \chi_2||_T, \qquad \sup_{\varepsilon>0} ||\nabla^{\varepsilon}_{\chi}\Gamma(\psi)||_T \le L||\chi||_T \tag{44}$$

$$\sup_{\varepsilon>0} ||\gamma_\varepsilon(\psi,\chi_1) - \gamma_\varepsilon(\psi,\chi_2)||_T \le L||\chi_1 - \chi_2||_T, \qquad \sup_{\varepsilon>0} ||\gamma_\varepsilon(\psi,\chi)||_T \le L||\chi||_T\,. \tag{45}$$

Proof. The first and third inequalities follow directly from the Lipschitz continuity of the RM stated in (39) and the definitions of $\nabla^{\varepsilon}_{\chi}\Gamma(\psi)$ and γ_ε given in (9) and (40), respectively. The second and fourth bounds follow simply by choosing $\chi_1 = \chi$ and $\chi_2 = 0$ in the first and third bounds, respectively, and noting that $\nabla^{\varepsilon}_{0}\Gamma(\psi) = \gamma_\varepsilon(\psi,0) = 0$. □

The proof of monotonicity will make repeated use of the following elementary inequality, whose simple proof is included for completeness.

Lemma 3.2 *Any two real-valued functions f and g that are defined on $[0,\infty)$ satisfy, for every $T < \infty$,*

$$\overline{f}(T) \vee 0 - \overline{g}(T) \vee 0 \le \overline{f-g}(T) \vee 0 \le |\overline{f-g}(T)|. \tag{46}$$

Proof. If $\overline{f} \le 0$, then the left-hand side of (46) is non-positive and so the first inequality in (46) holds trivially. On the other hand, if $\overline{f} > 0$, let $t_n \in [0,T]$ be such that $\overline{f} \le f(t_n) + \frac{1}{n}$. Then we have

$$\overline{f} \vee 0 - \overline{g} \vee 0 \le f(t_n) - \overline{g} \vee 0 + \frac{1}{n} \le f(t_n) - g(t_n) + \frac{1}{n} \le \overline{f-g} \vee 0 + \frac{1}{n}.$$

Since n is arbitrary, this shows that the first inequality in (46) holds. The second inequality in (46) is trivially satisfied. □

Lemma 3.3 (Monotonicity) *Given $\psi, \chi \in \mathcal{D}_{\lim}$, let $\gamma_\varepsilon \doteq \gamma_\varepsilon(\psi,\chi)$ be defined by (40). Then for $i = 1,\ldots,K$, γ_ε^i is monotonically nonincreasing as $\varepsilon \downarrow 0$, so that*

$$0 < \varepsilon_1 \le \varepsilon_2 \qquad \textit{implies} \qquad \gamma_{\varepsilon_1}^i(s) - \gamma_{\varepsilon_2}^i(s) \le 0, \qquad s \in [0,\infty). \tag{47}$$

Moreover, for every $t \ge 0$, the limit $\gamma(t) \doteq \gamma(\psi,\chi)(t) = \lim_{\varepsilon \downarrow 0} \gamma_\varepsilon(t)$ exists.

Proof. Let $0 < \varepsilon_1 \le \varepsilon_2$ and fix $i \in \{1,\ldots,K\}$ and $s \in [0,\infty)$. Using the representation (43) for γ_ε^i and making repeated use of the inequality (46), we obtain for $t \in [0,s]$,

$$\begin{aligned}
\gamma_{\varepsilon_1}^i(t) - \gamma_{\varepsilon_2}^i(t) &= \overline{-\varepsilon_1^{-1}\xi^i - \chi^i + [P\gamma_{\varepsilon_1}]^i}(t) \vee 0 - \overline{-\varepsilon_1^{-1}\xi^i}(t) \vee 0 \\
&\quad - \overline{-\varepsilon_2^{-1}\xi^i - \chi^i + [P\gamma_{\varepsilon_2}]^i}(t) \vee 0 + \overline{-\varepsilon_2^{-1}\xi^i}(t) \vee 0 \\
&= \overline{-\varepsilon_1^{-1}\xi^i - \chi^i + [P\gamma_{\varepsilon_1}]^i}(t) \vee 0 - \overline{-\varepsilon_2^{-1}\xi^i - \chi^i + [P\gamma_{\varepsilon_2}]^i}(t) \vee 0 \\
&\quad -(\varepsilon_1^{-1} - \varepsilon_2^{-1})\left(\overline{-\xi^i}(t) \vee 0\right) \\
&\le \overline{-(\varepsilon_1^{-1} - \varepsilon_2^{-1})\xi^i + [P\gamma_{\varepsilon_1}]^i - [P\gamma_{\varepsilon_2}]^i}(t) \vee 0 - \overline{-(\varepsilon_1^{-1} - \varepsilon_2^{-1})\xi^i}(t) \vee 0 \\
&\le \overline{[P\gamma_{\varepsilon_1}]^i - [P\gamma_{\varepsilon_2}]^i}(t) \vee 0,
\end{aligned}$$

where we have used the fact that $(\varepsilon_1^{-1} - \varepsilon_2^{-1}) > 0$ in the penultimate line. By Remark 1.1, there exists a diagonal matrix A with $A_{ii} > 0$ for $i = 1,\ldots,K$, and $\delta > 0$ such that the matrix $\tilde{P} \doteq A^{-1}PA$ satisfies $\max_{i=1,\ldots,K} \sum_{j=1}^K \tilde{P}_{ij} \le 1 - \delta$. Define $\tilde{\gamma} \doteq A^{-1}\gamma$. Then, $\tilde{P}$ is nonnegative (since P is nonnegative), $P\gamma = A\tilde{P}\tilde{\gamma}$ and by the inequality derived above, we obtain for every $t \in [0,s]$,

$$\begin{aligned}
\tilde{\gamma}_{\varepsilon_1}^i(t) - \tilde{\gamma}_{\varepsilon_2}^i(t) = \frac{1}{A_{ii}}\left[\gamma_{\varepsilon_1}^i(t) - \gamma_{\varepsilon_2}^i(t)\right] &\le \overline{\frac{1}{A_{ii}}[A\tilde{P}\tilde{\gamma}_{\varepsilon_1}]^i - [A\tilde{P}\tilde{\gamma}_{\varepsilon_2}]^i}(s) \vee 0 \\
&= \overline{[\tilde{P}\tilde{\gamma}_{\varepsilon_1}]^i - [\tilde{P}\tilde{\gamma}_{\varepsilon_2}]^i}(s) \vee 0 \\
&\le \left(\sum_{j=1}^K \tilde{P}_{ij}\right) \max_{k=1,\ldots,K} \overline{\tilde{\gamma}_{\varepsilon_1}^k - \tilde{\gamma}_{\varepsilon_2}^k}(s) \vee 0 \\
&\le (1-\delta) \max_{k=1,\ldots,K} \overline{\tilde{\gamma}_{\varepsilon_1}^k - \tilde{\gamma}_{\varepsilon_2}^k}(s) \vee 0.
\end{aligned}$$

Taking the supremum of the left-hand side of the above inequality over $t \in [0,s]$ and then the maximum over $i = 1,\ldots,K$ yields the relation

$$\max_{k=1,\ldots,K} \overline{\tilde{\gamma}_{\varepsilon_1}^k - \tilde{\gamma}_{\varepsilon_2}^k}(s) \le (1-\delta) \max_{k=1,\ldots,K} \overline{\tilde{\gamma}_{\varepsilon_1}^k - \tilde{\gamma}_{\varepsilon_2}^k}(s) \vee 0,$$

which implies $\max_{k=1,\ldots,K} \overline{\tilde{\gamma}_{\varepsilon_1}^k - \tilde{\gamma}_{\varepsilon_2}^k}(s) \le 0$. Since $\gamma^i = A_{ii}\tilde{\gamma}^i$ and $A_{ii} > 0$, this implies (47).

Now the uniform boundedness of the sequence $\{\gamma_\varepsilon\}$ proved in Lemma 3.1 shows that for each $s \in [0,\infty)$, there exists a subsequence (which could depend on s) of $\{\gamma_\varepsilon(s)\}$ that converges to a limit. The monotonicity property shows that this limit, which we denote by $\gamma(s)$, is independent of the subsequence. Thus, γ is the real-valued function that equals the pointwise limit of the sequence of functions $\{\gamma_\varepsilon\}$, as $\varepsilon \downarrow 0$. □

3.2.3 Properties of the pointwise limit In this section we first show that the limit γ of Lemma 3.3 lies in $\mathcal{D}_{\lim}$. Note that this is not apriori obvious even if ψ and χ are assumed to be continuous (which would in turn imply that the functions $\gamma_\varepsilon = \gamma_\varepsilon(\psi,\chi), \varepsilon > 0$, are continuous) since the limit of a monotone non-increasing sequence of real-valued continuous functions $\{f_n\}$ need not in general lie in $\mathcal{D}_{\lim}$. For instance, define $f_n(t) \doteq \sin(1/t)$ if $t \in [1/(2n\pi + \pi/2), \infty)$ and $f_n(t) \doteq 1$ otherwise, $n \in \mathbb{N}$, and let $f(t) \doteq \sin(1/t)$ if $t \in (0,\infty)$ and $f(0) \doteq 1$. As $n \to \infty$, the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise monotonically down to f, but f does not lie in $\mathcal{D}_{\lim}$ since it has no right limit at 0. However, the γ_ε possess special properties by virtue of the fact they are defined via ORPs, which allow us to show that γ must lie in $\mathcal{D}_{\lim}$. The case when $\chi \in \mathcal{BV}$ is proved in Lemma 3.5, which makes use of some general properties of functions, which are summarised in Lemma 3.4 and whose proof is relegated to Section 5.1. The case of general $\chi \in \mathcal{D}_{\lim}$ is dealt with subsequently, in the proof of Theorem 1.1(i) below. Recall that $|f|_T$ denotes the total variation of the function f on the interval $[0,T]$.

LEMMA 3.4 *Any two real-valued functions f and g that are defined on $[0,\infty)$ satisfy*

$$\left|\overline{f} \vee 0 - \overline{g} \vee 0\right|_T \leq |f-g|_T. \tag{48}$$

Moreover, consider a sequence of functions $\{f_n\} \subset \mathcal{D}_{\lim}$ that converges pointwise to a function f. If $\sup_n |f_n|_T < \infty$ for every $T<\infty$ then $f \in \mathcal{D}_{\lim}$.

LEMMA 3.5 (UNIFORMLY BV) *Given an H-R reflection matrix R and associated RM Γ, $\psi \in \mathcal{D}_{\lim}$ and $\chi \in \mathcal{BV}$, let $\gamma_\varepsilon \doteq \gamma_\varepsilon(\psi,\chi)$ be defined by (40). Then for every $T \in [0,\infty)$,*

$$\sup_{\varepsilon>0} |\gamma_\varepsilon|_T < \infty. \tag{49}$$

Moreover $\gamma \doteq \gamma(\psi,\chi)$, the pointwise limit of $\gamma_\varepsilon(\psi,\chi)$ as $\varepsilon \downarrow 0$, lies in $\mathcal{D}_{\lim}$.

PROOF. Fix $T<\infty$, let $P=I-R$ and fix $\varepsilon>0$. By the representation (43) for γ^i_ε, the inequality (48), with $f=-\varepsilon^{-1}\xi^i-\chi^i+[P\gamma_\varepsilon]^i$ and $g=-\varepsilon^{-1}\xi^i$, the triangle inequality and the fact that P is non-negative, we obtain

$$\left|\gamma^i_\varepsilon\right|_{\mathrm{T}} = \left|\overline{-\varepsilon^{-1}\xi^i-\chi^i+[P\gamma_\varepsilon]^i} \vee 0 - \overline{-\varepsilon^{-1}\xi^i} \vee 0\right|_{\mathrm{T}} \quad \leq \quad \left|-\chi^i+[P\gamma_\varepsilon]^i\right|_{\mathrm{T}} \quad \leq \quad \left|\chi^i\right|_{\mathrm{T}} + \textstyle\sum_{j=1}^K P_{ij}|\gamma^j_\varepsilon|_{\mathrm{T}}.$$

By Remark 1.1 there exists a diagonal matrix A (with $A_{ii}>0$) and $\delta>0$ such that the matrix $\tilde{P} \doteq A^{-1}PA$ satisfies $\max_{i=1,\ldots,K}\sum_{j=1}^K \tilde{P}_{ij} \leq 1-\delta$. Multiplying both sides of the last display by A_{ii} and substituting for P in terms of $\tilde{P}$ (note that $A_{jj}\tilde{P}_{ij}=A_{ii}P_{ij}$), we obtain the inequality

$$A_{ii}\left|\gamma^i_\varepsilon\right|_{\mathrm{T}} \leq A_{ii}\left|\chi^i\right|_{\mathrm{T}} + \sum_{j=1}^K \tilde{P}_{ij}A_{jj}\left|\gamma^j_\varepsilon\right|_{\mathrm{T}},$$

which implies that

$$\max_{i=1,\ldots,K} A_{ii}\left|\gamma^i_\varepsilon\right|_{\mathrm{T}} \leq \max_{i=1,\ldots,K} A_{ii}\left|\chi^i\right|_{\mathrm{T}} + (1-\delta)\max_{i=1,\ldots,K} A_{ii}\left|\gamma^i_\varepsilon\right|_{\mathrm{T}}.$$

On rearrangement, this yields

$$\max_{i=1,\ldots,K} A_{ii}|\gamma^i_\varepsilon|_{\mathrm{T}} \leq \frac{\max_{i=1,\ldots,K} A_{ii}|\chi^i|_{\mathrm{T}}}{\delta},$$

from which we conclude that

$$\sup_{\varepsilon>0}|\gamma_\varepsilon|_{\mathrm{T}} \leq K\sup_{\varepsilon>0}\max_{i=1,\ldots,K}|\gamma^i_\varepsilon|_{\mathrm{T}} \leq \frac{K\max_{i=1,\ldots,K}A_{ii}}{\delta\min_{i=1,\ldots,K}A_{ii}}|\chi|_{\mathrm{T}} < \infty,$$

where the last inequality follows because of the assumption that $\chi \in \mathcal{BV}$. This proves (49).

