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**1, Diestel 9.3:** An *arithmetic progression* is an increasing sequence of numbers of the form $a, a+d, a+2d, a+3d\ldots$. *Van der Waerden's theorem* says that no matter how we partition the natural numbers into two classes, one of these classes will contain arbitrarily long arithmetic progressions. Must there even be an infinite arithmetic progression in one of the classes?

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No, there need not be an infinitely long arithmetic progression. Consider the following coloring of $\mathbb{N}$: in step 1, color the first natural number red. In step 2, color the next two natural numbers blue. In general, in step $i$ color the first $i$ uncolored natural numbers red if $i$ is odd and blue if $i$ is even.

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, \ldots$$

Note that the last number colored before switching colors is $\frac{n(n+1)}{2}$ for any $n$.

Assume for the sake of contradiction that there is a monochromatic arithmetic progression of infinite length, say $a, a+d, a+2d, \ldots$.

First assume $a \leq \frac{d(d-1)}{2}$. Thus the first term of the arithmetic progression is a natural number we colored in the first $d-1$ steps of our coloring. Therefore there must be exactly one term in our arithmetic progression, say $a+jd$, that was colored during the $d^{\text{th}}$ step of our coloring. (Note that we cannot "jump over" this step because we only add $d$ to each term of the progression and we colored $d$ numbers in step $d$.) But then $a+(j+1)d$ was colored during step $d+1$ of our coloring and got the opposite color of $a+jd$, contradicting that the arithmetic progression is monochromatic.

Now assume $a > \frac{d(d-1)}{2}$. Then $a$ was colored after step $d-1$ of our coloring, say in step $k$. There are only $d$ numbers between any two terms of the arithmetic progression, but as we are further than step $d$ of our coloring, if two adjacent terms of the sequence have the same color then each number between them must share the same color. However, the numbers we colored in step $k$ can only contain at most $\lceil \frac{k}{d} \rceil$ of the terms from the arithmetic progression, at which point it must use numbers colored in step $k+1$; those numbers are of the opposite color and we reach a contradiction.

As we reach a contradiction in both cases, we conclude the given coloring does not have an infinite length arithmetic progression.

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**2:** i) Find, with proof, a 2-coloring of the edges of $K^6 - e$ such that no $K^3$ subgraph is monochromatic (has all of its edges colored with one color).

ii) Show that it is possible to remove 5 edges from $K^{10}$ and 2-color the remaining 40 edges without introducing a monochromatic $K^3$.

iii) Show that removing 4 (or fewer) edges from $K^{10}$ and 2-coloring the remaining edges will always introduce a monochromatic $K^3$.

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i) Consider the complete graph on [6] with the edge 16 deleted. Color the edges $\{12, 23, 34, 45, 51, 56, 26\}$ red, and the remaining edges blue. Students should draw a picture or make a remark that indicates that the result has no monochromatic triple.

ii) Add vertices $\{7, 8, 9, 0\}$ to the above construction, with the edges $\{27, 38, 49, 50\}$ deleted. Color with red the corresponding 2 edges for each new vertex as before, as well as the 5-cycle $\{67, 78, 89, 90, 06\}$. Students should draw a picture or make a remark that indicates that the result has no monochromatic triple.

iii) By Turan's Theorem, any graph on 10 vertices with more edges than the complete, balanced, 5-partite graph on 10 vertices must contain a $K^6$. Since this Turan graph has 40 edges, we conclude that a graph resulting from removing 4 (or fewer) edges from $K^{10}$ must contain a $K^6$. Since $R(3, 3) = 6$, there will be a monochromatic triangle among the 2-coloring of the edges of that $K^6$.

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**3, Diestel 9.9:** Prove the following result of Schur: for every $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that, for every partition of $\{1, \ldots, n\}$ into $k$ sets, at least one of the subsets contains numbers $x, y, z$ such that $x + y = z$.

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Let $n = R(\overbrace{3, 3, \ldots, 3}^{k})$. Suppose $\{1, \ldots, n\}$ is partitioned into $k$ sets. Assign each set a color.

Associate with each element of $\{1, \ldots, n\}$ a vertex in $K^n$. Color the edges of $K^n$ by assigning the edge connecting vertices $i$ and $j$ the same color that $|j - i|$ has according to the set into which it was partitioned.

By Ramsey's theorem this coloring of $K^n$ contains a monochromatic triangle $\{i, j, k\}$. Without loss of generality, $i < j < k$. Then $(k - i), (k - j)$ and $(j - i)$ all got the same color and thus are in the same part of the partition. As

$$(k - j) + (j - i) = (k - i)$$

letting $x = (k - j), y = (j - i)$ and $z = (k - i)$ satisfies the theorem.

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**4, Diestel 9.13:** Let $m, n \in \mathbb{N}$, and assume that $m-1$ divides $n-1$. Show that every tree $T$ of order $m$ satisfies $R(T, K_{1,n}) = m+n-1$.

