1, Diestel 3.4: Let X and X' be minimal separators in G such that X meets (intersects non-trivially) at least two components of $G - X'$. Show that X' meets all the components of $G - X$, and that X meets all the components of $G - X'$.

We assume that G is connected. By minimality, every vertex in X (resp, X') is adjacent to a vertex in each of the components of $G - X$ (resp, $G - X'$). Now let $x, y \in X$ with $x \in C_1$, $y \in C_2$, where C_1 and C_2 are components of $G - X'$. Let C be any component of $G - X$. x and y are each adjacent to vertices in C, so there is a path from x to y which uses only vertices of C in its interior. Since X' separates x and y, there must be a vertex from X' on the path, which must also be in C. Since C was arbitrary, we conclude that X' meets all components of $G - X$.

Now let D_1 and D_2 be components of $G - X$, with $x, y \in X'$ such that $x \in D_1$ and $y \in D_2$. If X does not meet some component D of $G - X'$, then as above, we find a path from x to y using interior vertices from D. This path doesn't use any vertices of X, which contradicts x and y being in distinct components of $G - X$. Thus, X meets all components of $G - X'$.

2, Diestel 3.7: Show the block graph of any connected graph is a tree.

Claim: The block graph of a connected graph is connected
Proof: Let G be a connected graph and let B be the block graph of G. G is non-empty and the trivial graph has no cutvertices, so there exists a non-empty block in G, and so B is non-empty. Let $v, w \in B$ be arbitrary. If v is a cutvertex in G, then v is in some block of G, b (because $G[v]$ is a connected subgraph without a cutvertex). By the definition of B, $vb \in E(B)$. We can do the same for w if w is a cutvertex. The, it suffices to find a v–w walk in B when v and w are both arbitrary blocks (because if they are cutvertices we can take edges to blocks and then the walk between those blocks, and similarly if only one is a cutvertex).

If v and w are both vertices representing blocks, then these blocks are nonempty (because they are maximal, and in the earlier paragraph we showed that G has a non-empty block). So, let $v' \in V(G)$ be a vertex in block v and let $w' \in V(G)$ be a vertex in block w. There exists a v'–w' path in G, so we call one such path P'. We define W, and prove that W is a v–w walk in B. Then, W proves the existence of a v–w path in B, so we conclude that B is connected.

Let $b_0 = v$ and $P_0 = P$, then define b_i and P_i recursively so that P_{i+1} is a maximal subpath of P_i that does not begin with an edge in b_i. Then, b_{i+1} is the block that has the first edge of P_{i+1}. Eventually P_k is trivial (because for $i \geq 1$ each P_{i+1} has fewer edges than P_i) and we define $b_k = w$ instead. This definition uses the comment from Diestel that each edge is in a unique block.

To form W, insert between each b_i and b_{i+1} the vertex a_i that is in both b_i and b_{i+1}. Because a_i is in both b_i and b_{i+1}, each edge traversed in our walk is indeed an edge of B, so

$$W = b_0 a_0 b_1 a_1 \cdots a_{k-1} b_k$$

is a b_0–b_k walk, and because $b_0 = v$ and $b_k = w$, it is a v–w walk. It remains to be shown that a_i as defined exists.

Diestel comments that blocks intersect in at most one vertex, and that this vertex is a cutvertex, hence the usage of the definite article "the" above, but we must confirm that b_i and b_{i+1} do indeed intersect. If $0 < i < k$, b_i has some edge that occurs in P_i, and then all edges of P_i are in b_i until an edge in b_{i+1} is reached. Because P_i is a path, there is a vertex in both the last consecutive edge of b_i and the first edge of b_{i+1}, and because this vertex is in edges of both b_i and b_{i+1}, it is in b_i and b_{i+1}. Finally, in the case $i = 0$, if P begins with an edge in b_0 then the proof is the same as for other i. Otherwise, the first edge must be in b_1, and so v' is in $v = b_0$ (by definition) and is in an edge of b_1 (because P is a path), and so is in b_1. □

Claim: Block graphs are acyclic.
Proof: Let G be a graph, B the block graph of G, and C a cycle in B. Name

$$C = b_0 a_1 b_1 a_2 b_2 \cdots a_k b_k$$

where $b_0 = b_k$ and note that cycles have at least 3 vertices so $k \geq 2$.We prove that $H = \cup_{i=1}^{k} b_i$ is a connected subgraph with no cutvertices. Because $b_1 \neq b_2$, this contradicts the maximality of b_1 or b_2.

That H is connected is fairly clear: each b_i is connected, and each a_i for is in b_i and b_{i-1}, so given vertices $v \in b_i$ and $w \in b_j$ with $i \leq j < k$, we know v is connected to a_{i+1} (they're both in b_i) which is connected to a_{i+2} (they're both in b_{i+1}), etc. until we have a_j connected to w (both in b_j). Connectedness is transitive, so v is connected to w.

In fact, the above can show that H has no cutvertices. Observe that each b_i is still connected after the deletion of a vertex (because it is a cutvertex), so the only possibility of eliminating a v, w path is by deleting some a_i. Yet, the above argument never uses a_k, so we may reindex, rotating the cycle of blocks so that $a_i = a_k$, and we see that the remainder of H is still connected. $\square$

The block graph is a tree by the above claims and definition.