Now, γ is also the pointwise limit of any subsequence $\{\gamma_{\varepsilon_n}\}_{n\in\mathbb{N}}$ and, clearly, $\sup_{n\in\mathbb{N}}|\gamma_{\varepsilon_n}|_T<\infty$ for every $T<\infty$. Lemma 3.4 then allows us to conclude that $\gamma \in \mathcal{D}_{\lim}$. □

We now establish the first property of Theorem 1.1. The remaining properties are established in Section 3.4.2.

PROOF OF THEOREM 1.1(i). For any $\psi,\chi \in \mathcal{D}_{\lim}$, Lemma 3.3 establishes the existence of the pointwise limit $\gamma(\psi,\chi)$. Next, we show that $\gamma(\psi,\chi)$ lies in $\mathcal{D}_{\lim}$. Since $\mathcal{D}_{c,\lim}$ is dense in $\mathcal{D}_{\lim}$ with respect to the topology of uniform convergence on compact sets (see, for example, Whitt [32]) and clearly, $\mathcal{D}_{c,\lim} \subset \mathcal{BV}$, there exists a sequence $\{\chi_n\}_{n\in\mathbb{N}} \subset \mathcal{BV}$ such that χ is the limit (in this topology) of χ_n, as $n\to\infty$. Since $\chi_n \in \mathcal{BV}$, each $\gamma(\psi,\chi_n)$ lies in $\mathcal{D}_{\lim}$ by Lemma 3.5. On the other hand, Lemma 3.1 shows that $\gamma(\psi,\chi_n) \in \mathcal{D}_{\lim}$ converges, as $n\to\infty$, to $\gamma(\psi,\chi)$ in the uniform topology on every bounded interval. Since $\mathcal{D}_{\lim}$ is complete with respect to this topology, we deduce that $\gamma \doteq \gamma(\psi,\chi) \in \mathcal{D}_{\lim}$. The relation (41) then shows that the pointwise limit $\nabla_\chi\Gamma(\psi)$ of $\nabla^\varepsilon_\chi\Gamma(\psi)$ exists and is equal to $\chi+R\gamma$. Thus, in particular, $\nabla_\chi\Gamma(\psi) \in \mathcal{D}_{\lim}$.

For any fixed $\psi \in \mathcal{D}_{\lim}$, the Lipschitz continuity of the map $\chi \mapsto \nabla_\chi \Gamma(\psi)$ is a direct consequence of (44). Lastly, since $\nabla_\chi \Gamma(\psi) = \chi + R\gamma(\psi, \chi)$, in order to establish (15) it suffices to show that for $\alpha, \beta > 0$, $\gamma(\beta\psi, \alpha\chi) = \alpha\gamma(\psi, \chi)$. From (38) it is clear that for $\beta > 0$, $\Theta(\beta\psi) = \beta\Theta(\psi)$. Fix $\varepsilon > 0$. By (40) we then see that

$$\gamma_\varepsilon(\beta\psi, \alpha\chi) = \varepsilon^{-1}\left[\Theta(\beta\psi + \varepsilon\alpha\chi) - \Theta(\beta\psi)\right] = \beta\varepsilon^{-1}\left[\Theta\left(\psi + \varepsilon\frac{\alpha}{\beta}\chi\right) - \Theta(\psi)\right].$$

Setting $\tilde{\varepsilon} = \varepsilon\alpha/\beta$, we can rewrite the above equation as

$$\gamma_\varepsilon(\beta\psi, \alpha\chi) = \alpha\tilde{\varepsilon}^{-1}\left[\Theta(\psi + \tilde{\varepsilon}\chi) - \Theta(\psi)\right] = \alpha\gamma_{\tilde{\varepsilon}}(\psi, \chi).$$

Taking limits as $\varepsilon \to 0$, and noting that then $\tilde{\varepsilon} \to 0$, we obtain the desired relation $\gamma(\beta\psi, \alpha\chi) = \alpha\gamma(\psi, \chi)$. □

3.3 The Generalized One-dimensional Derivative

3.3.1 A representation for the multi-dimensional derivative In the last section we showed that given $\psi, \chi \in \mathcal{D}_{\lim}$, the directional derivative has the form $\nabla_\chi \Gamma(\psi) = \chi + R\gamma$, where $\gamma \doteq \gamma(\psi, \chi) \in \mathcal{D}_{\lim}$ is the pointwise limit of the monotonically non-increasing sequence $\{\gamma_\varepsilon(\psi, \chi)\}$. From the expression (43) for $\gamma_\varepsilon^i \doteq \gamma_\varepsilon^i(\psi, \chi)$, it is clear that for every $t \in [0, \infty)$,

$$\begin{aligned} \gamma^i(t) &= \lim_{\varepsilon\downarrow 0}\left[\overline{-\varepsilon^{-1}\xi^i - \chi^i + [P\gamma_\varepsilon]^i}(t) \vee 0 - \overline{-\varepsilon^{-1}\xi^i}(t) \vee 0\right] \\ &= \lim_{\varepsilon\downarrow 0}\left[F^i\left(\varepsilon^{-1}\xi + \chi - P\gamma_\varepsilon, 0\right)(t) - F^i\left(\varepsilon^{-1}\xi, 0\right)(t)\right]. \end{aligned} \tag{50}$$

Since P is non-negative, it follows that $P\gamma$ is the pointwise limit of the monotonically non-increasing sequence $\{P\gamma_\varepsilon\}$. Thus γ^i has a representation as a one-dimensional pointwise limit of the form

$$\lim_{\varepsilon\downarrow 0}\left[\overline{\varepsilon^{-1}f + g_\varepsilon} \vee 0 - \overline{\varepsilon^{-1}f} \vee 0\right], \tag{51}$$

where $f \doteq -\xi^i$ and $g_\varepsilon \doteq -\chi^i + [P\gamma_\varepsilon]^i$ lie in $\mathcal{D}_{\lim}(\mathbb{R})$ and g_ε monotonically converges pointwise down to the function $g = -\chi^i + [P\gamma]^i$ in $\mathcal{D}_{\lim}(\mathbb{R})$. If, instead, $g_\varepsilon \equiv g$ were independent of ε, then (51) would reduce to a limit of the form

$$\lim_{\varepsilon\downarrow 0}\left[\overline{\varepsilon^{-1}f + g} \vee 0 - \overline{\varepsilon^{-1}f} \vee 0\right]. \tag{52}$$

Under the assumption that $f, g \in \mathcal{C}(\mathbb{R})$ and the limit has a finite number of discontinuities on any compact interval, the limit in (52) was shown in [17] to be equal to $\max_{s\in\Phi_f(t)} g(s)$, where Φ_f is as defined in (11). This representation was later generalized to the case $f, g \in \mathcal{D}_r(\mathbb{R})$ in [33, Theorem 9.3.1]. It may be natural to conjecture that the limit in (51) is equal to the limit $\max_{s\in\Phi_f(t)} g(s)$ of (52), but where $g \in \mathcal{D}_{\lim}(\mathbb{R})$ is now the pointwise limit of the sequence $\{g_\varepsilon\}_{\varepsilon>0}$. If that were true, then the limit in (51) could be identified simply by generalizing the results in [17, 33] to the case when $f, g \in \mathcal{D}_{\lim}(\mathbb{R})$. However, it turns out that the topology of pointwise convergence $g_\varepsilon \downarrow g$ is too weak for such a conjecture to hold in general (see Remark 3.1(3) below for examples when the two limits fail to coincide). Thus a more careful analysis is required in order to determine the correct limit in (51). This is carried out in Section 3.3.2. Fortunately, it turns out that the conjecture is true for the important case when $f \in \mathcal{C}(\mathbb{R})$ and $g_\varepsilon \in \mathcal{C}(\mathbb{R})$ for all $\varepsilon > 0$, and in this case the one-dimensional limit takes a rather nice form (see Theorem 3.2). In the multi-dimensional case, when $P \neq 0$, (50) leads to a finite system of coupled equations that implicitly describe γ. The additional justification required to establish that this system of equations uniquely determines γ is provided in Section 3.4.

3.3.2 The form of the generalized one-dimensional derivative In order to describe the limit in (51) we need to first introduce some definitions. For $f, g, g_1, g_2 \in \mathcal{D}_{\lim}(\mathbb{R})$, define

$$H(f, g, g_1, g_2)(t) \doteq \begin{cases} 0 & \text{for } t \in \mathcal{T}_\ell(f), \\ S(f, g, g_1, g_2)(t) \vee 0 & \text{for } t \in \mathcal{T}_m(f), \\ S(f, g, g_1, g_2)(t) & \text{for } t \in \mathcal{T}_u(f), \end{cases} \tag{53}$$

where

$$\mathcal{T}_\ell = \mathcal{T}_\ell(f) \doteq \{t \in [0, \infty) : \bar{f}(t) < 0\}, \tag{54}$$

$$\mathcal{T}_m = \mathcal{T}_m(f) \doteq \{t \in [0, \infty) : \bar{f}(t) = 0\}, \tag{55}$$

$$\mathcal{T}_u = \mathcal{T}_u(f) \doteq \{t \in [0, \infty) : \bar{f}(t) > 0\}, \tag{56}$$

and

$$S(f,g,g_1,g_2)(t) \doteq \sup_{s\in\Phi_f^L(t)} \{g_1(s)\} \vee \sup_{s\in\Phi_f(t)} \{g(s)\} \vee \sup_{s\in\tilde{\Phi}_f^R(t)} \{g_2(s)\}, \tag{57}$$

with Φ_f, Φ_f^L and $\tilde{\Phi}_f^R$ defined as in (11), (12) and (13), respectively. Moreover, let

$$S_1(f,g) \doteq S(f,g,g,g) \quad \text{and} \quad H_1(f,g) \doteq H(f,g,g,g) \tag{58}$$

and, likewise, let

$$S_2(f,g) \doteq S(f,g,g_l,g_r) \quad \text{and} \quad H_2(f,g) \doteq H(f,g,g_l,g_r), \tag{59}$$

where g_l and g_r are the left and right regularizations of g, as defined in (6). It is easy to see that for $f \in \mathcal{D}_{\lim}(\mathbb{R})$ and $t \in [0,\infty)$, $\Phi_f^L(t) \cup \Phi_f(t) \cup \tilde{\Phi}_f^R(t) \neq \emptyset$ and hence $S(f,g,g_1,g_2)$, $S_1(f,g)$ and $S_2(f,g)$ are always finite. The following theorem provides a useful characterization of the generalized one-dimensional derivative.

Theorem 3.2 (Generalization of the one-dimensional derivative) *Consider a sequence $\{g_\varepsilon\} \subseteq \mathcal{D}_{\lim}(\mathbb{R})$ such that $\sup_{\varepsilon>0} \|g_\varepsilon\|_N < \infty$ for every $N \in [0,\infty)$ and for every $s \in [0,\infty)$*

$$\varepsilon_1 \le \varepsilon_2 \quad \Rightarrow \quad g_{\varepsilon_1}(s) \le g_{\varepsilon_2}(s).$$

Moreover, let $g, g^{*,l}, g^{*,r} \in \mathcal{D}_{\lim}(\mathbb{R})$ be such that $g_\varepsilon \downarrow g \in \mathcal{D}_{\lim}(\mathbb{R})$, $g_{\varepsilon,l} \downarrow g^{*,l}$ and $g_{\varepsilon,r} \downarrow g^{*,r}$ pointwise as $\varepsilon \downarrow 0$, where $g_{\varepsilon,l}$ and $g_{\varepsilon,r}$ are respectively the left and right regularizations of g_ε. For $f \in \mathcal{D}_{\lim}(\mathbb{R})$, if

$$\tilde{\gamma}_\varepsilon \doteq \overline{\varepsilon^{-1}f + g_\varepsilon} \vee 0 - \overline{\varepsilon^{-1}f} \vee 0 \tag{60}$$

then $\tilde{\gamma}_\varepsilon \to \tilde{\gamma} \in \mathcal{D}_{\lim}(\mathbb{R})$ pointwise as $\varepsilon \downarrow 0$, where the generalized derivative takes the form

$$\tilde{\gamma} \doteq H(f,g,g^{*,l},g^{*,r}), \tag{61}$$

and H is given by (53). Moreover, if $\{g_\varepsilon\}_{\varepsilon>0} \subset \mathcal{C}(\mathbb{R})$ then the generalized derivative takes the simpler form

$$\tilde{\gamma} = H_1(f,g), \tag{62}$$

and if in addition $f \in \mathcal{C}(\mathbb{R})$, then $\tilde{\gamma} = H_1(f,g) = H_2(f,g)$ and

$$S_1(f,g) = S_2(f,g) = \sup_{s\in\Phi_f(t)} [g(s)]. \tag{63}$$

Lastly, if $f \in \mathcal{D}_{\lim}(\mathbb{R})$ and $g_\varepsilon \to g$ in the uniform topology, then

$$\tilde{\gamma} = H_2(f,g). \tag{64}$$

The proof of Theorem 3.2 is relegated to Section 5.2.2. Here, we make some observations on the theorem.

Remark 3.1 (The generalized one-dimensional derivative)

(i) For $f, g \in \mathcal{D}_{\lim}$, the limit in (52) is equal to the function $H_2(f,g)$ defined in (59). Indeed, when $g_\varepsilon = g$ is independent of ε, then clearly $g^{*,l} = g_l$ and $g^{*,r} = g_r$ (see Lemma 5.3(ii)) and by (59) we have $H(f,g,g^{*,l},g^{*,r}) = H(f,g,g_l,g_r) = H_2(f,g)$. If, in addition, $f, g \in \mathcal{C}(\mathbb{R})$ then $\Phi_f^L(t) \cup \Phi_f(t) \cup \Phi_f^R(t) = \Phi_f(t)$ and $g(s-) = g(s) = g(s+)$, so that

$$S_2(f,g)(t) = \sup_{s\in\Phi_f(t)} [g(s)]. \tag{65}$$

Thus Theorem 3.2 contains as a special case the results in [17, Lemma 5.2] and [33, Theorem 9.3.1].

