Proofs in Diestel for Midterm 1 - We covered the following proofs in class, so you should review them.

Proposition 1.2.1. The number of vertices of odd degree in a graph is always even.

Proposition 1.3.1. Every graph G has a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$ (given that $\delta(G) \geqslant 2$).

Proposition 1.3.2. Every graph G containing a cycle satisfies $g(G) \leqslant 2\,\mathrm{diam}(G) + 1$.

Proposition 1.4.2. If G is non-trivial, then

$$\kappa(G) \leqslant \lambda(G) \leqslant \delta(G).$$

Corollary 1.5.3. A connected graph with n vertices is a tree if and only if it has $n - 1$ edges.

Proposition 1.6.1. A graph is bipartite if and only if it contains no odd cycle.

Theorem 1.8.1. **(Euler 1736)** A connected graph is Eulerian (i.e. has an Eulerian tour) if and only if all of its vertices have even degree.

Theorem 2.1.1. (König 1931) The maximum cardinality of a matching in a bipartite graph G is equal to the minimum cardinality of a vertex cover of its edges.

Theorem 2.1.2. (Hall 1935) A bipartite graph G with bipartition $\{A, B\}$ has a matching of A if and only if $|N(S)| \geqslant |S|$ for all $S \subseteq A$.

Corollary 2.1.3. Every k-regular bipartite graph ($k \geqslant 1$) has a 1-factor.

Corollary 2.1.5. (Petersen 1891) Every regular graph of positive even degree has a 2-factor.

Theorem 2.2.1. (Tutte 1947) A graph G has a 1-factor if and only if $q(G - S) \leqslant |S|$ for all $S \subseteq V(G)$.

Corollary 2.2.2. (Petersen 1891) Every bridgeless cubic graph (i.e. 3-regular graph) has a 1-factor.

Proposition 3.1.1. A graph is 2-connected if and only if it can be constructed from a cycle by successively adding H-paths to graphs H already constructed.

Lemma 3.1.2. Let G be any graph.

- Any cycle of G is contained in a single block of G.
- Any bond of G is contained in a single block of G.