Mock OH Questions

These three questions with refsols should be used to prepare for mock office hours, which will happen after you teach a mock recitation in your interview. While you should be familiar with all of them, we will pick only one to ask you about. Expect your interviewers to ask questions as if they were a student coming to your OH for that particular question, and answer as you would if you were the TA helping them. Good luck!

Problem 1.

Count the number of 5 card hands with cards from exactly 3 unique ranks.

Solution.

We partition on whether the ranks are of form $R_1R_1R_2R_2R_3$ or $R_1R_1R_1R_2R_3$

- In the first case, our process is as follows:
  1. choose the two ranks with two cards $\binom{13}{2}$
  2. choose the third rank $\binom{11}{1}$
  3. choose the suits for the lowest rank with two cards $\binom{4}{2}$
  4. choose the suits for the other rank with two cards $\binom{4}{2}$
  5. choose the suit for the last rank 4

  We have a k step process, so our result in this case is $\binom{13}{2}\binom{11}{1}\binom{4}{2}\binom{4}{2} * 4$ by MP.
- In the second case, our process is as follows:
  1. choose the ranks with one cards $\binom{13}{2}$
  2. choose the third rank $\binom{11}{1}$
  3. choose the suit for the lowest rank with one card 4
  4. choose the suit for the other rank with one card 4
  5. choose the suits for the last rank with three cards$\binom{4}{3}$

  We have a k step process, so our result in this case is $\binom{13}{2}\binom{11}{1} * 4 * 4\binom{4}{3}$ by MP.

The two cases are **disjoint** as the number of ranks with 3 cards at that rank are different in both cases.

It is **exhaustive** as we must have at least 1 card for each of the three ranks, then the remaining two cards can either be different ranks of the three specified or the same rank of the three specified which must be one of the two cases.

As a result, we can use AP to argue that the solution is

$$\binom{13}{2}\binom{11}{1} * 4 * 4\binom{4}{3} + \binom{13}{2}\binom{11}{1}\binom{4}{2}\binom{4}{2} * 4$$

Problem 2.

Show that, given a function $f : A \to B$ with $A, B \neq \emptyset$:

$$f \text{ injective} \implies (\forall C, D \subseteq A)\ (f[C] \cap f[D] = f[C \cap D]).$$

(This is a bi-implication, but we will only focus on this direction for the problem.

Solution.

Let f be injective. Then $f(x) = f(y) \implies x = y$. Consider $C, D \subseteq A$.

First we show $f[C \cap D] \subseteq f[C] \cap f[D]$. Let $y \in f[C \cap D]$. Then there exists $x \in C \cap D$ with $f(x) = y$. Then $x \in C \implies f(x) \in f[C] \wedge x \in D \implies f(x) \in f[D]$. So $f(x) \in f[C] \cap f[D]$, and since $y = f(x), y \in f[C] \cap f[D]$.

For the next direction, let $y \in f[C] \cap f[D]$. Then there exists $x \in C$ with $f(x) = y$, and exists $x' \in D$ with $f(x') = y$. But note that $f(x) = f(x') \implies x = x'$ by the injectivity of f, so we have that $x \in C \wedge x \in D \implies x \in C \cap D$. So $f(x) \in f[C \cap D]$ and thus $y \in f[C \cap D]$.

Problem 3.

Consider all infinite binary strings $b_0b_1b_2b_3\ldots$ with property $\forall i, j \in \mathbb{N},\ i \le j \implies b_i \le b_j$. Determine if this set of binary strings is countable.

Solution.

This set is countable. We will map this set to the naturals. Note that every element in this set must have form $0000\ldots, 1111\ldots$, or $0^i11111\ldots$ for some $i \in \mathbb{N}^+$. We will prove that this is the case via a double containment argument.

A string that is not one of these must have a 1 at position i but then have a zero at some later position $j > i$. This directly breaks the rule, so it is not in the set.

Next, $000\ldots, 111\ldots$ satisfies the rule because $\forall i, j \in \mathbb{N},\ b_i = b_j$, and $0^k111\ldots$ satisfies the rule since for all $i < j$, we either have $b_i = b_j = 0$ for $j \le k$, $b_i = 0 \le 1 = b_j$ for $i \le k < j$, and we have $b_i = b_j = 1$ for $k < i$ so have considered all cases.

Next to show that this set has an injection to the naturals. I will show that $S = \{0^k1111\ldots \mid k \in \mathbb{N}^+\}$ has an injection to the naturals via $f(0^k11111) = k$. It is easy to show that this is an injection. Let $f(b_0b_1b_2\ldots) = f(b'_0b'_1b'_2\ldots) = k$. Then $b_0b_1b_2\ldots = 0^k11111\ldots = b'_0b'_1b'_2\ldots$.

Since the set we are trying to count is $\{0^k1111\ldots \mid k \in \mathbb{N}^+\} \cup \{0000\ldots, 11111\ldots\}$. We proved that the first set is countable, and the second set is finite, so their union must also be countable.