(ii) The notation $\tilde{\Phi}_f^R$ rather than Φ_f^R is used in the definitions of S, S_1 and S_2 in order to emphasize that $t \notin \tilde{\Phi}_f^R(t)$, in contrast with the sets $\Phi_f^L(t)$ and $\Phi_f(t)$, which could contain t. In the definition for $S_2(f,g)$ in [33, Theorem 9.3.1], however, the set $\tilde{\Phi}_f^R$ is replaced by the set

$$\Phi_f^R(t) \doteq \{s \in [0,t] : f(s+) = \overline{f}(t)\}, \tag{66}$$

which could contain t. This gives the correct expression when $g \in \mathcal{D}_r(\mathbb{R})$, which is the setting considered in [33]. However, the following example shows that when $g \in \mathcal{D}_{\lim}(\mathbb{R})$, S_2 must be defined with $\tilde{\Phi}_f^R$ rather than with Φ_f^R, even when $f \in \mathcal{C}(\mathbb{R})$.
Example 1. Let $f(s) \doteq s1_{[0,1)}(s) + 1_{[1,2]}(s)$ for $s \in [0,2]$, and for every $\varepsilon > 0$, let $g_\varepsilon(s) = g(s) \doteq 1_{(1,2]}$ for $s \in [0,2]$. Then f is continuous and g is left continuous and has finite right limits. Moreover, from the definition of f it follows that $\Phi_f^L(1) = \Phi_f(1) = \{1\}$ and $\tilde{\Phi}_f^R(1) = \emptyset$, while $\Phi_f^R(1) = \{1\}$. By the definitions of S and S_2 given in (57) and (59), respectively, we then have $S_2(f,g)(1) = g(1-) \vee g(1) = 0$, while for the modified case (i.e., with $\tilde{\Phi}_f^R$ replaced by Φ_f^R in the definition of S_2) we see that $S_2(f,g)(1) = g(1-) \vee g(1) \vee g(1+) = 1$. However, by direct verification it is easy to see in this simple example that

$$\lim_{\varepsilon \downarrow 0} \left[\overline{\varepsilon^{-1} f + g_\varepsilon}(1) - \overline{\varepsilon^{-1} f}(1) \right] = \lim_{\varepsilon \downarrow 0} \left[\overline{\varepsilon^{-1} f + g}(1) - \overline{\varepsilon^{-1} f}(1) \right] = g(1) = 0.$$

(iii) When f and g_ε, $\varepsilon > 0$, are continuous and $g_\varepsilon \downarrow g$ as $\varepsilon \downarrow 0$, it follows from Theorem 3.2 (specifically, (62) and (63)) that $\tilde{\gamma} = H_1(f,g) = H_2(f,g)$. Since the limit in (51) is given by $\tilde{\gamma}$ and, by Remark 3.1(i) above, $H_2(f,g)$ is the limit in (52), the two limits in (51) and (52) coincide in this special case. However, the following two examples demonstrate that these two limits need not be equal for general $f, g, g_\varepsilon \in \mathcal{D}_{\lim}$. In particular, Example 2 illustrates why the family of functions $\{g_\varepsilon\}_{\varepsilon>0}$ needs to be continuous, while Example 3 shows why f must be continuous, for the two limits to coincide.
Example 2. Let $f(s) \doteq s$ and $g(s) \doteq 1$ for $s \in [0,2]$. Also, for $\varepsilon > 0$, let

$$g_\varepsilon(t) \doteq \begin{cases} 1 & \text{for } t \in [0, 1-\varepsilon), \\ 2 & \text{for } t \in [1-\varepsilon, 1), \\ 1 & \text{for } t \in [1,2]. \end{cases}$$

Then clearly f, $g \in \mathcal{C}(\mathbb{R})$ and each g_ε lies in $\mathcal{D}_r(\mathbb{R})$. Moreover, $g_\varepsilon \downarrow g$ pointwise, $\Phi_f^L(1) = \Phi_f(1) = \{1\}$, $\tilde{\Phi}_f^R(1) = \emptyset$ and the fact that $g_{\ell,\varepsilon}(1) = g_\varepsilon(1-) = 2$ for every $\varepsilon > 0$ implies $g^{*,l}(1) = 2$. Since $\mathcal{T}_u(f) = (0,2]$, Theorem 3.2 shows that at $t = 1$, the limit in (51), is equal to $S(f,g,g^{*,l},g^{*,r})(1) = g^{*,l}(1) \vee g(1) = 2$. On the other hand, by Remark 3.1(i), the limit in (52) at $t = 1$ equals $H_2(f,g)(1) = S_2(f,g)(1) = S(f,g,g_\ell,g_r)(1) = g(1-) \vee g(1) = 1 \neq 2$.
Example 3. Define $f(s) \doteq s1_{[0,1)}$, $g(s) \doteq 1_{[1,2]}(s)$ for $s \in [0,2]$ and, for $\varepsilon > 0$, let

$$g_\varepsilon(s) \doteq \begin{cases} 0 & \text{for } s \in [0, 1-\varepsilon), \\ \dfrac{s-(1-\varepsilon)}{\varepsilon} & \text{for } s \in [1-\varepsilon, 1), \\ 1 & \text{for } s \in [1,2]. \end{cases}$$

Then clearly each g_ε lies in $\mathcal{C}(\mathbb{R})$ and as $\varepsilon \downarrow 0$, g_ε converges pointwise monotonically down to the function $g \in \mathcal{D}_{\lim}(\mathbb{R})$. Moreover, $f \in \mathcal{D}_{\lim}(\mathbb{R})$, $\Phi_f^L(1) = \{1\}$ and $\Phi_f(1) = \tilde{\Phi}^R(1) = \emptyset$. By Remark 3.1(i) above and the fact that $\mathcal{T}_u(f) = (0,2]$, the limit in (52) at $t = 1$ is given by $H_2(f,g)(1) = S_2(f,g)(1) = g(1-) = 0$. On the other hand, since each g_ε is continuous, by (62) of Theorem 3.2, the limit in (51) is equal to $H_1(f,g)(1) = S_1(f,g)(1) = g(1) = 1 \neq 0$.

3.4 The Multi-dimensional Derivative The main result of this section is the proof of Theorem 1.1, which is given in Section 3.4.2. First, in Section 3.4.1, we use the results of Sections 3.2 and 3.3 to obtain an autonomous characterization of $\gamma(\psi,\chi)$ when either $\psi, \chi \in \mathcal{C}$ or when $\psi \in \mathcal{D}_c$ and $\chi \in \mathcal{D}_{\lim}$.

3.4.1 An autonomous characterization Fix $\psi, \chi \in \mathcal{D}_{\lim}$ and, as usual, let γ_ε and ξ be defined,respecitvely, via (40) and (42). Also, for $i = 1, \ldots, K$, let χ_l^i and χ_r^i be the left and right regularizations of χ^i, let $\gamma^{*,l}$ and $\gamma^{*,r}$ be the limits of the left and right regularized sequences $\{\gamma_{\varepsilon,l}\}$ and $\{\gamma_{\varepsilon,r}\}$, respectively, and let γ be the pointwise limit of $\{\gamma_\varepsilon\}$ (which exists by Lemma 3.3). In addition, let the functions H and H_j, $j = 1,2$, be defined as in Section 3.3.2. Combining the characterization (50) of γ^i as a generalized one-dimensional derivative with Theorem 3.2, it follows that for $i = 1, \ldots, K$,

$$\gamma^i(\psi,\chi) = H\left(-\xi^i, -\chi^i + [P\gamma]^i, -\chi_l^i + [P\gamma^{*,l}]^i, -\chi_r^i + [P\gamma^{*,r}]^i\right). \tag{67}$$

Since, in general, $\gamma^{*,l}$ and $\gamma^{*,r}$ depend on the structure of the sequence $\{\gamma_\varepsilon(\psi,\chi\}$, and are therefore not uniquely determined by $\gamma(\psi,\chi)$, this does not lead to an autonomous characterization of $\gamma(\psi,\chi)$. However, we now show that under additional assumptions on ψ and χ, $\gamma^{*,l}$ and $\gamma^{*,r}$ are uniquely determined by $\gamma(\psi, \chi)$. Specifically, consider the case when $\psi, \chi \in \mathcal{C}$. Then, by Theorem 3.1, $\theta, \theta_\varepsilon \in \mathcal{C}$ and consequently $-\xi$ and $\gamma_\varepsilon \in \mathcal{C}$. Likewise, if $\psi \in \mathcal{D}_c$ and $\chi \in \mathcal{D}_{\lim}$, Theorem 3.1 shows that $\xi \in \mathcal{D}_c$. So it follows from Theorem 3.2 that $\gamma \doteq \gamma(\psi, \chi)$ satisfies

$$\gamma^i = H_j\left(-\xi^i, -\chi^i + [P\gamma]^i\right), \qquad i = 1, \ldots, K, \tag{68}$$

with $j = 1$ when $\psi, \chi \in \mathcal{C}$ and $j = 2$ when $\psi \in \mathcal{D}_c$ and $\chi \in \mathcal{D}_{\lim}$. In both cases, the system of equations (67) reduces to an autonomous set of equations.

Lemma 3.6 *Given an H-R matrix R, $P \doteq I - R$, and $\psi, \chi \in \mathcal{D}_{\lim}$, for $j = 1, 2$, the system of equations (68) has a unique solution $\gamma_{(j)} \doteq \gamma_{(j)}(\psi, \chi) \in \mathcal{D}_{\lim}$. Moreover, for $j = 1, 2$, given any $\gamma_{0,j} \in \mathcal{D}_{\lim}$, if the sequence $\{\gamma_{n,j}\}$ is defined recursively by*

$$\gamma_{n+1,j} \doteq H_j\left(-\xi^i, -\chi^i + [P\gamma_{n,j}]^i\right)$$

then for every $N < \infty$, $||\gamma_{(j)} - \gamma_{n,j}||_N \to 0$ as $n \uparrow \infty$.

Proof. Fix $\psi, \chi \in \mathcal{D}_{\lim}$, and $N < \infty$, let ξ be defined via (42) and recall from Lemma 3.1 that $\gamma(\psi, \chi)$ is uniformly bounded on $[0, N]$ in $\mathcal{D}_{\lim}$ (with respect to the sup norm). For $j = 1, 2$, consider the mapping $\Lambda_j : \mathcal{D}_{\lim} \mapsto \mathcal{D}_{\lim}$ defined, for $\gamma \in \mathcal{D}_{\lim}$, by

$$[\Lambda_j(\gamma)]^i \doteq H_j\left(-\xi^i, \chi^i + [P\gamma]^i\right), \qquad i = 1, \ldots, K.$$

For $j = 1, 2$ (and fixed ξ, χ), from the definition of H_j it is clear that Λ_j maps bounded sets of $\mathcal{D}_{\lim}$ to bounded sets of $\mathcal{D}_{\lim}$. We shall show below that each Λ_j is a contraction mapping on $\mathcal{D}_{\lim}$. Since $\mathcal{D}_{\lim}$ endowed with the sup norm metric is a complete metric space, the existence of a unique fixed point for Λ_j and convergence of iterations of the map Λ_j from any starting point to this unique fixed point then follows from standard theorems [30, Theorems 5.2.1 and 5.2.3].

To establish the contraction property we first consider the case when the maximum row sum of the matrix P is equal to $1 - \delta < 1$. The general case can then be handled in the usual way using diagonal similarity transforms (as in, for example, the proof of Lemma 3.5). Let $\gamma_1, \gamma_2 \in \mathcal{D}_{\lim}$. Then the definition of H_1, along with Lemma 3.2, shows that

$$\begin{aligned}
||[\Lambda_1(\gamma_1)]^i - [\Lambda_1(\gamma_2)]^i||_N &= \max_{i=1,\ldots,K} ||H_1^i\left(-\xi^i, -\chi^i + [P\gamma_1]^i\right) - H_1^i\left(-\xi^i, -\chi^i + [P\gamma_2]^i\right)||_N \\
&\leq \max_{i=1,\ldots,K} ||[P\gamma_1]^i - [P\gamma_2]^i||_N \\
&\leq \max_{i=1,\ldots,K} \sum_{k=1}^K P_{ik} ||\gamma_1^k - \gamma_2^k||_N \\
&\leq (1-\delta) \max_{k=1,\ldots,K} ||\gamma_1^k - \gamma_2^k||_N,
\end{aligned}$$

which proves the contraction property since $1 - \delta < 1$. The proof for Λ_2 follows analogously and is thus omitted. □

The next lemma, which establishes a useful equivalence, will make use of the following consequence of the definition (42) of ξ, the fact that $P = I - R$ and Theorem 3.1: for $t \in [0, \infty)$,

$$\phi^i(t) = \psi^i(t) + [R\theta]^i(t) = \psi^i(t) - [P\theta]^i(t) + \theta^i(t) = \xi^i(t) + \overline{-\xi^i}(t) \vee 0. \tag{69}$$

Lemma 3.7 *The set of equations in (68) with $j = 1$ coincides with the set of equations in (17). In particular, for every t such that $\overline{-\xi^i}(t) \geq 0$, $\Phi^i(t) = \Phi_{-\xi^i}(t)$.*

Proof. The equivalence is easily deduced from the following observations. Note that (69) implies that $\overline{-\xi^i}(t) < 0$ if and only if $\inf_{s\in[0,t]} \phi^i(s) > 0$ (which also implies $\theta^i(t) = 0$); $\overline{-\xi^i}(t) = 0$ if and only if $\inf_{s\in[0,t]} \phi^i(s) = 0$ and $\theta^i(t) = 0$; and $\overline{-\xi^i}(t) > 0$ if and only if $\theta^i(t) > 0$. Now, $\inf_{s\in[0,t]} \phi^i(s) > 0$ for all $s \in [0, t_\ell^i)$ and $\phi^i(t_\ell^i) = 0$ where for the last equality, we used the fact that if ψ lies in $\mathcal{D}_r$ (respectively, $\mathcal{C}$), then ϕ, θ, ξ also lie in $\mathcal{D}_r$ (respectively, $\mathcal{C}$). Hence, $\mathcal{T}_\ell(-\xi^i) = [0, t_\ell^i)$. A similar reasoning shows that $(t_u^i, \infty) \subseteq \mathcal{T}_u(-\xi^i) \subseteq [t_u^i, \infty)$ and $t_u^i \in \mathcal{T}_u(-\xi^i)$ if and only if $\theta^i(t_u^i) > 0$, which can take place only if θ^i is not left continuous at t_u^i. In particular, this shows $\mathcal{T}_u(-\xi^i) = (t_u^i, \infty)$ if ψ is continuous. Lastly, from (69) it also follows that $\Phi^i(t) = \Phi_{-\xi^i}(t)$ for all t such that $\overline{-\xi^i}(t) \geq 0$ or, equivalently, for all $t \in \mathcal{T}_u(-\xi^i) \cup \mathcal{T}_m(-\xi^i)$. □

3.4.2 Proof of Theorem 1.1 Property (i) of Theorem 1.1 was proved at the end of Section 3.2.3, where it was also shown that for $\psi, \chi \in \mathcal{D}_{\lim}$, $\nabla_\chi \Gamma(\psi) = \chi + R\gamma$. Now, suppose $\psi, \chi \in \mathcal{C}$. From Theorem 3.2 (see also the discussion at the beginning of Section 3.4.1) and Lemma 3.6 it follows that $\gamma(\psi, \chi) = \gamma_{(1)}$ where $\gamma_{(1)}$ is the unique solution in $\mathcal{D}_{\lim}$ to the corresponding system of equations (68). However, by Lemma 3.7, these equations are equivalent to the equations in (68) with $j = 1$. Moreover, by Lemma 3.3 $\gamma_{(1)}$ is the decreasing limit of continuous functions γ^ε. This immediately implies that the convergence is uniform on compact subsets of continuity points of the limit, and that $\gamma_{(1)}$ is upper semicontinuous (see, e.g., [17]). This completes the proof of the theorem.