3, Diestel 3.8: Let G be a k-connected graph, and let xy be an edge of G. Show that G/xy is k-connected if and only if $G - \{x, y\}$ is $(k-1)$-connected.

Let G be a k-connected graph and let xy be an edge in G.

Assume G/xy is k-connected. As G/xy is k-connected, it has at least $k+1$ vertices, so G has at least $k+2$ vertices and $G - \{x, y\}$ has at least k vertices. Any separator of G/xy contains at least k vertices, at most one of which is v_{xy}, so any separator of $G/xy - v_{xy}$ has size at least $k-1$. As $G/xy - v_{xy} = G - \{x, y\}$, we conclude $G - \{x, y\}$ is $(k-1)$-connected.

Now assume $G - \{x, y\}$ is $(k-1)$-connected. Let $X \subseteq G/xy$. Assume that $|X| < k$. If $v_{xy} \notin X$, then $X \subseteq G$ but X is not a separator of G as it has size less than k and G is k-connected. So $G - X$ is connected, and as contracting and edge cannot disconnect a graph, $G/xy - X$ is not disconnected. If $v_{xy} \in X$, let $X' = X \setminus \{v_{xy}\}$. Then $|X'| \subseteq G - \{x, y\}$, but $|X| < k - 1$ so X' does not separate $G - \{x, y\}$. As $G/xy - X = G - \{x, y\} - X'$, G/xy is also connected. In both cases, $G/xy - X$ is connected, so

$$|X| < k \Rightarrow X \text{ is not a separator of } G/xy,$$

so by the contrapositive,

$$X \text{ is a separator of } G/xy \Rightarrow |X| \geq k.$$

Finally, if $G - \{x, y\}$ is $(k-1)$-connected, it has at least k vertices, so G has at least $k+2$ vertices, so G/xy has at least $k+1$ vertices. We conclude G/xy is k-connected.

4, Diestel 3.9: (i) Let e be an edge in a 2-connected graph $G \neq K^3$. Show that either $G - e$ or G/e is again 2-connected.

(ii) Does every 2-connected graph $G \neq K^3$ have an edge e such that G/e is still 2-connected?

(i) Assume that $G - e$ is not 2-connected. We will show that G/e is 2-connected. Let v be a cutvertex of $G - e$. v is not an endpoint of e, since G has no cutvertices. By problem number 3 (Diestel 3.8) we only need to show that G, minus e's endpoints, is connected. Let C_1 and C_2 be the components of $(G - e) - v$ containing endpoints x and y of e, respectively. If z is an arbitrary vertex of $C_1 - x$, then, using the fact that x is not a cutvertex of G, we find a path from z to v in $G - x$. The path doesn't pass through y since x and y are in different components of $(G - e) - v$. We can do the same for vertices of $C_2 - y$, showing that $G - \{x, y\}$ is connected, as desired.

(ii) Consider the construction of G, starting with a cycle. If only edges between pairs of non-adjacent vertices were added to create G, then since G is not a 3-cycle, it has a cycle using all of the $|G| > 3$ vertices. Contracting an edge of that cycle leaves a 2-connected graph, and G is the result of adding edges between non-adjacent vertices of it, which is still 2-connected.

Otherwise, an H-path of length at least 2 was added at some point, so consider the last such path, $x_0, x_1, \ldots, x_k$. $G/\{x_0, x_1\}$ is created by adding edges between non-adjacent vertices of the graph where an H-path of length one less is added at the stage where $x_0, x_1, \ldots, x_k$ was added. The result of this process, namely $G/\{x_0, x_1\}$, is thus 2-connected.

5, Diestel 3.14: Show that every transitive graph G with $\kappa(G) = 2$ is a cycle. Hint: Exercise 3.4 is useful.

Let G be such a graph and consider a 2-separator $\{x, y\}$ such that $G - \{x, y\}$ has a component C of smallest possible cardinality. Note that if $|C| = 1$, then the result follows easily, because the vertex of C would have to have degree at most 2. Since it must have degree at least 2 (in order to ensure 2-connected), its degree must be 2. The graph is then connected and 2-regular, and must be a cycle.

We now assume for the sake of contradiction that $|C| > 1$.

Observe that x and y can't each be connected to all of the vertices of C. If so, then mapping x to an arbitrary vertex of C, would produce a minimal separator $\{\phi(x), \phi(y)\}$, which by the minimality of $|C|$, and problem 1, would leave x and y in different components upon deletion. This is contradicted by the remaining existence of a path from x to y through a different vertex of C.

Thus we may, without loss of generality, consider the minimal separator $\{\phi(x), \phi(y)\}$ obtained by mapping x to a vertex in C to which it is not adjacent. By problem 1, $\{x, \phi(x)\}$ separates G, with one of the components, say D, being C intersected with the component of $G - \{\phi(x), \phi(y)\}$ containing x. D is a non-empty (it contains a vertex from C to which x is adjacent) proper (it doesn't contain $\phi(x)$) subset of C, which contradicts the minimality of $|C|$.

Thus, $|C| = 1$, and the result follows.