21-127 review problems

1. (i) How many 5-card poker hands, dealt from a standard 52-card deck, have 5 different ranks represented? Justify your answer, but do not simplify.

(ii) How many ways are there to label the 4 corners of a square with integers from $\{0, 1, \ldots, 9\}$ such that adjacent corners get different integers? Note that the square is fixed, and we don't regard labelings to be the same if they are rotations or reflections of one another.

2. (i) How many functions from $[4]$ to $[6]$ are injections? Explain briefly.

(ii) How many $f : [7] \to [4]$ satisfy $f(i) \leq i$ for all $i \in [7]$? Explain briefly.

(iii) Define a *good word* as a sequence of letters that consists only of the letters A, B, and C - some of these letters may not appear in the sequence - and in which A is never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A. How many six-letter good words are there?

3. Show, by partitioning a set into four non-empty subsets, that for every positive integer n

$$\binom{2n+2}{n+1} = \binom{2n}{n+1} + \binom{2n}{n} + \binom{2n}{n} + \binom{2n}{n-1}.$$

Be explicit about the set, how it is partitioned, and the calculation of the sizes of the sets.

4. Provide brief justification for your answers.

(i) Determine the unknown digits a and b in 128a151b given that 128a151b is congruent to 1 mod 5 and 128a151b is congruent to 7 mod 11.

(ii) State Wilson's Theorem.

(iii) Find the remainder of 2^{121} when divided by 61.

(iv) Find the remainder of 2^{119} when divided by 55.

5. (i) Find, with proof, all integers x which satisfy $25\,x \equiv 3 \mod 52$.

(ii) Let p be a prime number larger than 2. Find, with proof, the remainder of $2 \cdot 4 \cdot 6 \cdots (2p-2)$ when divided by p.