4. Discontinuities of the Derivative for Continuous $\psi, \chi \in \mathcal{C}$ Throughout this section we fix an ORP with an H-R constraint matrix $R \in \mathbb{R}^{K\times K}$ and $\psi, \chi \in \mathcal{C}$ and, as usual, let ξ^i be defined as in (42). For conciseness, we denote the corresponding unique solution $\gamma_{(1)}$ to the set of equations (17) simply by γ. The main result of this section is the proof of Theorem 1.2, which is presented in Section 4.3. First, in Section 4.1 we derive necessary and sufficient conditions for the existence of discontinuities in γ^i in terms of properties of the set $\Phi_{-\xi^i}$. In Section 4.2, we provide equivalences between properties of the set $\Phi_{-\xi^i}$ and certain sets introduced in Definition 1.4, which allow a more physically intuitive description of when discontinuities may occur.

4.1 Classification of the Discontinuities of $\gamma_{(1)}$ Theorem 4.1 provides necessary and sufficient conditions for the existence of discontinuities in γ^i in terms of properties of the set functions $\Phi_{-\xi^i}(\cdot)$. We first introduce some additional notation. For $i = 1, \ldots, K$, define

$$\mathcal{A}^i \doteq \{s \in [t_l^i, t_u^i] : -\chi^i(s) + [P\gamma]^i(s) > 0 \text{ and } -\xi^i(s) = 0\}, \tag{70}$$

and let

$$\tilde{t}_u^i \doteq \inf\{t : t \in \mathcal{A}^i\} \wedge t_u^i \tag{71}$$

Note that due to the convention that $\inf(\emptyset) = \infty$, if $\mathcal{A}^i = \emptyset$ then $\tilde{t}_u^i = t_u^i$. In Lemma 4.1, we first establish some properties of γ^i that will be used to prove Theorem 4.1.

Lemma 4.1 *For $i = 1, \ldots, K$, the following properties hold.*

(i) $\gamma^i(t) = 0$ *for* $t \in [0, \tilde{t}_u^i)$.

(ii) *If* $\mathcal{A}^i = \emptyset$ *then* $\gamma^i(\tilde{t}_u^i) = \gamma^i(t_u^i) = 0$. *On the other hand, if* $\mathcal{A}^i \neq \emptyset$ *then*

$$\gamma^i(\tilde{t}_u^i) = -\chi^i(\tilde{t}_u^i) + [P\gamma]^i(\tilde{t}_u^i) \geq 0. \tag{72}$$

Moreover, $\tilde{t}_u^i \in LDisc(\gamma^i)$ *implies* $\tilde{t}_u^i \in \Phi_{-\xi^i}(\tilde{t}_u^i)$ *and (72) holds with a strict inequality.*

(iii) *For* $t \in (\tilde{t}_u^i, \infty)$,

$$\gamma^i(t) = \sup_{s\in\Phi_{-\xi^i}(t)} \left[-\chi^i(s) + [P\gamma]^i(s)\right]. \tag{73}$$

(iv) *For* $t \in (\tilde{t}_u^i, \infty)$, *if* $\{t\} \neq \Phi_{-\xi^i}(t)$ *then t is a point of left increase for* γ^i *and*

$$\gamma^i(t-) = \sup_{s\in\Phi_{-\xi^i}(t)\setminus\{t\}} \left[-\chi^i(s) + [P\gamma]^i(s)\right], \tag{74}$$

whereas if $\{t\} = \Phi_{-\xi^i}(t)$ *then*

$$\gamma^i(t-) = -\chi^i(t) + [P\gamma]^i(t-) \leq -\chi^i(t) + [P\gamma]^i(t) = \gamma^i(t). \tag{75}$$

(v) *For* $t \in [0, \infty)$, $t \in Disc(\gamma^i)$ *implies* $t \in \Phi_{-\xi^i}(t)$ *and*

$$\gamma^i(t) = \left[-\chi^i(t) + [P\gamma]^i(t)\right] \vee \gamma^i(t-). \tag{76}$$

(vi) *For* $t \in [\tilde{t}_u^i, \infty)$, *if* $\Phi_{-\xi^i}(r) \cap [0, t] = \emptyset$ *for some* $r > t$ *then*

$$\gamma^i(t+) = \gamma^i(t), \tag{77}$$

whereas if

$$\Phi_{-\xi^i}(s) \subseteq (t, s] \quad \textit{for all } s > t, \tag{78}$$

then

$$\gamma^i(t+) = -\chi^i(t) + [P\gamma]^i(t+). \tag{79}$$

Proof. Fix $i \in \{1, \ldots, K\}$. For $t \in [t^i_\ell, t^i_u]$, the equality $\Phi^i(t) = \Phi_{-\xi^i}(t)$ established in Lemma 3.7 and the fact that $\overline{-\xi}(t) = 0$ show that $-\xi^i(s) = 0$ for $s \in \Phi^i(t)$. Property (i) and the first assertion of property (ii) of the lemma then follow directly from the characterization (17) for γ^i. Now suppose $\mathcal{A}^i \neq \emptyset$. Then, $\tilde{t}^i_u < t^i_u$ and there must exist a sequence $\{s_n\}_{n\in\mathbb{N}} \subset [\tilde{t}^i_u, t^i_u]$ such that $s_n \downarrow \tilde{t}^i_u$, as $n \to \infty$, and for every $n \in \mathbb{N}$, $-\xi^i(s_n) = 0$ and $-\chi^i(s_n) + [P\gamma]^i(s_n)] > 0$. Since $s_n \leq t^i_u$, this implies $\overline{-\xi^i}(s_n) = 0$. and hence $s_n \in \Phi_{-\xi^i}(s_n)$. By (17), it then follows that

$$\gamma^i(s_n) = \sup_{s \in \Phi_{-\xi^i}(s_n)} \left[-\chi^i(s) + [P\gamma]^i(s)\right] \vee 0 \geq \left[-\chi^i(s_n) + [P\gamma]^i(s_n)\right] > 0.$$

Taking limits as $n \to \infty$ in the last expression, we obtain

$$\gamma^i(\tilde{t}^i_u+) \geq -\chi^i(\tilde{t}^i_u) + [P\gamma]^i(\tilde{t}^i_u+) \geq 0. \tag{80}$$

Now $-\chi^i(s) + [P\gamma]^i(s) \leq 0$ for $s < \tilde{t}^i_u$ and, due to the continuity of ξ, $\xi^i(\tilde{t}^i_u) = 0$, $\tilde{t}^i_u \in \Phi_{-\xi^i}(\tilde{t}^i_u)$. Together with (80) and the uppersemicontinuity of γ, this implies that

$$0 \leq \gamma^i(\tilde{t}^i_u+) \leq \gamma^i(\tilde{t}^i_u) = \sup_{s \in \Phi_{-\xi^i}(\tilde{t}^i_u)} [-\chi^i(s)+[P\gamma]^i(s)] \vee 0 = [-\chi^i(\tilde{t}^i_u)+[P\gamma]^i(\tilde{t}^i_u)] \vee 0 = [-\chi^i(\tilde{t}^i_u)+[P\gamma]^i(\tilde{t}^i_u)],$$

where the nonnegativity of the last term follows from the last inequality in (80) and the upper semicontinuity of $P\gamma$. This establishes (72). Now, suppose $\tilde{t}^i_u \in LDisc(\gamma^i)$. Then it must be that $\mathcal{A}^i \neq \emptyset$, and so the above argument also establishes the last statement of property (ii).

For property (iii), it suffices to consider $t \in (\tilde{t}^i_u, t^i_u]$ because the representations (73) and (17) coincide for $t \in (t^i_u, \infty)$ by Lemma 3.7. Since $-\overline{\xi^i}(t) = 0$ for $t \in (\tilde{t}^i_u, t^i_u]$, $\Phi_{-\xi^i}(\cdot)$, and therefore γ^i is monotonically non-decreasing on that interval. When combined with the representation (17) for γ^i and the fact that $\gamma^i(\tilde{t}^i_u) \geq 0$ by property (ii), this yields the representation (73).

For the fourth property, fix $t > \tilde{t}^i_u$ and suppose $\{t\} \neq \Phi_{-\xi^i}(t)$. Since $\Phi_{-\xi^i}(t) \cap [0, t)$ is non-empty, this implies there exists $s < t$ such that $\overline{-\xi^i}(s) = \overline{-\xi^i}(t)$. Therefore, for every $r \in [s, t]$, $\Phi_{-\xi^i}(r) = \Phi_{-\xi^i}(t) \cap [0, r]$. As a consequence,

$$\begin{aligned} \gamma^i(t-) = \lim_{r\uparrow t} \lim_{s \in \Phi_{-\xi^i}(r)} \left[-\chi^i(s) + [P\gamma]^i(s)\right] &= \lim_{r\uparrow t} \sup_{s \in \Phi_{-\xi^i}(t)\cap[0,r]} \left[-\chi^i(s) + [P\gamma]^i(s)\right] \\ &= \sup_{s \in \Phi_{-\xi^i}(t)\cap[0,t)} \left[-\chi^i(s) + [P\gamma]^i(s)\right], \end{aligned}$$

which proves (74). Now, consider the case when $\{t\} = \Phi_{-\xi^i}(t)$. Let s_n be an increasing sequence such that $s_n \uparrow t$, and let $u_n \in [0, s_n]$ satisfy

$$u_n = \min\{u \in [0, s_n] : \overline{-\xi^i}(u) = \overline{-\xi^i}(s_n)\}.$$

We claim that then $u_n \uparrow t$. Indeed, since u_n is uniformly bounded, there exists a subsequence (which we denote again by u_n) that converges to a limit $u_* \in [0, t]$. Since ξ^i is continuous, clearly $\overline{-\xi^i}(s_n) \to \overline{-\xi^i}(t)$ and hence $\overline{-\xi^i}(u_n) \to \overline{-\xi^i}(t)$, as $n \to \infty$. Therefore, $u_* \in \Phi_{-\xi^i}(t)$. Due to the assumption $\Phi_{-\xi^i}(t) = \{t\}$, we conclude that $u_* = t$. Also, observe that

$$\gamma^i(s_n) = \max_{s \in \Phi_{-\xi^i}(s_n)} [-\chi^i(s) + [P\gamma]^i(s)] = \max_{s \in \Phi_{-\xi^i}(s_n)\cap[u_n, s_n]} [-\chi^i(s) + [P\gamma]^i(s)].$$

Take limits as $n \uparrow \infty$ on both sides of the above equality and use the fact that $u_n \uparrow t$ to obtain

$$\gamma^i(t-) = -\chi^i(t) + [P\gamma]^i(t-) \leq -\chi^i(t) + [P\gamma]^i(t) = \gamma^i(t),$$

where the inequality is a consequence of the upper semicontinuity of $[P\gamma]^i$ and, due to (73), the last equality is a trivial consequence of the fact that $\{t\} = \Phi_{-\xi^i}(t)$. This proves (75).

Due to properties (i) and (ii) and the fact that $\tilde{t}^i_u \in \Phi_{-\xi^i}(\tilde{t}^i_u)$, to establish property (v), the relation (76) needs to be verified only for $t \in (\tilde{t}^i_u, \infty)$. First, we establish the contrapositive of the first statement. Suppose $t \notin \Phi_{-\xi^i}(t)$. Then, since $\xi \in \mathcal{C}$, there must exist $\delta > 0$ such that for $s \in [t - \delta, t + \delta]$, $-\xi^i(s) < \overline{-\xi^i}(t)$ and $\Phi_{-\xi^i}(s) = \Phi_{-\xi^i}(t)$, which in turn means that $\gamma^i(s) = \gamma^i(t)$, thus showing that γ^i is continuous at t. Hence $t \in Disc(\gamma^i)$ only if $t \in \Phi_{-\xi^i}(t)$. Along with the relations (73)–(75), this yields (76) and thus property (v) follows.