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To establish the lower bound, we exhibit a graph on $m+n-2$ vertices that contains no $T$ whose complement contains no $K_{1,n}$. Since $T$ has order $m$, we can take a disjoint union of $K^{m-1}$s to get a graph that is $T$-free (since $T$ is connected, it would have to be a subgraph of some component, but no component has enough vertices). Since $m-1 \mid n-1$, $m-1 \mid n+m-2 = (n-1)+(m-1)$, so we let $c = \frac{m+n-2}{m-1}$ and define

$$G = \bigcup_{1}^{c} K^{m-1}.$$

We have proven that $G$ is $T$-free, and are left to show that its complement contains no $K_{1,n}$. $G$ is $m-2$-regular, so $\overline{G}$ is $|\overline{G}| - 1 - (m-2) = (m+n-3) - (m-2) = n-1$-regular, and as we have observed in Extremal Graph Theory, a graph $\overline{G}$ is $K_{1,n}$-free iff $\Delta(G) \leq n-1$, so $G$ displays the lower bound.

For the upper bound, suppose a graph $G$ has at least $m+n-1$ vertices and $K_{1,n} \not\subseteq \overline{G}$. By the extremal observation above, $\Delta(\overline{G}) \leq n-1$. Then

$$\delta(G) = (|G|-1) - \Delta(\overline{G}) \geq (m+n-2) - (n-1) = m-1.$$

Therefore, it is enough to show that if $\delta(G) \geq m-1$ and $T$ has order $m$, then $T \subseteq G$. In Diestel's proofs this is done with a theorem from chapter 1, but we prove it directly here by induction on $m$. The base case is trivial. Let $x$ be a leaf of $T$. Name $x$'s neighbor $y$, and let $T' = T - x$. Then $T'$ is order $m-1$ and $\delta(G) \geq m-1 > m-2$, so by the induction hypothesis $T' \subseteq G$. Consider a mapping of vertices of $T'$ to vertices of $G$ that shows that $T'$ is a subgraph, and call $y'$ the vertex of $G$ to which $y$ is mapped. $d(y) \geq \delta(G) \geq m-1$, and $|T'| = m-1$, so $y$ has a neighbor to which $T'$ does not map any vertices, call it $x'$. Mapping $x$ to $x'$ (and all other vertices as in the mapping of $T'$ to $G$ proves $T \subseteq G$.

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**5, Diestel 9.16:** Show that given any two graphs $H_1$ and $H_2$, there exists a graph $G = G(H_1, H_2)$ such that, for every vertex-coloring of $G$ with colors 1 and 2, there is either an induced copy of $H_1$ colored 1 or an induced copy of $H_2$ colored 2 in $G$.

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Consider the graph $H$ which is the union of two disjoint copies of $H_1$ and $H_2$. By Theorem 9.3.1, there is a graph $G$ such that any 2-coloring of the edges of $G$ yields an induced copy of $H$ in color 1 or color 2. If $H$ occurs induced in color 1, then there is an induced $H_1$ in color 1. If $H$ occurs induced in color 2, then there is an induced $H_2$ in color 2, as desired.

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**6:** Let $V_n$ be the set of all binary n-tuples.

The hamming distance between a pair of n-tuples is the number of coordinates in which they differ. A right triangle is a set of 3 $n$-tuples such that the hamming distance between some pair is equal to the sum of the hamming distances between the other two pairs. Find, with proof, the smallest value of $n$ such that every 2-coloring of the edges of the complete graph on $V_n$ contains a right triangle with all three edges of the same color.

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We claim the smallest value of $n$ such that every 2-coloring of the edges of the complete graph on $V_n$ contains a right triangle with all three edges of the same color is 5.

For $n < 5$, color all edges of Hamming distance one or four red and all edges of Hamming distance two or three blue. Assume for the sake of contradiction this coloring contains a monochromatic right triangle. If the triangle is red, it contains only edges of length one and four, but this is impossible as $1+1, 1+4, 4+4 \notin \{1, 4\}$. If the triangle is blue, it contains only edges of length two and three, but this is also impossible as $2+2, 2+3, 3+3 \notin \{2, 3\}$. We conclude this coloring contains no monochromatic right triangle.

Now we prove every coloring of the complete graph on $V_5$ contains a monochromatic right triangle. Consider the vertices corresponding to the following six points:

$$(0,0,0,0,0), (1,0,0,0,0), (1,1,0,0,0), (1,1,1,0,0), (1,1,1,1,0), (1,1,1,1,1)$$

We claim any three of these points form a right triangle. This fact is clear because the difference between two of these coordinates is the difference in the number of coordinates that are set to 1 and each coordinate has a distinct number of 1's.

Note that these six vertices form a $K^6$ subgraph in $K^{V_5}$. As $R(3) = 6$, any coloring of $K^{V_5}$ must color this $K^6$ and this have a monochromatic triangle which, as argued above, is a right triangle.

We conclude $n = 5$.