Oddtown

A town of n people form clubs in such a way that every club has an odd number of people while every two clubs have an even (possibly 0) number of people in common. What is the maximum possible number of clubs that can be formed?

At most n clubs can be formed. The construction is simple: each person forms their own club. Now we prove that we can do no better. Number the people $1, \dots, n$, and suppose there are m clubs numbered $1, \dots, m$, and associate to each club a characteristic vector $v_i \in \mathbb{F}_2^n$ where the jth entry in v_i is 1 if and only if person j belongs to club i. Then the condition translates to

$$v_i \cdot v_i = 1 \quad \text{and} \quad v_i \cdot v_j = 0 \text{ for } i \neq j.$$

We want to show that m, the number of vectors (which correspond to clubs), cannot exceed n. Phrasing the combinatorial situation in the language of vectors reveals a natural way to approach this: We claim that $v_1, \dots, v_m$ are linearly independent. Indeed, the v_i form an orthonormal set, so $A^T A = I \in \mathbb{F}_2^{m \times m}$ where $A \in \mathbb{F}_2^{n \times m}$ is the matrix whose columns are the v_i. We deduce via the rank inequality that $\operatorname{rank} A \geq m$, implying that the $v_1, \dots, v_m$ are independent; thus $m \leq n$.

Eventown

A town of n people form clubs in such a way that every club has an even (possibly 0) number of people and every two clubs have an even (possibly 0) number of people in common. What is the maximum possible number of clubs that can be formed?

Though we've only changed one word in the problem statement, the bound on the number of clubs turns out now to be exponential! At most $2^{\lfloor n/2 \rfloor}$ clubs can be formed, achieved by having the n people partner up (possibly leaving one unmatched person whom we cavalierly throw out in the construction) and making the clubs all $2^{\lfloor n/2 \rfloor}$ subsets of the $\lfloor n/2 \rfloor$ pairs. Now we prove the bound, starting the same way as in our approach to oddtown. We consider the m characteristic vectors $v_1, \dots, v_m$ corresponding to the clubs. This time, the vectors satisfy the condition

$$v_i \cdot v_j = 0 \text{ for } \textit{all } i \text{ and } j.$$

Let $S \subseteq \mathbb{F}_2^n$ be the subspace spanned by $\{v_1, \dots, v_m\}$, and let $S^\perp$ be its orthogonal complement. The condition implies that $S \subseteq S^\perp$. In particular, $\dim S \leq \dim S^\perp$. Then we may conclude, via rank-nullity, that $\dim S \leq \lfloor n/2 \rfloor$. This implies the bound.

Mod p town

Let p be a prime. A town of n people form clubs in such a way that the number of members of each club is not divisible by p while the number of people in common between any two *clubs is divisible by p (possibly 0). What is the maximum possible number of clubs that can be formed?*

Notice that if $p = 2$, then this is just oddtown. Fortunately, the proof is exactly the same for other p. Define characteristic vectors $v_1, \ldots, v_m \in \mathbb{F}_p^n$ just as before which satisfy

$$v_i \cdot v_i \neq 0 \quad \text{and} \quad v_i \cdot v_j = 0 \text{ for } i \neq j.$$

Let $A \in \mathbb{F}_p^{n \times m}$ be the matrix whose columns are the v_i. Then $A^T A \in \mathbb{F}_p^{m \times m}$ is diagonal and in particular has rank m. Therefore the rank of A is at least m, implying that $\{v_1, \ldots, v_m\}$ is an independent set, so $m \leq n$. Equality holds when each club consists of a single member.

Mod p^k town

Let p be a prime and $1 \leq k \in \mathbb{N}$. A town of n people form clubs in such a way that the number of members of each club is not divisible by p^k while the number of people in common between any two clubs is divisible by p^k (possibly 0). What is the maximum possible number of clubs that can be formed?