To establish property (vi), first fix $t \in [0, \infty)$, and note that given a family of non-empty sets $A_u, u > t$, with the property that $A_u \subseteq (t, u]$,

$$\lim_{u\downarrow t} \sup_{s \in A_u} \left[-\chi^i(s) + [P\gamma]^i(s)\right] = -\chi^i(t) + [P\gamma]^i(t+). \tag{81}$$

Suppose $\Phi_{-\xi^i}(r) \cap [0,t] \neq \emptyset$ for some $r > t$. In this case, $\overline{-\xi^i}(r) = \overline{-\xi^i}(t)$. Hence, for all $u \in [t,r]$, $\Phi_{-\xi^i}(u) \cap [0,t] = \Phi_{-\xi^i}(t)$ and $\Phi_{-\xi^i}(u) \cap (t,u] = \Phi_{-\xi^i}(r) \cap (t,u]$. The representation (73) for γ^i then shows that for every $u \in [t,r]$,

$$\gamma^i(u) = \sup_{s\in\Phi_{-\xi^i}(t)}[-\chi^i(s)+[P\gamma]^i(s)] \vee \sup_{s\in\Phi_{-\xi^i}(r)\cap(t,u]}[-\chi^i(s)+[P\gamma]^i(s)] = \gamma^i(t) \vee \sup_{s\in\Phi_{-\xi^i}(r)\cap(t,u]}[-\chi^i(s)+[P\gamma]^i(s)].$$

Taking limits as $u \downarrow t$ and invoking (81) and using the upper semicontinuity of γ^i we have $\gamma^i(t) \geq \gamma^i(t+) \geq \gamma^i(t)$, which implies $\gamma^i(t+) = \gamma^i(t)$. On the other hand, if (78) holds then (79) is a direct consequence of (73) and (81) and the proof of the lemma is complete. □

We now state and prove the main result of the section.

THEOREM 4.1 (DISCONTINUITIES OF $\gamma_{(1)}$) *Let $\nabla\Gamma \doteq \nabla_\chi\Gamma(\psi) = \chi + [R\gamma]^i$. Then, for $i = 1,\ldots,K$, $\gamma^i(t) = 0$ for $t \in [0,\tilde{t}^i_u)$, and the following three properties hold for every $t \in [t^i_u,\infty)$.*

(i) $t \in LDisc(\gamma^i)$ if and only if $t \in \Phi_{-\xi^i}(t)$ and one of the following conditions holds.

L1. $t = \tilde{t}^i_u$ and $0 < -\chi(\tilde{t}^i_u) + [P\gamma]^i(\tilde{t}^i_u)$. In this case, $\nabla\Gamma^i(\tilde{t}^i_u) = 0$.

L2. $t > \tilde{t}^i_u$, $\Phi_{-\xi^i}(t) \neq \{t\}$ and the following equality is satisfied:

$$\sup_{s\in\Phi_{-\xi^i}(t)\setminus\{t\}}[-\chi^i(s) + [P\gamma]^i(s)] < -\chi^i(t) + [P\gamma]^i(t). \tag{82}$$

In this case, $\nabla\Gamma^i(t) = 0$. Moreover, if t is not isolated in $\Phi_{-\xi^i}(t)$ then $t \in LDisc([P\gamma]^i)$ and $\nabla\Gamma^i(t-) \geq \nabla\Gamma^i(t)$.

L3. $t > \tilde{t}^i_u$, $\{t\} = \Phi_{-\xi^i}(t)$ and $t \in LDisc([P\gamma]^i)$. In this case, $\nabla\Gamma^i(t-) = \nabla\Gamma^i(t) = 0$.

(ii) $t \in RDisc(\gamma^i)$ if and only if $t \in \Phi_{-\xi^i}(t)$, $\Phi_{-\xi^i}(s) \subset (t,s]$ for all $s > t$ and one of the following conditions is satisfied:

R1. $\Phi_{-\xi^i}(t) \neq \{t\}$, $[P\gamma]^i$ is right continuous at t and

$$\sup_{s\in\Phi_{-\xi^i}(t)\setminus\{t\}}[-\chi^i(s) + [P\gamma]^i(s)] > -\chi^i(t) + [P\gamma]^i(t). \tag{83}$$

In this case, γ^i is left continuous at t and $\nabla\Gamma^i(t) > \nabla\Gamma^i(t+) = 0$.

R2. $\Phi_{-\xi^i}(t) \neq \{t\}$ and $t \in RDisc([P\gamma]^i)$. In this case $\nabla\Gamma^i(t) \geq \nabla\Gamma^i(t+) = 0$.

R3. $\{t\} = \Phi_{-\xi^i}(t)$ and $t \in RDisc([P\gamma]^i)$. In this case, $\nabla\Gamma^i(t) = \nabla\Gamma^i(t+) = 0$.

(iii) $t \in LDisc(\gamma^i) \cap RDisc(\gamma^i) = SDisc(\gamma^i)$ if and only if $t \in \Phi_{-\xi^i}(t)$, $\Phi_{-\xi^i}(s) \subset (t,s]$ for all $s > t$ and one of the following conditions holds:

S1. $t \in RDisc([P\gamma]^i)$, and either $t = \tilde{t}^i_u$, or $\{t\} \neq \Phi_{-\xi^i}(t)$ and (82) holds. In the latter case, $\nabla\Gamma^i(t) = \nabla\Gamma^i(t+) = 0$.

S2. $\{t\} = \Phi_{-\xi^i}(t)$ and $t \in LDisc([P\gamma]^i) \cap RDisc([P\gamma]^i)$. In this case, $\nabla\Gamma^i(t-) = \nabla\Gamma^i(t) = \nabla\Gamma^i(t+) = 0$.

PROOF. In the proof below, we will make repeated use of the fact that $\nabla\Gamma^i(t) = \chi^i(t) + [R\gamma]^i(t) = \chi^i(t) - [P\gamma]^i(t) + \gamma^i(t)$ proved in Theorem 1.1(ii), without explicit reference. The fact that $\gamma^i(t) = 0$ for $t \in [0,\tilde{t}^i_u)$ follows from property (i) of Lemma 4.1, and this also implies that $\gamma^i(\tilde{t}^i_u-) = 0$. By the definition of $\tilde{t}^i_u$, $-\xi(\tilde{t}^i_u) = 0$ or, equivalently, $\tilde{t}^i_u \in \Phi_{-\xi^i}(\tilde{t}^i_u)$. By property (ii) of Lemma 4.1, $\gamma^i(\tilde{t}^i_u) > 0 = \gamma^i(\tilde{t}^i_u-)$ if and only if $\gamma^i(\tilde{t}^i_u) = -\chi(\tilde{t}^i_u) + [P\gamma]^i(\tilde{t}^i_u) > 0$, which proves assertion L1 of the theorem.

Now, fix $t \in (\tilde{t}^i_u,\infty)$. Then Lemma 4.1(v) shows that if $t \in LDisc(\gamma^i)$ then $t \in \Phi_{-\xi^i}(t)$ and $\gamma^i(t) = \chi^i(t) + [P\gamma]^i(t) = 0$, so that in this case $\nabla\Gamma^i(t) = 0$. We consider two exhaustive sub-cases, namely when $t \in \Phi_{-\xi^i}(t) \neq \{t\}$ and $\{t\} = \Phi_{-\xi^i}(t)$. In the first case, the fact that (82) holds if and only if $t \in LDisc(\gamma^i)$ follows from (74) and (76). If t is not isolated in $\Phi_{-\xi^i}(t)$, then there must exist a sequence $\{s_n\}_{n\in\mathbb{N}} \subseteq \Phi_{-\xi^i}(t)$ with $s_n \uparrow t$, and by (74) we have

$$\gamma^i(t-) = \sup_{s\in\Phi_{-\xi^i}(t)\setminus\{t\}}[-\chi^i(s) + [P\gamma]^i(s)] \geq \lim_{n\to\infty} -\chi^i(s_n) + [P\gamma]^i(s_n) = -\chi^i(t) + [P\gamma]^i(t-).$$

Thus, we have shown that $\nabla\Gamma^i(t-) \geq 0 = \nabla\Gamma^i(t)$. On the other hand, recalling that γ^i is upper semicontinuous and $t \in LDisc(\gamma^i)$, we also have

$$-\chi^i(t) + [P\gamma]^i(t) = \gamma^i(t) > \gamma^i(t-) \geq -\chi^i(t) + [P\gamma]^i(t-),$$

which implies $t \in LDisc([P\gamma]^i)$. Now, consider the second case when $\{t\} = \Phi_{-\xi^i}(t)$. Then (75) shows that $t \in LDisc(\gamma^i)$ if and only if $[P\gamma]^i$ is discontinuous at t, as stated in L3. This completes the proof of Theorem 4.1(i).

Next, consider the right discontinuities of γ^i. Clearly, $RDisc(\gamma^i) \subset [\tilde{t}^i_u, \infty)$. Properties (v) and (vi) of Lemma 4.1 show that $t \in \Phi_{-\xi^i}(t)$ and $\Phi_{-\xi^i}(s) \subset (t, s]$ for all $s > t$ are necessary conditions for $t \in RDisc(\gamma^i)$, and, moreover, that then $\gamma^i(t+) = -\chi^i(t) + [P\gamma]^i(t+)$, so that $\nabla\Gamma^i(t+) = 0$. Furthermore, due to the upper semicontinuity of $[P\gamma]^i$, it follows that

$$\begin{aligned} \gamma^i(t) = [-\chi^i(t) + [P\gamma]^i(t)] \vee \sup_{s \in \Phi_{-\xi^i}(t) \setminus \{t\}} [-\chi^i(s) + [P\gamma]^i(s)] \quad &\geq \quad -\chi^i(t) + [P\gamma]^i(t) \\ &\geq \quad \chi^i(t) + [P\gamma]^i(t+) = \gamma^i(t+). \end{aligned} \tag{84}$$

Now, suppose that $[P\gamma]^i$ is right continuous. Then $\gamma^i(t+) = -\chi^i(t) + [P\gamma]^i(t)$ and so $t \in RDisc(\gamma^i)$ if and only if $\Phi_{-\xi^i}(t) \neq \{t\}$ and (83) holds. A simple rearrangement of terms also shows that in this case $\nabla\Gamma^i(t) > 0$ and, since the conditions (82) and (83) are mutually exclusive, it follows that γ^i is left continuous at t. On the other hand, if $t \in RDisc([P\gamma]^i)$, then the second inequality in (84) is strict and so we always have $t \in RDisc(\gamma^i)$. Moreover, if $\gamma^i(t) = -\chi^i(t) + [P\gamma]^i(t)$ (as would be the case if $\{t\} = \Phi_{-\xi^i}(t)$), then $\nabla\Gamma^i(t) = \nabla\Gamma^i(t+) = 0$ (as stated in R3), and otherwise $\nabla\Gamma^i(t) > \nabla\Gamma^i(t+) = 0$ (from which R2 follows).

Finally, we analyze the separated discontinuities of γ^i. Note that since $\gamma^i \in \mathcal{D}_{usc}$, it follows that $SDisc(\gamma^i) = LDisc(\gamma^i) \cap RDisc(\gamma^i)$. From properties (i) and (ii) of this theorem, which were proved above, it is clear that, since the condition R1 is incompatible with both L1–L3, the only ways in which a separated discontinuity can occur is if (a) R2 and either L1 or L2 are satisfied or (b) L3 and R3 hold, from which property (iii) of the theorem follows immediately. $\square$

4.2 Alternative Description of the Regimes of (ϕ, θ) In Definition 1.4, the regimes of (ϕ, θ) were described in terms of properties of the solution (ϕ, θ) to the ORP for an input ψ. On the other hand, as shown in Theorem 4.1, the analysis of the discontinuities of γ lead naturally to conditions involving the sets $\Phi_{-\xi^i}(t)$ defined in Section 3.3.2. The following lemma provides a link between these two sets of conditions.

Physical Description	Definition in terms of (ϕ, θ)	Equivalent Condition in terms of $\Phi_{-\xi^i}$
Overloaded	$\phi^i(t) > 0$	$t \notin \Phi_{-\xi^i}(t)$
Underloaded	$\phi^i(t) = 0$ $\Delta\theta^i(t-) \neq 0$, $\Delta\theta^i(t+) \neq 0$	$\{t\} = \Phi_{-\xi^i}(t)$ and $\Phi_{-\xi^i}(s) \subset (t, s] \quad \forall s > t$
Critical	$\phi^i(t) = 0$ and either $\Delta\theta^i(t-) = 0$ or $\Delta\theta^i(t+) = 0$	$t \in \Phi_{-\xi^i}(t)$ and either $\Phi_{-\xi^i}(t) \neq \{t\}$ or $\exists s > t$ such that $\Phi_{-\xi^i}(s) \cap [0, t] \neq \emptyset$
End of Overloading	$\phi^i(t) = 0$ and $\exists \delta > 0$ such that $\phi^i(s) > 0$ for $s \in (t - \delta, t]$	$t \in \Phi_{-\xi^i}(t)$, $\Phi_{-\xi^i}(t) \neq \{t\}$, t is isolated in $\Phi_{-\xi^i}(t)$
Start of Underloading	$\phi^i(t) = 0$ $\Delta\theta^i(t-) = 0$ and $\Delta\theta^i(t+) \neq 0$	$t \in \Phi_{-\xi^i}(t), \Phi_{-\xi^i}(t) \neq \{t\}$ $\Phi_{-\xi^i}(s) \subset (t, s] \quad \forall s > t$

Table 1: Equivalent Descriptions of the Regimes of (ϕ, θ)

Lemma 4.2 *For $i \in \{1, \ldots, K\}$, let $\Phi_{-\xi^i}(t)$ be as defined by (14) and let t^i_l be defined by (18). Then for $t \in [t^i_l, \infty)$, the equivalences in Table 1 are satisfied.*

Proof. The lemma follows essentially from the property proved in Lemma 3.7 that for $t \geq t^i_l$, $\Phi^i(t) = \Phi_{-\xi^i}(t)$, where Φ^i is as defined in (11). The first equivalence (for the overloaded state) is then an immediate consequence of the fact that $t \in \Phi^i(t)$ if and only if $\phi^i(t) = 0$. We now show that $\{t\} = \Phi_{-\xi^i}(t)$ if and only if $\phi^i(t) = 0$ and $\Delta\theta^i(t-) \neq 0$. Indeed, $\{t\} = \Phi_{-\xi^i}(t)$ implies $-\xi^i(t) = \overline{-\xi^i}(t)$ and $\xi^i(s) < \overline{-\xi^i}(t)$

for all $s < t$. In turn, this holds if and only if $\theta^i(s) = \overline{-\xi^i}(s) < \overline{-\xi^i}(t) = \theta^i(t)$ for every $s < t$. Since θ^i is non-decreasing, this is equivalent to the condition that θ^i is not flat to the left of t. By a similar reasoning, it is easy to see that $\Delta\theta^i(t+) \neq 0$ if and only if $\Phi_{-\xi^i}(s) \subset (t, s]$ for all $s > t$. Combining the above equivalences, all the results of Table 1 can be obtained in a straightforward manner. The few remaining details are left to the reader. □

4.3 Discontinuities of the Derivative We now combine the results of Theorem 4.1 and Lemma 4.2 to identify necessary conditions for discontinuities in $\nabla\Gamma$ to occur. We first establish a simple corollary of Theorem 4.1 and then present the proof of Theorem 1.2.

