Combinatorics Practice Exam

Please enjoy the following eight problems.

Q1. This problem counts various "labeled" trees on the vertex set $[100] = \{1, 2, \ldots, 100\}$.

(i) How many trees on $[100]$ are there with $\deg(1) = \deg(2) = 30$?

(ii) How many trees on $[100]$ have exactly two vertices of degree 30?

Q2. Recall that a *tournament* is an orientation of the edges of a complete graph, and that a tournament is *transitive* if whenever uv and vw are directed edges, so is uw. Construct a tournament having 7 vertices which has no transitive subtournament on 4 vertices.

Q3. Compute a "closed form" OGF for the sequence $a_n = \binom{2n}{n}$.

Q4. Suppose that $n \geq 2$. Compute the chromatic polynomial of the graph G_n obtained by deleting one edge from the complete graph K_n.

Q5. We work over the finite field $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$.

(i) Prove that no 1-error-correcting code $C \subseteq (\mathbb{F}_5)^6$ has more than 625 codewords.

(ii) Construct a matrix H whose kernel $C = \{\mathbf{v} \in (\mathbb{F}_5)^6 : H\mathbf{v} = \mathbf{0}\}$ is a 1-error-correcting code of size 625.

Q6. Suppose that $(P, \leq)$ is a finite poset with zeta function ζ. Recall that a *chain* from x to y in P is a sequence $x = v_0 < v_1 < \cdots < v_k = y$. Show that the total number of chains from x to y is given by $(2I - \zeta)^{-1}(x, y)$.

Q7. Suppose that $(P, \leq)$ is a finite poset with smallest and largest elements, and let μ denote its Möbius function. Show that $\sum_x \sum_y \mu(x, y) = 1$. Here the sum is taken over all $x, y \in P$.

Q8. This problem concerns integer partitions.

(i) Show that for all positive integers n, the number of partitions of n with no part of size 1 equals the number of partitions whose largest two parts have the same size.

(ii) Show that for all positive integers n, the number of partitions of n with no part of size 1 equals $p(n) - p(n-1)$.