21-127 Practice Problems

1. Give counterexamples for each of the following statements.
Briefly explain why your counterexamples are, in fact, counterexamples.

(i) $(\forall x \in \mathbb{R})\,(\exists! y \in \mathbb{R})\,(xy = 0)$.

(ii) $(\forall \text{ sets } A,\, B)\quad (A \times B = B \times A \Leftrightarrow A = B)$.

(iii) $(\forall \text{ sets } A,\, B)\quad (\mathcal{P}(A) \setminus \mathcal{P}(B) = \mathcal{P}(A \setminus B))$.

2. (i) Match each propositional formula on the left to the propositional formula on the right with which it is equivalent. You can draw lines between equivalent formulae and there is no need to justify or show work.

p	$q \vee (p \wedge q)$
q	$p \vee ((\neg p) \wedge q)$
$p \vee q$	$p \Rightarrow q$
$p \wedge q$	$p \wedge ((\neg p) \vee q)$
$(\neg p) \vee q$	$p \wedge (p \vee q)$

(ii) Show that $p \Rightarrow (q \vee r)$ is equivalent to $(p \Rightarrow q) \vee (p \Rightarrow r)$.

3. For each of the following statements, prove it or prove its negation.

(i) $(\forall \text{ sets } A,\, B)(\forall\ A_1, A_2 \subseteq A)(\forall\ f : A \to B) \quad (f[A_1] \setminus f[A_2] = f[A_1 \setminus A_2])$.

(ii) $(\forall \text{ sets } A,\, B)(\forall\ B_1, B_2 \subseteq B)(\forall\ f : A \to B) \quad (f^{-1}[B_1] \cap f^{-1}[B_2] = f^{-1}[B_1 \cap B_2])$.

4. Let $\mathbb{R}^+$ be the set of positive real numbers. Consider

$$f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+ \text{ defined by } f(x, y) = (x^2 - y^2,\, x + y).$$

(i) Prove that f is an injection.

(ii) Prove that f is **NOT** a surjection.

5. Given sets A, B, C and D prove that $(A \cap B) \setminus (C \cap D)$ is a subset of $(A \setminus C) \cup (B \setminus D)$. Exhibit sets A, B, C and D which show that the two quantities needn't be equal.

Bonus. Let T_n be the number of ways of tiling a row of n squares using dominoes (dominoes cover 2 adjacent squares) and squares.

Note that $T_1 = 1$, $T_2 = 2$, and $T_3 = 3$. What is T_{10}?