LEMMA 4.3 *For $i = 1, \ldots, K$ and $t \geq 0$, we have the following two properties.*

(*i*) $[P\gamma]^i$ *is left continuous at t if the following condition is satisfied: $t \notin \{\tilde{t}_u^k, k = 1, \ldots, K\}$ and*

$$\textit{there is no critical chain preceding } i \textit{ at time } t \tag{85}$$

(*ii*) $[P\gamma]^i$ *is right continuous at t if the following condition is satisfied:*

$$\textit{there is no sub-critical chain preceding } i \textit{ at time } t. \tag{86}$$

PROOF. Fix $t \in [0, \infty)$ and $i \in \{1, \ldots, K\}$ and suppose (85) holds. Define $\mathcal{E}$ to be the class of empty chains $i = j_0, j_1, \ldots, j_m$ preceding i at time t, and let

$$\tilde{\mathcal{E}} \doteq \{ij_1, \ldots, j_m \in \mathcal{E} : t \in LDisc(\gamma^{j_m})\}.$$

We will argue by contradiction to show that in fact $\tilde{\mathcal{E}} = \emptyset$. Since (85) is satisfied, by Definition 1.5(i) of a critical chain it is clear that there are no cyclic chains in $\mathcal{E}$. Thus, the maximum length of any chain in $\mathcal{E}$ is bounded by K. Suppose $\tilde{\mathcal{E}} \neq \emptyset$ and let $m \in \{1, \ldots, K-1\}$ be the largest integer such that there exists a chain $i, j_1, \ldots, j_m \in \tilde{\mathcal{E}}$. Consider the set $\mathcal{B} \doteq \{k : P_{j_m k} > 0\}$ and $\tilde{B} \doteq \{k \in \mathcal{B} : \phi^k(t) > 0\}$. By Lemma 4.2, if $k \in \tilde{B}$ then $t \notin \phi_{-\xi^k}(t)$ and so Theorem 4.1(i) shows that $t \notin LDisc(\gamma^k)$. On the other hand, if $k \in B \setminus \tilde{B}$ then $i, j_1, \ldots j_m, j_k$ is an empty chain and the maximality of m allows us to conclude once again that $t \notin LDisc(\gamma^{j_k})$. Together, this implies that $[P\gamma]^{j_m} = \sum_{k \in B} P_{j_m k}\gamma^k$ is left continuous at t. Since $t \notin \{\tilde{t}_u^k, k = 1, \ldots, K\}$, by Theorem 4.1(i) it is possible for $t \in LDisc(\gamma^{j_m})$ only if t is isolated in $\Phi_{-\xi_{j_m}}(t) \neq \{t\}$. However, by Lemma 4.2 this corresponds to j_m being at the end of overloading, which in turn implies that $i, j_1, \ldots, j_m$ is a critical chain, which is not possible due to (85). Thus, it must be that $\tilde{\mathcal{E}} = \emptyset$. Now, for all k with $P_{ik} > 0$, either $\phi^k(t) = 0$, in which case jk is an empty chain and $\tilde{\mathcal{E}} = \emptyset$ implies $t \notin LDisc(\gamma^k)$; or $\phi^k(t) > 0$, which is equivalent to $t \notin \Phi_{-\xi^k}(t)$ by Lemma 4.2. In the latter case, Theorem 4.1(i) shows that $t \notin LDisc(\gamma^k)$. Together, this leads to the desired conclusion that $[P\gamma]^i = \sum_{k:P_{ik}>0} P_{ik}\gamma^k$ is left continuous at t.

The proof of the second assertion is similar. Define $\tilde{\mathcal{E}}$ in an analogous fashion, but with LDisc replaced by RDisc. Arguing as above, but this time using property (ii) instead of property (i) of Theorem 4.1, it is possible to conclude that if $i, j_1, \ldots, j_m$ is a maximal chain in $\tilde{\mathcal{E}}$ then $[P\gamma]^{j_m}$ is right continuous at t. By Theorem 4.1(ii) this can only occur if $t \in \Phi_{-\xi^{j_m}}(t)$ and $\Phi_{\xi^{j_m}}(s) \subset (t, s)$ for all $s > t$ (note that here we do not need to assume that $t \notin \{\tilde{t}_u^k, k \in 1, \ldots, K\}$). By Lemma 4.2 this corresponds to j_m being the start of underloading and thus contradicts (86). This shows that $\tilde{\mathcal{E}} = \emptyset$ in this case as well, and the rest of the proof proceeds exactly as before. □

PROOF OF THEOREM 1.2. Fix $t \in (0, \infty)$ and for simplicity, denote $\gamma_{(1)}$ simply by γ. By (16), $\nabla\Gamma^i = \chi^i + [R\gamma]^i = \chi^i + \gamma^i - [P\gamma]^i$. Therefore, $t \in LDisc(\nabla\Gamma^i)$ only if either $t \in LDisc([P\gamma]^i)$ or $t \in LDisc(\gamma^i)$. Suppose $t \notin \{\tilde{t}_u^k, k = 1, \ldots, K\}$ and $t \in LDisc([P\gamma]^i)$. By Lemma 4.3(i), there must exist a critical chain preceding i and, moreover, by L3 of Theorem 4.1(i) for $t \in LDisc(\nabla\Gamma^i)$, one cannot have $\Phi_{-\xi^i}(t) = \{t\}$. In particular, due to Lemma 4.2, this implies that i cannot be underloaded. This corresponds to condition (Lb). If, in addition, i is overloaded, this means $t \notin \Phi_{-\xi^i}(t)$ by Lemma 4.2, and Theorem 4.1(i) then dictates that γ^i is left continuous at t. The inequality in (23) is then a direct result of the upper semicontinuity of $[P\gamma]^i$. Next, suppose that $t \notin LDisc([P\gamma]^i)$ (this holds, for example, if there is no critical chain that precedes i), but $t \in LDisc(\gamma^i)$. Invoking Theorem 4.1(i) once again (this time condition L2) it follows that $t \in LDisc(\nabla\Gamma^i)$ only if t is isolated in $\Phi_{-\xi^i}(t) \neq \{t\}$, and in this case

(22) holds. By Lemma 4.2, the latter condition corresponds to the end of overloading, and this establishes condition (La). This completes the proof of necessary conditions for left discontinuities.

The corresponding proof for right discontinuities is analogous. If i is underloaded at time t then Theorem 4.1(ii) shows that $t \notin RDisc(\nabla\Gamma^i)$. Thus, suppose that i is not underloaded at time t. If $t \in RDisc([P\gamma]^i)$, then by Lemma 4.3(ii), there must exist a sub-critical chain preceding i. This corresponds condition R(b). If, in addition, i is overloaded then $t \notin \Phi_{-\xi^i}(t)$ and so Theorem 4.1(ii) implies that $t \notin RDisc(\gamma^i)$. The inequality (25) then follows from the upper semicontinuity of $[P\gamma]^i$. On the other hand, if $t \notin RDisc([P\gamma]^i)$ then for $t \in RDisc(\nabla\Gamma^i)$ it must be that $t \in RDisc(\gamma^i)$. By Theorem 4.1(ii), this can only occur if condition R1 holds, which implies $t \in \Phi_{-\xi^i}(t) \neq t$ and $\Phi_{-\xi^i}(s) \subset (t,s]$ for all $s > t$. As shown in Lemma 4.2, this is equivalent to the statement that i is at the start of underloading. This scenario is addressed in condition R(a).

Now, for a left and right discontinuity to hold simultaneously, we can either have scenario L(a) and R(b) holding at the same time, which corresponds to LR(a), or conditions L(b) and R(a) being satisfied, which corresponds to LR(b) or conditions L(b) and R(b) holding, which corresponds to LR(c). The remaining combination, L(a) and R(a) is excluded because, comparing conditions L2 and R1 of Theorem 4.1, it is clear that it is not possible to have both a left and right discontinuity at the end of overloading and start of underloading. This completes the proof of the theorem. □

5. Proofs of Auxiliary Results We now provide the proof of Lemma 3.4 in Section 5.1 and the characterization of the generalized one-dimensional derivative (i.e., the proof of Theorem 3.2) in Section 5.2.2.

5.1 Proof of Lemma 3.4 We start with the proof of (48). Note that $\overline{f} \vee 0 - \overline{g} \vee 0$ is the difference of two monotonic functions and thus lies in $\mathcal{BV}$. Therefore, for every $n \in \mathbb{N}$, there exists a partition $\pi_n \doteq \{0 = t_1^n < t_2^n < \cdots < t_{k_n}^n = T\}$ of $[0,T]$ such that

$$\left|\overline{f} \vee 0 - \overline{g} \vee 0\right|_T \leq \sum_{i=1}^{k_n} \alpha_i^n + \frac{1}{n},$$

where

$$\alpha_i^n \doteq \left|\overline{f}(t_i^n) \vee 0 - \overline{g}(t_i^n) \vee 0 - \overline{f}(t_{i-1}^n) \vee 0 + \overline{g}(t_{i-1}^n) \vee 0\right|.$$

For any function h on $[0,T]$, let $h^{(i)}(s) \doteq h(t_{i-1}^n + s) - \overline{h}(t_{i-1}^n)$ for $s \in [0, T - t_{i-1}^n]$, and note that

$$\overline{h}(t_i^n) = \overline{h}(t_{i-1}^n) + \sup_{s \in [0, t_i^n - t_{i-1}^n]} \left[h^{(i)}(s)\right] \vee 0. \tag{87}$$

We can apply (87) with $h = \overline{f} \vee 0$ and $h = \overline{g} \vee 0$, to rewrite α_i^n in the form

$$\alpha_i^n = \left|\sup_{s \in [0, t_i^n - t_{i-1}^n]} f^{(i)}(s) \vee 0 - \sup_{s \in [0, t_i^n - t_{i-1}^n]} g^{(i)}(s) \vee 0\right| \leq \left|\sup_{s \in [0, t_i^n - t_{i-1}^n]} \left(f^{(i)}(s) - g^{(i)}(s)\right) \vee 0\right|,$$

where the inequality is obtained by first replacing f, g and the interval $[0,T]$ in (46) by $f^{(i)}, g^{(i)}$ and the interval $[0, t_i^n - t_{i-1}^n]$, respectively, and then taking absolute values on both sides. Substituting $h = (f-g) \vee 0$ in (87), we can reexpress the right-hand side of the last display to obtain

$$\alpha_i^n \leq \left|\overline{f-g}(t_i^n) \vee 0 - \overline{f-g}(t_{i-1}^n) \vee 0\right|.$$

Thus, we have

$$\left|\overline{f} \vee 0 - \overline{g} \vee 0\right|_T - \frac{1}{n} \leq \sum_{i=1}^{k_n} \alpha_i^n \leq \sum_{i=1}^{k_n} \left|\overline{f-g}(t_i^n) \vee 0 - \overline{f-g}(t_{i-1}^n) \vee 0\right| \leq \left|\overline{f-g}\right|_T.$$

Sending $n \to \infty$, one obtains (48).

Next, let f be the pointwise limit of a sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ in $\mathcal{D}_{\lim}$. To complete the proof of the lemma, it suffices to establish the contrapositive that if $f \notin \mathcal{D}_{\lim}$, then there exists $T < \infty$ such that $\sup_n |f_n|_T = \infty$. If $f \notin \mathcal{D}_{\lim}$ then there must either exist $t \in (0,\infty)$ such that f has no finite left limit at t or there must exist $t \in [0,\infty)$ such that f has no finite right limit at t. We only consider the case when f does not have a finite left limit at a certain $t \in (0,\infty)$ since the other case follows by a similar argument. In this case, by the Cauchy condition there must exist $\delta > 0$ and sequences $\{s_i\}_{i\in\mathbb{N}}$ and $\{s'_i\}_{i\in\mathbb{N}}$ such that $s_i \uparrow t$, $s'_i \uparrow t$ and for every $i \in \mathbb{N}$,

$$|f(s_i) - f(s'_i)| \geq 4\delta. \tag{88}$$

Further, we can assume without loss of generality that $s_1 < s'_1 < s_2 < s'_2 < \ldots$. Since $f_n \to f$ pointwise, given any $m \in \mathbb{N}$ there exists $N < \infty$ such that for all $n \geq N$, $|f_n(s) - f(s)| < \delta$ for all $s \in \{s_i, s'_i, i = 1,\ldots,m\}$. Combining this inequality with (88) we conclude that $|f_n(s_i) - f_n(s'_i)| > \delta$ for every $i = 1,\ldots,m$ and $n \geq N$, and hence that

$$|f_n|_t \geq \sum_{i=1}^m |f_n(s_i) - f_n(s'_i)| \geq m\delta.$$

Taking first the limit superior in n of the left-hand side, and then letting m go to infinity, we conclude that $\limsup_n |f_n|_t = \infty$ and hence $\sup_n |f_n|_t = \infty$. This proves the contrapositive, and hence the lemma.

5.2 Proof of the Representation for the One-dimensional Derivative We now establish the representation for the generalized one-dimensional derivative stated in Section 3.3.2. We will start with three preliminary lemmas in Section 5.2.1 and present the proof in Section 5.2.2.

5.2.1 Preliminary lemmas The first two results summarize some elementary properties of functions and the third result identifies conditions under which the functions $g^{*,\ell}$ and $g^{*,r}$ of Theorem 3.2 can be completed determined by g, and do not rely on the family of functions $\{g_\varepsilon\}_{\varepsilon>0}$.