Immediately the p^k modulus obstructs us from our usual approach since $\mathbb{Z}_{p^k}$ no longer plays nicely with division. This problem shows how our choice of field can be important: The idea is to work over $\mathbb{Q}$. Let $v_1, \ldots, v_m \in \mathbb{Q}^n$ be the characteristic vectors of the m clubs. Note that these are still 0-1 vectors, but we view them as elements of the vector space $\mathbb{Q}^n$. Analogous to previous iterations, we have

$$v_i \cdot v_i \neq 0 \pmod{p^k} \quad \text{and} \quad v_i \cdot v_j = 0 \pmod{p^k} \text{ for } i \neq j.$$

We claim that they are linearly independent over $\mathbb{Q}$. Indeed, suppose for the sake of contradiction that there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{Q}$, not all zero, which satisfy

$$\alpha_1 v_1 + \cdots + \alpha_m v_m = 0 \in \mathbb{Q}^n. \tag{$*$}$$

By multiplying through by a common denominator, we may assume without loss of generality that $\alpha_i \in \mathbb{N}$. Moreover, by dividing through by any common factors, we may further assume that $\gcd(\alpha_1, \ldots, \alpha_m) = 1$. Taking the dot product of v_1 and $(*)$ yields

$$\alpha_1 v_1 \cdot v_1 + \alpha_2 v_1 \cdot v_2 + \cdots + \alpha_m v_1 \cdot v_m = 0.$$

Note that p^k divides each term $\alpha_j v_1 v_j$ for $j \neq 1$. Therefore p^k must divide $\alpha_1 v_1 \cdot v_1$ as well. But then $p \mid \alpha_1$ since $v_1 \cdot v_1$ is not divisible by p^k. Of course, there is nothing special about α_1; by taking the dot product of $(*)$ and any v_i we can conclude that $p \mid \alpha_i$. This contradicts the assumption that $\gcd(\alpha_1, \ldots, \alpha_n) = 1$.

Reverse oddtown

A town of n people form clubs in such a way that every club has an even (possibly 0) number of people while every two clubs have an odd number of people in common. What is the maximum possible number of clubs that can be formed?

The setup is the same. Suppose there are m clubs with characteristic vectors $v_1, \ldots, v_m \in \mathbb{F}_2^n$ which satisfy

$$v_i \cdot v_i = 0 \quad \text{and} \quad v_i \cdot v_j = 1 \text{ for } i \neq j.$$

Let $A \in \mathbb{F}_2^{n\times m}$ be the matrix whose columns are the v_i. The condition implies the following:

- $\operatorname{rank} A \leq n-1$ since a priori $\operatorname{rank} A \leq n$ due to its dimension, but we also have $A^T\mathbf{1} = 0$ (where $\mathbf{1} \in \mathbb{F}_2^n$ is the all-ones vector) because each row v_i of A^T has an even number of 1s.

- $A^TA = J_m$, where $J_m \in \mathbb{F}_2^{m\times m}$ is the matrix with 0s on the diagonal and all 1s elsewhere.

Over $\mathbb{F}_2$, the matrix J_m satisfies the following nice property.

Lemma

If m is even, then $\operatorname{rank} J_m = m$. If m is odd, then $\operatorname{rank} J_m = m-1$.

Proof. Suppose first that m is even. We will show that the det J_m is nonzero. Indeed, using the transversal expansion of the determinant this follows directly from the fact that d_m, the mth derangement, is odd (it can be proven inductively that the derangements alternate in parity). Now suppose m is odd. Then the first $m-1$ columns of J_m are linearly independent since J_{m-1} is full rank, and it is easy to see that the mth column is the sum of the first $m-1$ columns. □

By the rank inequality and the observations made above, we have

$$n-1 \geq \operatorname{rank} A \geq \operatorname{rank}(A^TA) = \operatorname{rank} J_m \geq m-1,$$

whence $m \leq n$. If n is odd, it is possible to achieve exactly n clubs simply by considering J_n to be the incidence matrix. If n is even, then the bound above shows that $m = n$ is impossible, for $\operatorname{rank} J_m$ would be $m = n$ in this case, giving $n-1 \geq n$ which is absurd. However, it is possible to form $n-1$ clubs by copying the construction for an $n-1$ person town.