1. Give counterexamples for each of the following statements.
Briefly explain why your counterexamples are, in fact, counterexamples.

(i) $(\forall x \in \mathbb{R})\,(\exists! y \in \mathbb{R})\,(xy = 0)$.

Consider $x = 0$. Then there are at least two different values of y such that $xy = 0$, namely $y = 1$ and $y = 2$.

(ii) $(\forall$ sets $A, B)$ $\;(A \times B = B \times A \Leftrightarrow A = B)$.

Consider $A = \emptyset$ and $B = \{1\}$. Then $A \times B = B \times A = \emptyset$, but $A \neq B$.

(iii) $(\forall$ sets $A, B)$ $\;(\mathcal{P}(A) \setminus \mathcal{P}(B) = \mathcal{P}(A \setminus B))$.

Consider $A = B = \emptyset$. Then the left hand side is $\emptyset$, while the right hand side is $\{\emptyset\}$, which is not the empty set.

2. (i) Match each propositional formula on the left to the propositional formula on the right with which it is equivalent. You can draw lines between equivalent formulae and there is no need to justify or show work.

p	$q \vee (p \wedge q)$
q	$p \vee ((\neg p) \wedge q)$
$p \vee q$	$p \Rightarrow q$
$p \wedge q$	$p \wedge ((\neg p) \vee q)$
$(\neg p) \vee q$	$p \wedge (p \vee q)$

Solution. The statements on the left are equivalent (in order) to the fifth, first, second, fourth and third statements on the right.

(ii) Show that $p \Rightarrow (q \vee r)$ is equivalent to $(p \Rightarrow q) \vee (p \Rightarrow r)$.

Solution. One can display a truth table verifying that the formulae have the same truth values for each of the 8 possible truth assignments to p, q, and r, or one can note that $p \Rightarrow (q \vee r)$ is false if and only if p is true and $q \vee r$ is false, ie when p is true and both q and r are false. Similarly, $(p \Rightarrow q) \vee (p \Rightarrow r)$ is false if and only if both $p \Rightarrow q$ and $p \Rightarrow r$ are false, ie when p is true and both q and r are false. Since the necessary and sufficient conditions for falsity are identical, the formulae are equivalent.

3. For each of the following statements, prove it or prove its negation.

(i) $(\forall \text{ sets } A,\, B)(\forall\, A_1, A_2 \subseteq A)(\forall\, f : A \to B) \quad (f[A_1] \setminus f[A_2] = f[A_1 \setminus A_2])$.

Solution. The statement is false. Consider $f : \{1,2\} \to \{1\}$ defined by $f(1) = 1$ and $f(2) = 1$. Then

$$f[\{1\}] \setminus f[\{2\}] = \emptyset \neq \{1\} = f[\{1\} \setminus \{2\}].$$

(ii) $(\forall \text{ sets } A,\, B)(\forall\, B_1, B_2 \subseteq B)(\forall\, f : A \to B) \quad (f^{-1}[B_1] \cap f^{-1}[B_2] = f^{-1}[B_1 \cap B_2])$.

Solution. The statement is true. Let the sets A, B, subsets $B_1, B_2 \subseteq B$, and the function $f : A \to B$ be given. Then

$$\begin{gathered} x \in f^{-1}[B_1] \cap f^{-1}[B_2] \\ \Leftrightarrow (x \in f^{-1}[B_1]) \wedge (x \in f^{-1}[B_2]) \\ \Leftrightarrow ((x \in A) \wedge (f(x) \in B_1)) \wedge ((x \in A) \wedge (f(x) \in B_2)) \\ \Leftrightarrow (x \in A) \wedge ((f(x) \in B_1) \wedge (f(x) \in B_2)) \\ \Leftrightarrow (x \in A) \wedge (f(x) \in B_1 \cap B_2) \\ \Leftrightarrow x \in f^{-1}[B_1 \cap B_2]. \end{gathered}$$

4. Let $\mathbb{R}^+$ be the set of positive real numbers. Consider

$$f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+ \text{ defined by } f(x, y) = (x^2 - y^2,\ x + y).$$

(i) Prove that f is an injection.

Solution. Let $(x, y), (x', y') \in \mathbb{R}^+ \times \mathbb{R}^+$, and assume that $f(x, y) = f(x', y')$. Then $x^2 - y^2 = (x')^2 - (y')^2$ and $x + y = x' + y'$. Factoring the first equation yields $(x - y)(x + y) = (x' - y')(x' + y')$. Using the fact that the factors on the right are equal and non-zero, this implies that $x - y = x' - y'$. Thus, $2x = (x + y) + (x - y) = (x' + y') + (x' - y') = 2x'$, from which we conclude $x = x'$. Substituting this back into $x + y = x' + y'$ yields $y = y'$ and hence $(x, y) = (x', y')$, as desired.

(ii) Prove that f is **NOT** a surjection.

Solution. The ordered pair $(1, 1)$ is not in the image of f. Assume for the sake of contradiction that there is $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $f(x, y) = (1, 1)$. Then $x^2 - y^2 = (x - y)(x + y) = 1$ and $x + y = 1$. Substituting the second equation into the first yields $x - y = 1$. From this, we have $2y = (x + y) - (x - y) = 1 - 1 = 0$, which implies that $y = 0$. This contradicts our assumption that y is a positive real number.

5. Given sets A, B, C and D prove that $(A \cap B) \setminus (C \cap D)$ is a subset of $(A \setminus C) \cup (B \setminus D)$. Exhibit sets A, B, C and D which show that the two quantities needn't be equal.

Solution. Let $x \in (A \cap B) \setminus (C \cap D)$. Then $x \in (A \cap B)$ and $x \notin (C \cap D)$. Since $x \in (A \cap B)$, $x \in A$ and $x \in B$. Since $x \notin (C \cap D)$, DeMorgan's Law implies that $x \notin C$ or $x \notin D$. If $x \notin C$, then $x \in A \setminus C$ and hence $x \in (A \setminus C) \cup (B \setminus D)$. If $x \notin D$, then $x \in B \setminus D$ and hence $x \in (A \setminus C) \cup (B \setminus D)$. In either case, $x \in (A \setminus C) \cup (B \setminus D)$, and the result is proven.

To show that the two quantities needn't be equal, let $A = \{1\}$ and let B, C, and D all be the empty set. In this case, $(A \cap B) \setminus (C \cap D)$ is the empty set, while $(A \setminus C) \cup (B \setminus D) = \{1\}$.

Bonus. Let T_n be the number of ways of tiling a row of n squares using dominoes (dominoes cover 2 adjacent squares) and squares.

Note that $T_1 = 1,\ T_2 = 2$, and $T_3 = 3$. What is T_{10}?

Solution. The tilings can be partitioned into two disjoint sets, depending on whether the rightmost tile in the tiling is a domino or a square. This yields the recurrence relation $T_n = T_{n-1} + T_{n-2}$ for all $n \geq 3$. Using this relation recursively, we see that $T_4 = 5$, $T_5 = 8$, $T_6 = 13$, $T_7 = 21$, $T_8 = 34$, $T_9 = 55$, and finally, $T_{10} = 89$.