LEMMA 5.1 *Consider a family of left (respectively, right) continuous functions $\{g_\varepsilon\}_{\varepsilon>0}$ that converges pointwise monotonically down to a function $g \in \mathcal{D}_{\lim}$ as $\varepsilon \downarrow 0$. If s is a point of left (respectively, right) continuity for g, then given any real numbers $\{s_\varepsilon\}_{\varepsilon>0}$ such that $s_\varepsilon \uparrow s$ (respectively, $s_\varepsilon \downarrow s$) as $\varepsilon \to 0$, it follows that*

$$\lim_{\varepsilon\downarrow 0} g_\varepsilon(s_\varepsilon) = g(s).$$

PROOF. Fix $s \in [0,\infty)$. Given any $\delta > 0$, by the pointwise convergence of $\{g_\varepsilon\}_{\varepsilon>0}$, there exists $\kappa_0 > 0$ such that for all $\kappa \in (0,\kappa_0)$, $|g_\kappa(s) - g(s)| < \delta/2$. Likewise, given any $\kappa > 0$, since either g_κ is left continuous and $s_\varepsilon \uparrow s$, or g_κ is right continuous and $s_\varepsilon \downarrow s$, there exists $\varepsilon_0(\kappa) < \kappa$ such that for all $\varepsilon \in (0,\varepsilon_0(\kappa))$, $|g_\kappa(s_\varepsilon) - g_\kappa(s)| < \delta/2$. Together, the last two inequalities show that given any $\delta > 0$ there exists $\kappa_0 > 0$ such that for all $\kappa < \kappa_0$ and $\varepsilon < \varepsilon_0(\kappa)$, $|g_\kappa(s_\varepsilon) - g(s)| < \delta$. Since g_ε converges pointwise monotonically down to g, this implies that

$$g(s_\varepsilon) \leq g_\varepsilon(s_\varepsilon) \leq g_\kappa(s_\varepsilon) \leq g(s) + \delta.$$

Taking limits as $\varepsilon \downarrow 0$ and using the left continuity of g and the fact that $s_\varepsilon \uparrow s$ (or the right continuity of g and the fact that $s_\varepsilon \downarrow s$), one concludes that

$$g(s) \leq \liminf_{\varepsilon\downarrow 0} g_\varepsilon(s_\varepsilon) \leq \limsup_{\varepsilon\downarrow 0} g_\varepsilon(s_\varepsilon) \leq g(s) + \delta.$$

The statement of the lemma follows on sending $\delta \downarrow 0$. □

LEMMA 5.2 *Suppose $f \in \mathcal{D}_{\lim}$ and the family of functions $\{g_\varepsilon,\ \varepsilon > 0\} \subseteq \mathcal{D}_{\lim}$ is uniformly bounded, i.e.,*

$$L_N \doteq \sup_{\varepsilon>0} \|g_\varepsilon\|_N < \infty \quad \textit{for every} \ \ N \in [0,\infty). \tag{89}$$

Then the following properties hold for any $t \in [0,\infty)$.

(*i*) *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$*

$$\overline{f}(t) < 0 \quad \Rightarrow \quad \overline{\varepsilon^{-1}f + g_\varepsilon}(t) < 0 \tag{90}$$

and, likewise,

$$\overline{f}(t) > 0 \quad \Rightarrow \quad \overline{\varepsilon^{-1}f + g_\varepsilon}(t) > 0; \tag{91}$$

(*ii*) *For any* $\delta \in (0, L_t)$ *and* $s \in [0,t]$, *if*

$$\overline{\varepsilon^{-1} f + g_\varepsilon}(t) \leq \varepsilon^{-1} f(s) + g_\varepsilon(s) + \delta \tag{92}$$

for some $\varepsilon \in (0, \delta/6L_t)$, *then*

$$\overline{f}(t) \leq f(s) + \delta\,. \tag{93}$$

Proof. Fix $t \in [0,\infty)$, let $L \doteq L_t$ and choose $\varepsilon_0 = |\overline{f}(t)|/2L$. If $\overline{f}(t) < 0$, then for $\varepsilon \in (0, \varepsilon_0)$

$$\overline{\varepsilon^{-1} f + g_\varepsilon} \leq \varepsilon^{-1}\overline{f}(t) + \overline{g}_\varepsilon(t) = -\varepsilon^{-1}\left|\overline{f}(t)\right| + \overline{g}_\varepsilon(t) \leq -2L + L < 0\,,$$

which establishes (90). A similar argument establishes (91).

Suppose there exists $\varepsilon < \delta/6L_t$ and $s \in [0,t]$ such that (92) is satisfied. We now argue by contradiction to show that then (93) must hold. Indeed, if $f(s) < \overline{f}(t) - \delta$, then choose $\tilde{s} \in [0,t]$ such that $f(\tilde{s}) > \overline{f}(t) - \delta/2$. The last two inequalities together show that $f(\tilde{s}) > f(s) + \delta/2$ which, along with (92), the fact that $\delta < L_t$ and $\varepsilon < \delta/6L_t$, implies that

$$\begin{aligned}
\varepsilon^{-1} f(\tilde{s}) + g_\varepsilon(\tilde{s}) - \overline{\varepsilon^{-1} f + g_\varepsilon}(t) &\geq \varepsilon^{-1} f(\tilde{s}) + g_\varepsilon(\tilde{s}) - \left(\varepsilon^{-1} f(s) + g_\varepsilon(s)\right) - \delta \\
&\geq \varepsilon^{-1}\left[f(\tilde{s}) - f(s)\right] - 2L_t - \delta \\
&> \frac{\varepsilon^{-1}\delta}{2} - 3L_t \quad > \quad 0,
\end{aligned}$$

which contradicts the definition of the supremum since $\tilde{s} \in [0,t]$. Thus (93) must hold. □

The next lemma lists various conditions under which $g^{*,l}$ and $g^{*,r}$ of Theorem 3.2 do not depend on the entire sequence $\{g_\varepsilon\}_{\varepsilon>0}$, but can be determined just from g.

Lemma 5.3 *Let* $\{g_\varepsilon\}_{\varepsilon>0}$, $g, g^{*,\ell}$ *and* $g^{*,r}$ *be the functions described in Theorem 3.2. Then the following properties hold:*

(*i*) $g^{*,l} \geq g_l$ *and* $g^{*,r} \geq g_r$;

(*ii*) *If* $g_\varepsilon = g$ *is independent of* ε, *then* $g^{*,l} = g_l$ *and* $g^{*,r} = g_r$;

(*iii*) *If* $\{g_\varepsilon\} \subset \mathcal{C}$ *then* $g^{*,l} = g^{*,r} = g$;

(*iv*) *If* g_ε *converges to* g *in the uniform topology, i.e., for every* $N < \infty$ $\lim_{\varepsilon\downarrow 0} ||g_\varepsilon - g||_N = 0$, *then* $g^{*,l} = g_l$ *and* $g^{*,r} = g_r$.

Here, as usual g_l *and* g_r *are, respectively, the left and right regularisations of* g.

Proof. Fix $t \in [0,\infty)$. For every $\varepsilon > 0$, choose $t_\varepsilon \in (t-\varepsilon, t)$ such that $|g_\varepsilon(t_\varepsilon) - g_\varepsilon(t-)| < \varepsilon$. Then $t_\varepsilon \uparrow t$ as $\varepsilon \downarrow 0$ and the monotonicity of the sequence of functions g_ε ensures that $g(t_\varepsilon) \leq g_\varepsilon(t_\varepsilon) < g_\varepsilon(t-) + \varepsilon = g_{\varepsilon,\ell}(t) + \varepsilon$, where $g_{\varepsilon,\ell}$ is the left regularisation of g_ε. Taking limits as $\varepsilon \downarrow 0$, it follows that $g_l(t) = g(t-) \leq g_l^*(t)$. An analogous argument yields the inequality $g_r \leq g_r^*$, thus establishing the first property. The second property is a trivial consequence of the definitions and, due to the assumed monotonicity of the sequence $\{g_\varepsilon\}$. If g_ε is right continuous for every $\varepsilon > 0$, there exists a family of numbers $s_\varepsilon \in [s,\infty)$, $\varepsilon > 0$, such that $s_\varepsilon \downarrow s$ as $\varepsilon \downarrow 0$ and $|g_\varepsilon(s_\varepsilon) - g_{\varepsilon,r}(s_\varepsilon)| = |g_\varepsilon(s_\varepsilon) - g_\varepsilon(s+)| \leq \varepsilon$. Since $g_\varepsilon(s_\varepsilon) \to g(s)$ as $\varepsilon \downarrow 0$ by Lemma 5.1, this shows that $g^{*,r}(s) = \lim_{\varepsilon\downarrow 0} g_\varepsilon(s+) = g(s)$. The case when all the g_ε are left-continuous is exactly analogous, and so property 3 follows.

To prove the fourth property, fix $s \in [0,\infty)$ and for $\varepsilon \in (0,1]$, choose $s_\varepsilon \in [s, 2s]$ such that $|g_\varepsilon(s_\varepsilon) - g_\varepsilon(s+)| \leq \varepsilon$ and $s_\varepsilon \downarrow s$ as $\varepsilon \downarrow 0$. The uniform convergence of g_ε to g on the interval $[0, 2s]$ implies that given any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $n \in \mathbb{N}$, $g(s_\varepsilon) - \delta \leq g_\varepsilon(s_\varepsilon) \leq g(t_\varepsilon) + \delta$. This, in turn, implies that $g(s_\varepsilon) - \delta - \varepsilon \leq g_\varepsilon(s+) \leq g(s_\varepsilon) + \delta + \varepsilon$. Taking limits, first as $\varepsilon \downarrow 0$, to obtain the inequality

$$g_r(s) - \delta = g(s+) - \delta \leq g^{*,r}(s) \leq g(s+) + \delta = g_r(s) + \delta,$$

and then sending $\delta \downarrow 0$, we conclude that $g_r(s) = g^{*,r}(s)$. Since s is arbtirary, $g_r = g^{*,r}$ and an exactly analogous argument shows that $g_l = g^{*,l}$. □

5.2.2 Proof of Theorem 3.2 We are now ready to prove the characterization of the generalized one-dimensional derivative stated in Theorem 3.2.

First, note that the family of functions $\{g_\varepsilon\}_{\varepsilon>0}$ has a pointwise limit g as $\varepsilon \downarrow 0$ since for each $s \in [0,\infty)$, $\{g_\varepsilon(s)\}_{\varepsilon>0}$ is uniformly bounded and monotonically non-increasing. By the same token, since the left and right regularized sequences $\{g_{\varepsilon,l}\}_{\varepsilon>0}$ and $\{g_{\varepsilon,r}\}_{\varepsilon>0}$ inherit the uniform boundedness and monotonicity properties of $\{g_\varepsilon\}_{\varepsilon>0}$, the corresponding limits $g^{*,l}$ and $g^{*,r}$ are also well defined.

Fix $t \in \mathcal{T}_u(f)$ or, in other words, fix t such that $\overline{f}(t) > 0$. By assumption, $L \doteq \sup_{\varepsilon>0} ||g_\varepsilon||_t$ is finite and so, by relation (91) of Lemma 5.2, there exists $\varepsilon_0 > 0$ such that $\overline{\varepsilon^{-1}f + g_\varepsilon}(t) > 0$ for all $\varepsilon \in (0,\varepsilon_0)$. Hence, for all $\varepsilon \in (0,\varepsilon_0)$, we have $\tilde{\gamma}_\varepsilon(t) = \overline{\varepsilon^{-1}f + g_\varepsilon}(t) - \overline{\varepsilon^{-1}f}(t)$. Now, for each $\varepsilon \in (0, \varepsilon_0 \wedge 1/8)$, choose $s_\varepsilon \in [0,t]$ to satisfy

$$\left(\varepsilon^{-1}f + g_\varepsilon\right)(s_\varepsilon) \geq \overline{\varepsilon^{-1}f + g_\varepsilon}(t) - 8L\varepsilon. \tag{94}$$

Applying Lemma 5.2(ii), with $\delta = 8L\varepsilon \in (0,L)$ and $s = s_\varepsilon$, and using the definition of the supremum, it follows that $\overline{f}(t) - 8L\varepsilon \leq f(s_\varepsilon) \leq \overline{f}(t)$. Taking limits as $\varepsilon \downarrow 0$, this yields the equality

$$\lim_{\varepsilon\downarrow 0} f(s_\varepsilon) = \overline{f}(t). \tag{95}$$

Moreover, by (94) we clearly also have

$$\tilde{\gamma}_\varepsilon(t) = \overline{\varepsilon^{-1}f + g_\varepsilon}(t) - \overline{\varepsilon^{-1}f}(t) \leq g_\varepsilon(s_\varepsilon) + 8L\varepsilon + \varepsilon^{-1}\left[f(s_\varepsilon) - \overline{f}(t)\right] \leq g_\varepsilon(s_\varepsilon) + 8L\varepsilon,$$

and therefore

$$\limsup_{\varepsilon\downarrow 0} \tilde{\gamma}_\varepsilon(t) \leq \limsup_{\varepsilon\downarrow 0} g_\varepsilon(s_\varepsilon). \tag{96}$$

We now show that

$$\limsup_{\varepsilon\downarrow 0} g_\varepsilon(s_\varepsilon) \leq \tilde{\gamma}(t). \tag{97}$$

Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a sequence with $\varepsilon_n \downarrow 0$ as $n \to \infty$ such that

$$\lim_{n\uparrow\infty} g_{\varepsilon_n}(s_{\varepsilon_n}) = \limsup_{\varepsilon\downarrow 0} g_\varepsilon(s_\varepsilon). \tag{98}$$

