1: Let A and B be independent sets in G of maximum cardinality.

1. Prove that the subgraph of G induced by the partitions $A \setminus B$ and $B \setminus A$ is bipartite.
2. Prove that this subgraph has a perfect matching.
3. Does this still hold if A and B are maximal independent sets (over set inclusion)? Prove or disprove.

2: Give an example of each of the following, or prove that no such example exists

1. A bridgeless graph G with $\lambda(G) < 2$ and $|G| \geq 3$
2. A graph of order $n > 0$ with n bridges
3. A graph G with $\lambda(G) = 1$ and $\kappa(G) = 2$
4. A graph G with $\lambda(G) = 2$ and $\kappa(G) = 1$
5. A nonempty graph in which all vertices are cutvertices
6. A nonempty graph with radius n and order less than $2n$

3: Let G be a connected graph without a perfect matching. Furthermore, assume G is edge-maximal with this property. In other words, G does not have a perfect matching, but $G + e$ does have a perfect matching for any $e \notin E(G)$. Let S be a set of vertices such that every component of $G - S$ is complete and $(\forall s \in S)(\forall v \in G - s)(\{s, v\} \in E(G))$. Prove there is a set of vertices T such that $q(G - T) > |T|$.

4: Given a graph G that is disconnected, prove an upper bound on the diameter of $\overline{G}$

5: G has the property that every edge of G joins an odd degree vertex with an even degree vertex. Show that G has even size.

6: Given a bipartite graph G, prove that there exists a matching covering all vertices of degree $\Delta(G)$.

7: Prove that directed graphs are Eulerian if and only if they are connected and every vertex v has $d_{in}(v) = d_{out}(v)$. [Indegree and outdegree are defined as obvious, such as on Wikipedia; if this were on the exam we would give a formal definition.]

8: Let $G = (U, V, E)$ be a bipartite graph with bipartitions U and V such that for all $u \in U$ and all $v \in V$:

$$\deg(u) \geq \deg(v) \geq 1.$$

Prove that G has a matching of size $|U|$.

9: Prove that a graph G of order at least 3 is connected if and only if it contains distinct vertices u, v such that $G - u$ and $G - v$ are connected.

10: kn rooks are placed onto an $n \times n$ chess board in a way such that every file (column) of the chess board has k rooks, and every rank (row) has k rooks ($1 \leq k \leq n$). Prove that we can choose a set of n of these rooks such that no rooks in the set are attacking each other. Formally, this means no two rooks of the set are in the same rank or file.