Since $\{s_{\varepsilon_n}\}_{n\in\mathbb{N}} \subset [0,t]$ is uniformly bounded, it can be assumed without loss of generality (by choosing a subsequence if necessary) that there exists $s_0 \in [0,t]$ such that $\lim_{n\to\infty} s_{\varepsilon_n} = s_0$. By choosing a further subsequence if necessary, it can be assumed that either (i) $s_{\varepsilon_n} = s_0$ for all n sufficiently large, or (i) does not hold and either $s_{\varepsilon_n} \uparrow s_0$ or $s_{\varepsilon_n} \downarrow s_0$ as $n \to \infty$. If (i) holds, then (95) implies $f(s_0) = \overline{f}(t)$, so that $s_0 \in \Phi_f(t)$. In that case,

$$\limsup_{\varepsilon\downarrow 0} g_\varepsilon(s_\varepsilon) = \lim_{n\uparrow\infty} g_{\varepsilon_n}(s_0) = g(s_0) \leq \sup_{s\in\Phi_f(t)} g(s) \leq \tilde{\gamma}(t),$$

and (97) holds. On the other hand, suppose that (i) above does not hold, but instead $s_{\varepsilon_n} \uparrow s_0$ as $n \uparrow \infty$. Then $f(s_0-) = \overline{f}(t)$ by (95), and hence $s_0 \in \Phi_f^L(t)$. Fix $\delta > 0$ and given $\varepsilon_m > 0$, choose $N(m) \geq m$ such that for all $n \geq N(m)$, $g_{\varepsilon_m}(s_{\varepsilon_n}) \leq g_{\varepsilon_m}(s_0-) + \delta$. The fact that $\{g_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is a monotone non-increasing sequence as $n \uparrow \infty$ then shows that for all $n \geq N(m)$, $g_{\varepsilon_n}(s_{\varepsilon_n}) \leq g_{\varepsilon_m}(s_0-) + \delta$. Taking limits, first as $n \uparrow \infty$, and then as $m \uparrow \infty$, yields

$$\lim_{n\uparrow\infty} g_{\varepsilon_n}(s_{\varepsilon_n}) \leq \lim_{m\uparrow\infty} g_{\varepsilon_m}(s_0-) + \delta = \lim_{m\uparrow\infty} g_{\varepsilon_m,l}(s_0) + \delta = g^{*,l}(s_0) + \delta.$$

Sending $\delta \downarrow 0$ in the above display, using (98) and the fact that $s_0 \in \Phi_f^L(t)$, it follows that

$$\limsup_{\varepsilon\downarrow 0} g_\varepsilon(s_\varepsilon) = \lim_{n\uparrow\infty} g_{\varepsilon_n}(s_{\varepsilon_n}) \leq g^{*,l}(s_0) \leq \sup_{s\in\Phi_f^L(t)} g^{*,l}(s) \leq \tilde{\gamma}(t).$$

Lastly, if (i) does not hold but $s_{\varepsilon_n} \downarrow s_0$ as $n \uparrow \infty$, it must be that $s_0 \neq t$ (since $s_{\varepsilon_n} \in [0,t]$) and, due to (95), that $f(s_0+) = \overline{f}(t)$. Thus $s_0 \in \tilde{\Phi}_f^R(t)$, and arguments similar to those given above yield the relation

$$\limsup_{\varepsilon\downarrow 0} g_\varepsilon(s_\varepsilon) \leq \lim_{\varepsilon\downarrow 0} g_\varepsilon(s_0+) = \lim_{\varepsilon\downarrow 0} g_{\varepsilon,r}(s_0) = g^{*,r}(s_0) \leq \sup_{s\in\tilde{\Phi}_f^R(t)} g^{*,r}(s) \leq \tilde{\gamma}(t).$$

This establishes (97) which, when combined with (96), shows that

$$\limsup_{\varepsilon\downarrow 0} \tilde{\gamma}_\varepsilon(t) \leq \tilde{\gamma}(t). \tag{99}$$

In order to establish the reverse inequality, with lim sup replaced by lim inf, first note that for any $r \in \Phi_f(t)$,

$$\tilde{\gamma}_\varepsilon(t) = \overline{\varepsilon^{-1} f + g_\varepsilon}(t) - \varepsilon^{-1}\overline{f}(t) \geq \varepsilon^{-1} f(r) + g_\varepsilon(r) - \varepsilon^{-1}\overline{f}(t) = g_\varepsilon(r).$$

First take limits as $\varepsilon \downarrow 0$ and then take the supremum over $r \in \Phi_f(t)$ to obtain

$$\liminf_{\varepsilon \downarrow 0} \tilde{\gamma}_\varepsilon(t) \geq \sup_{r \in \Phi_f(t)} [g(r)]. \tag{100}$$

Next, if $\Phi_f^L(t) \neq \emptyset$, let $r \in \Phi_f^L(t)$ and for each $\varepsilon > 0$ choose $r_\varepsilon \in [r - \varepsilon, r]$ such that

$$\varepsilon^{-1}\left[f(r_\varepsilon) - \overline{f}(t)\right] > -\frac{\varepsilon}{2} \qquad \text{and} \qquad |g_\varepsilon(r_\varepsilon) - g_\varepsilon(r-)| < \frac{\varepsilon}{2}.$$

Then

$$\tilde{\gamma}_\varepsilon(t) \geq \varepsilon^{-1} f(r_\varepsilon) + g_\varepsilon(r_\varepsilon) - \varepsilon^{-1}\overline{f}(t) > g_\varepsilon(r-) - \varepsilon.$$

Take limits as $\varepsilon \downarrow 0$ and then take the supremum over $r \in \Phi_f^L(t)$ to arrive at the inequality

$$\liminf_{\varepsilon \downarrow 0} \tilde{\gamma}_\varepsilon(t) \geq \sup_{r \in \Phi_f^L(t)} \left[g^{*,l}(r)\right].$$

Analogous arguments can be used to show that

$$\liminf_{\varepsilon \downarrow 0} \tilde{\gamma}_\varepsilon(t) \geq \sup_{r \in \tilde{\Phi}_f^R(t)} \left[g^{*,r}(r)\right].$$

The last two displays, when combined with (100), yield the relation

$$\liminf_{\varepsilon \downarrow 0} \tilde{\gamma}_\varepsilon(t) \geq \sup_{r \in \Phi_f^L(t)} \left[g^{*,l}(r)\right] \vee \sup_{r \in \Phi_f(t)} [g(r)] \vee \sup_{r \in \tilde{\Phi}_f^R(t)} \left[g^{*,r}(r)\right] = \tilde{\gamma}(t).$$

Together with (99), this shows that $\lim_{\varepsilon \downarrow 0} \tilde{\gamma}_\varepsilon(t) = \tilde{\gamma}(t)$ for $t \in \mathcal{T}_u(f)$.

Note that the arguments above showed, in fact, that for any $t \in [0, \infty)$,

$$\lim_{\varepsilon \downarrow 0} \left[\overline{\varepsilon^{-1} f + g_\varepsilon}(t) - \overline{\varepsilon^{-1} f}(t)\right] = \left(\sup_{s \in \Phi_f^L(t)} \left[g^{*,l}(s)\right] \vee \sup_{s \in \Phi_f(t)} [g(s)] \vee \sup_{s \in \tilde{\Phi}_f^R(t)} \left[g^{*,r}(s)\right]\right). \tag{101}$$

If $t \in \mathcal{T}_m(f)$ then $\overline{f}(t) = 0$. Therefore, for such t, $\tilde{\gamma}_\varepsilon(t) = \overline{\varepsilon^{-1} f + g_\varepsilon}(t) \vee 0 = \left[\overline{\varepsilon^{-1} f + g_\varepsilon}(t) - \overline{\varepsilon^{-1}(f)}(t)\right] \vee 0$. Taking the maximum of both sides of (101) with zero and noting that in this case $\tilde{\gamma}(t)$ is defined to be the maximum of the right-hand side of (101) with zero yields the conclusion that $\lim_{\varepsilon \downarrow 0} \tilde{\gamma}_\varepsilon(t) = \tilde{\gamma}(t)$. On the other hand, if $t \in \mathcal{T}_\ell(f)$, then $\overline{f}(t) < 0$. Since the family $\{g_\varepsilon\}$ is uniformly bounded on $[0, t]$, relation (92) of Lemma 5.2 implies that for all ε sufficiently small $\overline{\varepsilon^{-1} f + g}(t) < 0$. Hence for all sufficiently small $\varepsilon > 0$, $\tilde{\gamma}_\varepsilon(t) = 0$. Since, by definition, $\tilde{\gamma}(t) = 0$ for $t \in \mathcal{T}_\ell(f)$, once again $\lim_{\varepsilon \downarrow 0} \tilde{\gamma}_\varepsilon(t) = \tilde{\gamma}(t)$. This completes the proof of (61) in Theorem 3.2.

When $\{g_\varepsilon\} \subset \mathcal{C}$, $g^{*,l} = g^{*,r} = g$ by Lemma 5.3(iii), and so $H(f, g, g^{*,l}, g^{*,r}) = H(f, g, g, g) = H_1(f, g)$ and the identity (62) follows. If, in addition, f is continuous and so $\Phi_f^L(t) \cup \tilde{\Phi}_f^R(t) \subseteq \Phi_f(t)$ and thus (63) holds. The last statement follows directly from Lemma 5.3(iv) and the definition of H_2.

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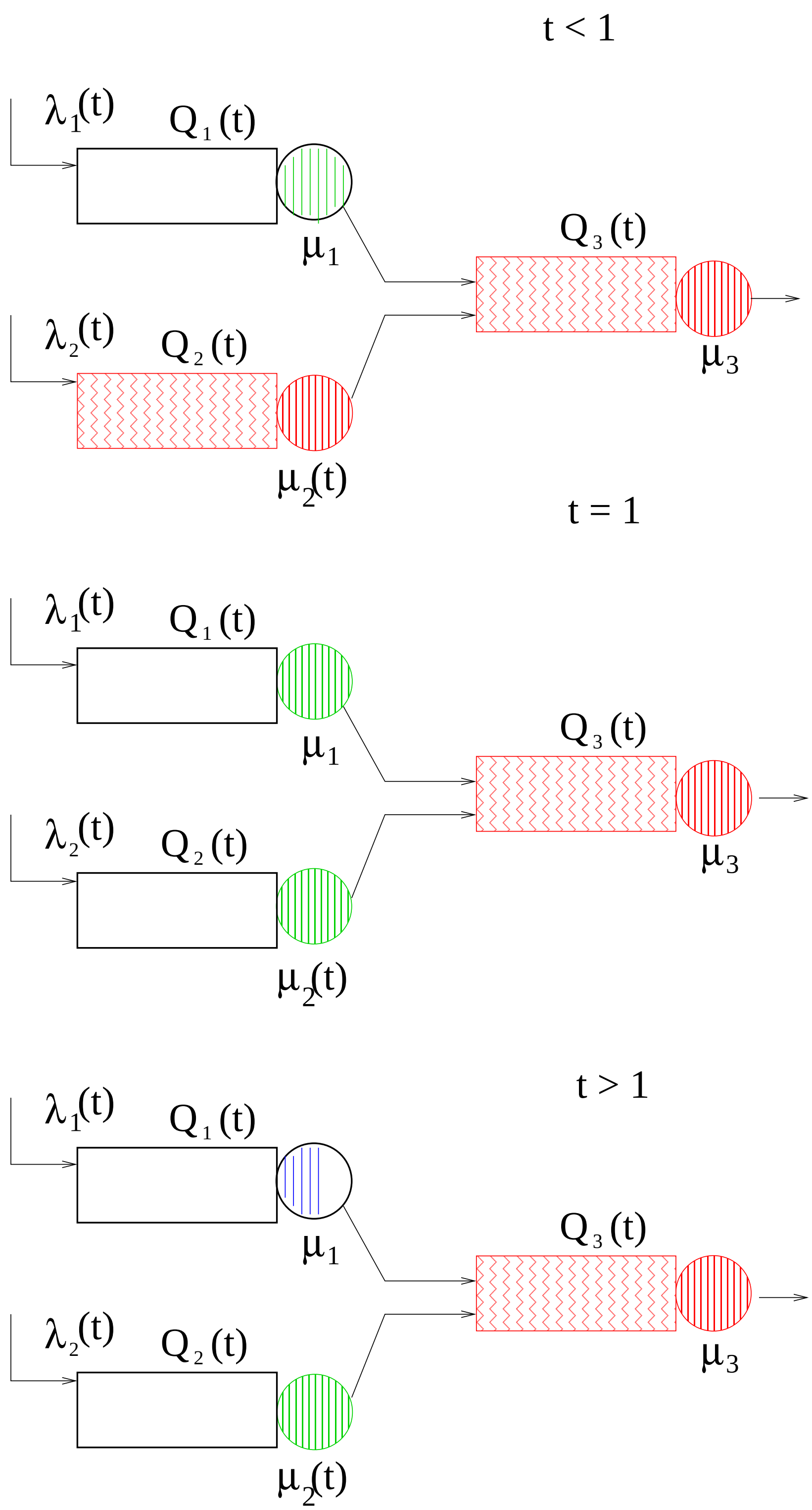

Figure 4: A queueing network with a merge and time-varying arrival and sevice rates giving rise to a separated discontinuity in the directional derivative at $t = 1$ (the colors red, green and blue represent, respectively, overloading, criticality and underloading).

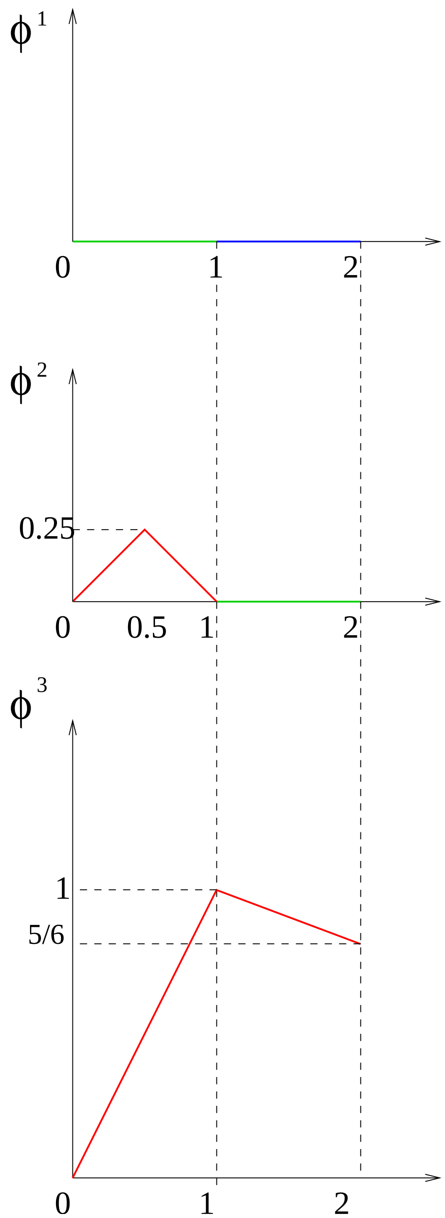

Figure 5: The fluid limit of the non-stationary merge queueing network (the colors red, green and blue represent, respectively, overloading, criticality and underloading).